reals [1] and from this work and the remarks in the introduction, a rotund Banach space over the reals satisfies the Menelaus Property.

Conversely, from Lemma 1, each two distinct points of M lie on a unique metric line and consequently M has the two-triple property [2, Theorem 21.3]. We show M satisfies the Young Postulate.

Let p, q, r be noncollinear points of M and let q', r' be the respective midpoints of p and q and p and r. Select a sequence $\{r_n\}$ of points between p and r with $\lim r_n = r'$ and $pr_n/pr > 1/2$. From Lemma 2, we obtain a sequence $\{s_n\}$ of points with r_n between q' and s_n and r between q and s_n, $n = 1, 2, \ldots$, and

$$(s_n r_n/r_n q')(pq'/pq)(rq/rs_n) = 1 . \tag{1}$$

But $pq'/pq = 1/2$ and $\lim s_n r_n/rs_n = 1$ since

$$(rs_n - rr_n)/rs_n \leqslant s_n r_n/rs_n \leqslant (rs_n + rr_n)/rs_n$$

and the extreme sides this inequality have limit 1. It follows from this and (1) that $\lim rq/r_n q' = 2$. However, by the continuity of the metric, $\lim rq/r_n q' = rq/r'q' = 2$, which implies $q'r' = qr/2$. Thus M satisfies the Young Postulate, so by the result of Andalafte and Blumenthal M is a real Banach space.

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Properties of a function of E. Marczewski

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Abstract. Let $E \subset R$ be measurable set with the positive measure $|E|$. H. Steinhaus has proved that there exists an interval $(0, \delta)$ such that

$$\alpha \in (0, \delta) \Rightarrow E \cap (E+\alpha) \neq \emptyset$$

where $E+\alpha = \{x\colon x-\alpha \in E\}$. Let $(0, \delta_E)$ denote the maximal of those intervals.

Then the function φ defined by the formula

$$\varphi(x) = \inf\{\delta_E\colon E \subset \langle 0, 1\rangle \wedge |E| = x\} \quad \text{for} \quad 0 \leqslant x \leqslant 1$$

can be expressed explicitly by the formula

$$\varphi(x) = \begin{cases} 0 & \text{for} \quad 0 \leqslant x \leqslant \tfrac{1}{2}, \\ \dfrac{x}{n} & \text{for} \quad \dfrac{n+1}{2n+1} < x \leqslant \dfrac{n}{2n-1}, \quad n \in N. \end{cases}$$

H. Steinhaus has proved the following

THEOREM. *Let $E \subset R$ be a measurable set such that $|E| > 0$. Then there is a number $\delta > 0$ such that $E \cap (E+\alpha) \neq \emptyset$ for all $\alpha \in (0, \delta)$.*

Put

$$\delta(E) = \sup\{\delta\colon 0 < \alpha < \delta \Rightarrow E \cap (E+\alpha) \neq \emptyset\} \tag{1}$$

for any measurable set $E \subset R$. Unless otherwise stated all the sets under considerations will be Lebesgue measurable and $|E|$ will denote the Lebesgue measure of E. The following function was defined by E. Marczewski:

$$\varphi(x) = \operatorname{int}\{\delta(E)\colon E \subset \langle 0, 1\rangle \wedge |E| = x\} \quad \text{for} \quad x \in \langle 0, 1\rangle . \tag{2}$$

In this paper we investigate the properties of function (2).

LEMMA 1. *If $E \subset (0, 1)$ is a measurable set and $E \cap (E+\alpha) = \emptyset$ then $|\alpha| > 2|E| - 1$.*

Proof. We may assume that $\alpha>0$. Then $E\cup(E+\alpha)\subset\langle 0,1+\alpha\rangle$ and that implies

$$1+\alpha\geqslant|E\cup(E+\alpha)| = |E|+|E+\alpha| = 2|E|.$$

This completes the proof.

PROPERTY 1. *$\varphi(x)\geqslant 2x-1$ for $0\leqslant x\leqslant 1$.*

Proof. If $2x-1\leqslant 0$ (hence $x\leqslant\frac{1}{2}$), then the asserted inequality is obviously valid. Let $E\subset(0,1)$ be an arbitrary set such that $|E| = x>\frac{1}{2}$. For every $\alpha\in(0,2x-1)$ it follows by Lemma 1 that $E\cap(E+\alpha)\neq\emptyset$. Thus $\delta(E)\geqslant 2x-1$. Since the set E was arbitrary, $\varphi(x)\geqslant 2x-1$ and Property 1 is proved.

The inequality "$\geqslant$" in Property 1 cannot be replaced by the inequality "$>$" since there exist numbers x for which $\varphi(x) = 2x-1$. This is true for points of the form $n/(2n-1)$:

PROPERTY 2. *$\varphi(x_n) = 2x_n-1 = x_n/n$ for $x_n = n/(2n-1)$ where $n\in N$.*

Proof. Set

$$E_n = \bigcup_{k=0}^{n}\left(\frac{2k}{2n-1}, \frac{2k+1}{2n-1}\right) \quad \text{for} \quad n = 1,2,\ldots$$

Obviously $E_n\subset\langle 0,1\rangle$ and $|E_n| = x_n$. Since E_n contains the interval $(0,1/(2n-1))$ we have $E_n\cap(E+\alpha)\neq\emptyset$ for every number from this interval. On the other hand, sets E_n and $E_n+1/(2n-1)$ are disjoint, which implies $\delta(E_n) = 1/(2n-1)$. Thus $\varphi(x_n)\leqslant\delta(E_n) = 1/(2n-1) = 2x_n-1$. But for Property 1 we also have $\varphi(x_n)\geqslant 2x_n-1$. Thus Property 2 is proved.

Remark. The proof of Property 2 follows from the inequality $\delta(E_n)\leqslant 1/(2n-1)$ only. The equality $\delta(E_n) = 1/(2n-1)$ with $|E_n| = n/(2n-1)$ proves the existence of sets for which $\varphi(|E|) = \delta(E)$.

It is obvious that the function (2) is nondecreasing, but it is not so trivial that the function $\psi(x) = \varphi(x)/x$ has the same property.

LEMMA 2. *If $0<x_1<x_2\leqslant 1$ then $\varphi(x_1)\colon x_1\leqslant\varphi(x_2)\colon x_2$.*

Proof. We first observe that if $E\subset(0,1)$ and $\Theta\in(0,1)$ then $E\cdot\Theta\subset(0,1)$, $|E\cdot\Theta| = |E|\cdot\Theta$ and $\delta(E\cdot\Theta) = \Theta\cdot\delta(E)$. Let $\Theta = x_1\colon x_2$. Thus

$$\begin{aligned}\varphi(x_1) &= \inf\{\delta(E)\colon E\subset(0,1), |E| = x_1\}\\ &\leqslant\inf\{\delta(\Theta E)\colon E\subset(0,1), |E| = x_2\}\\ &= \inf\{\Theta\delta(E)\colon E\subset(0,1), |E| = x_2\}\\ &= \Theta\cdot\varphi(x_2) = \frac{x_1}{x_2}\varphi(x_2).\end{aligned}$$

The lemma is proved.

PROPERTY 3. *The function (2) is nondecreasing on $\langle 0,1\rangle$. Moreover, if $\varphi(x_0)>0$ and $0<x_0<1$, then φ is increasing on $(x_0, 1)$.*

Property 3 is an immediate consequence of Lemma 2.

PROPERTY 4. *If $n\in N$ then $\varphi(x)\leqslant x/n$ for $0\leqslant x\leqslant x_n = n/(2n-1)$.*

Proof. It follows from Property 3 that $\psi(x)\leqslant\psi(x_n)$ for $0<x\leqslant x_n$. Since by Property 2 $\psi(x_n) = \varphi(x_n)\colon x_n = 1/n$, we have $\varphi(x)\colon x = \psi(x)\leqslant\psi(x_n) = 1/n$ and therefore $\varphi(x)\leqslant x/n$ for $x\leqslant x_n$.

LEMMA 3. *If $E\subset(0,1)$, $\alpha\geqslant 1/2n$ where $n\in N$ and $E\cap(E+\alpha) = \emptyset$ then $|E|\leqslant n\cdot\alpha$.*

Proof. Let I_k denote the interval $\langle\alpha k, \alpha(k+1)\rangle$. The sets E and $E+\alpha$ are disjoint and thus $(I_k\cap E)\cap(I_k\cap(E+\alpha)) = \emptyset$. Therefore

$$\begin{aligned}|E\cap(I_{2k-2}\cup I_{2k-1})| &= |E\cap I_{2k-2}|+|E\cap I_{2k-1}|\\ &= |(E+\alpha)\cap I_{2k-1}|+|E\cap I_{2k-1}|\\ &= |(E\cup(E+\alpha))\cap I_{2k-1}|\leqslant\alpha.\end{aligned}$$

Since for $\alpha\geqslant 1/2n$ we have $(0,1)\subset\bigcup_{k=1}^{n}(I_{2k-2}\cup I_{2k-1})$, it immediately follows that if $E\subset(0,1)$ then $E = \bigcup_{k=1}^{n}(E\cap(I_{2k-2}\cup I_{2k-1}))$. Thus

$$|E| = \sum_{k=1}^{n}|E\cap(I_{2k-2}\cup I_{2k-1})|\leqslant n\alpha.$$

COROLLARY 1. *If $\delta(E)\geqslant 1/(2n-1)$ for $E\subset(0,1)$, then $|E|\leqslant n\cdot\delta(E)$.*

Proof. $\delta(E)$ was defined by (1) as the supremum of such numbers δ that $E\cap(E+\alpha)\neq\emptyset$ while $0\leqslant\alpha<\delta$. Thus there exists such a nondecreasing sequence α_k with the limit $\delta(E)$ that $E\cap(E+\alpha_k) = \emptyset$ for $k = 1,2,\ldots$ Since $\alpha_k\geqslant\delta(E)\geqslant 1/2n$, it follows from Lemma 3 that $|E|\leqslant n\cdot\alpha_k$. By letting $k\to\infty$, we obtain $|E|\leqslant\cdot\delta(E)$ and the corollary is proved.

PROPERTY 5. *For every number $x\leqslant x_n = n/(2n-1)$, where $n\in N$, the inequality $\varphi(x)\geqslant 1/2n$ implies $\varphi(x) = x/n$.*

Proof. Since, for Property 4, $\varphi(x)\leqslant x/n$, it remains to prove the inequality $\varphi(x)\geqslant x/n$. Let $E\subset(0,1)$ and $|E| = x$. The assumption $\varphi(x)\geqslant 1/2n$ implies $\delta(E)\geqslant 1/2n$. Therefore by Corollary 1 it follows that $\delta(E)\geqslant|E|/n = x/n$. This implies the inequality

$$\varphi(x) = \inf\{\delta(E)\colon E\subset(0,1), E = |x|\}\geqslant\frac{x}{n},$$

which completes the proof of Property 5.

LEMMA 4. *If $E\subset(0,1)$, $0<\alpha<1/2n$, where $n\in N$ and $E\cap(E+\alpha) = \emptyset$, then $|E|\leqslant 1-n\alpha$.*

Proof. Since this proof is similar to that of Lemma 3, we will only write the following relations:

$$(0,1) = (0,2n\alpha)\cup\langle 2n\alpha, 1\rangle$$

and

$$|E| = |E\cap(0,2n\alpha)|+|E\cap(2n\alpha,1)|$$
$$= |E\cap(2n\alpha,1)|+\sum_{k=1}^{n}|E\cap(I_{2k-1}\cup I_{2k-2})|$$
$$\leqslant n\alpha+1-2n\alpha = 1-n\alpha .$$

COROLLARY 2. *If* $\delta(E)<1/2n$ *for a set* $E\subset(0,1)$ *then* $\delta(E)\leqslant(1-|E|)/n$.

Proof. Since $\delta(E)<1/2n$, there exists a number α with $\delta(E)<\alpha<1/2n$ and $E\cap(E+\alpha)=\emptyset$. Therefore $|E|\leqslant 1-n\alpha\leqslant 1-n\delta(E)$ and this implies $\delta(E)\leqslant(1-|E|)/n$.

PROPERTY 6. *If* $\varphi(x)<1/2n$ *for some* $x\in(0,1)$, *then* $\varphi(x)\leqslant(1-x)/n$.

Proof. Let $\varphi(x)<1/2n$. Then there exists such a set $E_0\subset(0,1)$ with $|E_0|=x$ that $\delta(E_0)<1/2n$. Then, by Corollary 2, it follows that $\delta(E_0)\leqslant(1-|E_0|)/n$ and therefore

$$\varphi(x)\leqslant\delta(E_0)\leqslant\frac{1-|E_0|}{n}=\frac{1-x}{n}.$$

Property 6 is proved.

We have obtained a collection of properties of the function (2), or more precisely some inequalities for the values $\varphi(x)$. In what follows we will prove that this collection of properties characterizes the function (2) and permits us to find an explicit expression for it. This results from the following theorem and its proof.

THEOREM. *The function* (2) *has the following expression*:

$$\varphi(x)=\begin{cases}0 & \text{for}\quad 0\leqslant x\leqslant\tfrac12,\\ \dfrac{x}{n} & \text{for}\quad \dfrac{n+1}{2n+1}<x\leqslant\dfrac{n}{2n-1},\ n\in N.\end{cases} \tag{3}$$

Proof. (A) For points of the form $x_n=n/(2n-1)$, (3) means the same as Property 2.

(B) Let $x_{n+1}<x<x_n$, where $x_k=k/(2k-1)$.

Since $\varphi(x_{n+1})=1/(2n+1)$, $\varphi(x_n)=1/(2n-1)$, the monotonicity of φ (Property 3) implies that

$$\frac{1}{2n+1}<\varphi(x)<\frac{1}{2n-1}.$$

Case 1. $\varphi(x)<1/2n$. In this case it follows from Property 6 that $\varphi(x)\leqslant(1-x)/n$. On the other hand, Property 1 implies that $\varphi(x)\geqslant 2x-1$. Thus $2x-1\leqslant(1-x)/n$, so that $x\leqslant(n+1)/(2n+1)$. Therefore Case 1 cannot occur.

Case 2. $\varphi(x)\geqslant 1/2n$. In this case Property 5 implies $\varphi(x)=x/n$. Since only Case 2 is possible, we have $\varphi(x)=x/n$ for $x_{n+1}<x\leqslant x_n$.

C) Let $0\leqslant x\leqslant\frac12$. For the monotonicity of φ we obtain $\varphi(x)\leqslant\varphi(x_n)\leqslant 1/2n$ for every $n\in N$. $\varphi(x)\leqslant 0$ and, because $\varphi(x)\geqslant 0$, it follows that $\varphi(x)=0$. The theorem is proved.

Remark. The assumption that $|E|$ denotes the Lebesgue measure is not necessary in all the foregoing definitions and properties. It can be replaced by the assumption that $|E|$ denotes an arbitrary additive and homogeneous measure such that $|(a,b)|=b-a$. In fact this extension of the considerations brings nothing new.

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