Properties of a function of E. Marczewski

by

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Abstract. Let $E \subseteq R$ be measurable set with the positive measure $|E|$. H. Steinhaus has proved that there exists an interval $(0, \delta)$ such that

$$\alpha \ast (0, \delta) = E \cap (E+\alpha) \neq \emptyset$$

where $E+\alpha = (x: x+\alpha \in E)$. Let $(0, \Delta)$ denote the maximal of those intervals.

Then the function $\varphi$ defined by the formula

$$\varphi(x) = \inf \{\delta: E \subset (0, 1) \land |E|=x\} \quad 0 < x < 1$$

can be expressed explicitly by the formula

$$\varphi(x) = \begin{cases} 0 & \text{for } 0 < x < 1, \\ \frac{x}{n} & \text{for } \frac{n}{2n+1} < x < \frac{n}{2n-1}, n \in N. \end{cases}$$

H. Steinhaus has proved the following

THEOREM. Let $E \subseteq R$ be a measurable set such that $|E| > 0$. Then there is a number $\delta > 0$ such that $E \cap (E+\alpha) \neq \emptyset$ for all $\alpha \in (0, \delta)$.

Put

$$\delta(E) = \sup \{\delta: 0 < \alpha < \delta \Rightarrow E \cap (E+\alpha) \neq \emptyset\}$$

for any measurable set $E \subseteq R$. Unless otherwise stated all the sets under considerations will be Lebesgue measurable and $|E|$ will denote the Lebesgue measure of $E$. The following function was defined by E. Marczewski:

$$\varphi(x) = \inf \{\delta(E): E \subset (0, 1) \land |E|=x\} \quad \text{for } x \in (0, 1).$$

In this paper we investigate the properties of function (2).

**Lemma 1.** If $E \subset (0, 1)$ is a measurable set and $E \cap (E+\alpha) = \emptyset$ then $|E| > 2|E|-1$. 

References


Proof. We may assume that \( a > 0 \). Then \( E \cup (E + a) \subset \langle 0, 1 + a \rangle \) and that implies

\[
1 + a \geq |E \cup (E + a)| = |E| + |E + a| = 2|E|.
\]

This completes the proof.

**Property 1.** \( \varphi(x) \geq 2x - 1 \) for \( 0 \leq x \leq 1 \).

Proof. If \( 2x - 1 \leq 0 \) (hence \( x \leq \frac{1}{2} \)), then the asserted inequality is obviously valid. Let \( E = (0, 1) \) be an arbitrary set such that \( |E| = x \geq \frac{1}{2} \). For every \( a \in (0, 2x - 1) \) it follows by Lemma 1 that \( E \cap (E + a) \neq \emptyset \). Thus \( \delta(E) \geq 2x - 1 \). Since the set \( E \) was arbitrary, \( \varphi(x) \geq 2x - 1 \) and Property 1 is proved.

The inequality "\( \geq \)" in Property 1 cannot be replaced by the inequality "\( > \)" since there exist numbers \( x \) for which \( \varphi(x) = 2x - 1 \). This is true for points of the form \( n/(2n - 1) \):

**Property 2.** \( \varphi(x_n) = 2x_n - 1 = x_n/n \) for \( x_n = n/(2n - 1) \) where \( n \in N \).

Proof. Set

\[
E_n = \bigcup_{k=1}^{n} \left( \frac{2k}{2n-1}, \frac{2k+1}{2n-1} \right)
\]

for \( n = 1, 2, ... \)

Obviously \( E_n \subset (0, 1) \) and \( |E_n| = x_{n} \). Since \( E_n \) contains the interval \( (0, 1/(2n-1)) \) we have \( E_n \cap (E + a) \neq \emptyset \) for every number from this interval. On the other hand, sets \( E_n \) and \( x_{n+1}/(2n-1) \) are disjoint, which implies \( \delta(E_n) = 1/(2n-1) \). Thus \( \varphi(x_n) = \delta(E_n) = 1/(2n-1) = 2x_n - 1 \). But for Property 1 we also have \( \varphi(x_n) \geq 2x_n - 1 \). Thus Property 2 is proved.

**Remark.** The proof of Property 2 follows from the inequality \( \delta(E_n) < 1/(2n-1) \) only. The only the equality \( \delta(E) = 1/(2n-1) \) with \( |E| = n/(2n-1) \) proves the existence of sets for which \( \varphi(E) = \delta(E) \).

It is obvious that the function \( \varphi(x) = \varphi(x)/x \) has the same property.

**Lemma 2.** If \( 0 < x_1 < x_2 < 1 \) then \( \varphi(x_1) \leq \varphi(x_2) \).

Proof. We first observe that if \( E = (0, 1) \) and \( \Theta \subset (0, 1) \) then \( E - \Theta \subset (0, 1) \), \( |E - \Theta| = |E| - \Theta \) and \( \delta(E - \Theta) = \Theta \cdot \delta(E) \). Let \( \Theta = x_1 \cdot x_2 \) and \( \varphi(x) = \inf \{ \delta(E): E = (0, 1), |E| = x \} \)

\[
\varphi(x) = \inf \{ \delta(E): E = (0, 1), |E| = x \} = \Theta \cdot \varphi(x_1) = \frac{x_1}{x_2} \varphi(x_1).
\]

The lemma is proved.

**Property 3.** The function \( \varphi(x) \) is nondecreasing on \( (0, 1) \). Moreover, if \( \varphi(x_0) > 0 \) and \( 0 < x_0 < 1 \), then \( \varphi(x) \) is increasing on \( (x_0, 1) \).

Property 3 is an immediate consequence of Lemma 2.

**Property 4.** If \( n \in N \) then \( \varphi(x) < x/n \) for \( 0 < x < x_n = n/(2n-1) \).

Proof. If it follows from Property 3 that \( \varphi(x_0) < x_0/n \) for \( 0 < x < x_n \). Since by Property 2 \( \varphi(x_0) = \varphi(x_1) = 1/n \), we have \( \varphi(x) = \varphi(x_0) < x_0/n \) and therefore \( \varphi(x) < x/n \) for \( x < x_n \).

**Lemma 3.** If \( E \subset (0, 1) \), \( \alpha > 1/2n \) where \( n \in N \) and \( E \cap (E + \alpha) = \emptyset \) then \( |E| < n/\alpha \).

Proof. Let \( I \) denote the interval \( (\alpha/2, (\alpha/2 + 1)) \). The sets \( E \) and \( E + \alpha \) are disjoint and thus \( I \cap E \cap (I \cap (E + \alpha)) = \emptyset \). Therefore

\[
|E \cap (I_{2k-1} \cup I_{2k-2})| = |E \cap I_{2k-1}| + |E \cap I_{2k-2}|
\]

\[
= |I_{2k-1} + I_{2k-2}| + |E \cap I_{2k-1}|
\]

\[
= |(E + \alpha) \cap I_{2k-1}| + |E \cap I_{2k-2}|
\]

Since \( \alpha > 1/2n \) we have \( (0, 1) \subset \bigcup_{k=1}^{n} (I_{2k-1} \cup I_{2k-2}) \), it immediately follows that if \( E \subset (0, 1) \) then \( E \subset \bigcup_{k=1}^{n} (I_{2k-1} \cup I_{2k-2}) \). Thus

\[
|E| = \sum_{k=1}^{n} |E \cap (I_{2k-1} \cup I_{2k-2})| < n/\alpha.
\]

**Corollary 1.** If \( \delta(E) \geq 1/(2n-1) \) for \( E \subset (0, 1) \), then \( |E| < n/\delta(E) \).

Proof. \( \delta(E) \) was defined by \( 0 < \alpha < \delta(E) \) while \( 0 < x < \delta(E) \). Thus there exist such \( a \) a nondecreasing sequence \( a_k \) with the limit \( \delta(E) \) that \( E \cap (E + a_k) = \emptyset \) for \( k = 1, 2, ... \). Since \( a_k \delta(E) \geq 1/2n \), it follows from Lemma 3 that \( |E| < n/\delta(E) \). By letting \( k \to \infty \), we obtain \( |E| < \delta(E) \) and the corollary is proved.

**Property 5.** For every number \( x \leq x_n = n/(2n-1) \), where \( n \in N \), the inequality \( \varphi(x) \geq 1/2n \) implies \( \varphi(x) > x/n \).

Proof. Since, for Property 4, \( \varphi(x) < x/n \) it remains to prove the inequality \( \varphi(x) > x/n \). Let \( E = (0, 1) \) and \( |E| = x \). The assumption \( \varphi(x) > x/n \) implies \( \delta(E) \geq 1/2n \). Therefore by Corollary 1 it follows that \( \delta(E) \geq |E|/n = x/n \). This implies the inequality

\[
\varphi(x) = \inf \{ \delta(E): E : (0, 1), |E| = x \} \geq x/n
\]

which completes the proof of Property 5.

**Lemma 4.** If \( E \subset (0, 1) \), \( 0 < a < 1/2n \), where \( n \in N \) and \( E \cap (E + a) = \emptyset \), then \( |E| < 1/\alpha \).

Proof. Since this proof is similar to that of Lemma 3, we will only write the following relations:

\[
(0, 1) = (0, 2na) \cup (2na, 1)
\]
and

\[ |E| = |E \cap (0, 2n] + |E \cap (2n, 1)] \]
\[ = |E \cap (2n, 1)] + \sum_{k=1}^{n} |E \cap (I_{2k-1} \cup I_{2k-2})| \]
\[ \leq \frac{1}{n} + 2 - 2n = 1 - \frac{1}{n}. \]

**Corollary 2.** If \( \delta(E) < 1/2n \) for a set \( E \subset (0, 1) \) then \( \delta(E) \leq (1-|E|)/|n| \).

**Proof.** Since \( \delta(E) < 1/2n \), there exists a number \( x \) with \( \delta(E) = n < 1/2n \) and \( E \cap (E+x) = \emptyset \). Therefore \( |E| \leq 1 - nx \leq 1 - n\delta(E) \) and this implies \( \delta(E) \leq (1-|E|)/|n| \).

**Property 6.** If \( \phi(x) < 1/2n \) for some \( x \in (0, 1) \), then \( \phi(x) \leq (1-x)/n \).

**Proof.** Let \( \phi(x) < 1/2n \). Then there exists such a set \( E_0 \subset (0, 1) \) with \( |E_0| = x \) that \( \delta(E_0) < 1/2n \). Then, by Corollary 2, it follows that \( \delta(E_0) \leq (1-|E_0|)/|n| \) and therefore

\[ \phi(x) \leq \delta(E_0) \leq \frac{1-|E_0|}{n} = \frac{1-x}{n}. \]

Property 6 is proved.

We have obtained a collection of properties of the function (2), or more precisely some inequalities for the values \( \phi(x) \). In what follows we will prove that this collection of properties characterizes the function (2) and permits us to find an explicit expression for it. This results from the following theorem and its proof.

**Theorem.** The function (2) has the following expression:

\[ \phi(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq \frac{1}{2}, \\ \frac{x}{n} & \text{for } \frac{n+1}{2n+1} \leq x \leq \frac{n}{2n-1}, \ n \in \mathbb{N}. \end{cases} \]

**Proof.** (A) For points of the form \( x_n = n/(2n-1) \), (3) means the same as Property 2.

(B) Let \( x_{n+1} < x < x_n \), where \( x_n = k/(2k-1) \).

Since \( \phi(x_{n+1}) = 1/(2n+1) \), \( \phi(x_n) = 1/(2n-1) \), the monotonicity of \( \phi \) (Property 3) implies that

\[ \frac{1}{2n+1} \leq \phi(x) \leq \frac{1}{2n-1}. \]

**Case 1.** \( \phi(x) < 1/2n \). In this case it follows from Property 6 that \( \phi(x) \leq (1-x)/n \).

On the other hand, Property 1 implies that \( \phi(x) \geq 2x - 1 \). Thus \( 2x - 1 \leq (1-x)/n \), so that \( x \leq (n+1)/(2n+1) \). Therefore Case 1 cannot occur.

**Case 2.** \( \phi(x) \geq 1/2n \). In this case Property 5 implies \( \phi(x) = x/n \). Since only Case 2 is possible, we have \( \phi(x) = x/n \) for \( x_{n+1} < x < x_n \).

C) Let \( 0 \leq x \leq \frac{1}{2} \). For the monotonicity of \( \phi \) we obtain \( \phi(x) \leq \phi(x_n) = 1/2n \) for every \( n \in \mathbb{N} \). \( \phi(x) \leq 0 \) and, because \( \phi(x) \geq 0 \), it follows that \( \phi(x) = 0 \). The theorem is proved.

**Remark.** The assumption that \( |E| \) denotes the Lebesgue measure is not necessary in all the foregoing definitions and properties. It can be replaced by the assumption that \( |E| \) denotes an arbitrary additive and homogeneous measure such that \( |a, b| = b - a \). In fact this extension of the considerations brings nothing new.

**Reference**


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4 — *Fundamenta Mathematicae C*