The Menelaus Property characterizes real rotund Banach spaces

by

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Abstract. A metric space satisfies the *Menelaus Property* provided for each triple of noncollinear points p, q, r, if q' and r' are between p and q and p and r respectively, and if s is collinear with q and r but not between them, then q', r' and s are collinear if and only if

$$(pr'/rr')(rs/sq)(qq'/pq') = 1 .$$

The main result of the paper is that a complete, convex, externally convex, metric space is a rotund Banach space over the reals if and only if it satisfies the Menelaus Property.

1. Introduction. The Theorem of Menelaus and its converse are of importance in euclidean plane geometry. These theorems state that three points on the sides of a triangle are collinear if and only if the product of the signed ratios in which the sides are divided by those three points is -1. Andalafte and Blumenthal [1], in the process of characterizing real Banach spaces, showed that a special case of the Theorem of Menelaus is a consequence of the Young Postulate which may be stated as follows.

THE YOUNG POSTULATE. If p, q, r are points of a metric space and if q', r', are the respective midpoints of p and q and p and r, then $q'r' = qr/2$.

Andalafte and Blumenthal proved the following special case of the Theorem of Menelaus is valid in a complete, convex, externally convex, metric space with the two-triple property which satisfies the Young Postulate.

THEOREM (Menelaus). *If p, q, r are noncollinear points, if q', r', are points between p and q and p and r, respectively, and if s is a point collinear with q and r such that q', r', s are collinear, then* $(pr'/rr')(rs/sq)(qq'/pq') = 1$.

Letting p, q, r be vertices of an equilateral triangle in the euclidean plane and q', r', s the respective midpoints of p and q, p and r and r and q we have

$$(pr'/rr')(rs/sq)(qq'/pq') = 1 ,$$

but q', r', s are not collinear. Thus the converse of the above theorem is not valid, even in a euclidean space. The reason for this is the fact that they were working in

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a metric space and did not consider signed distances. Even though the situation looks hopeless, we do the same in this paper.

First we obseive that the Andalafte–Blumenthal proof of the above theorem [1, p. 32] actually shows that s is not between q and r. Moreover, it is an easy exercise to show that their theorems of Section 3 imply the following converse of the Theorem of Menelaus.

THEOREM. *If p, q, r are noncollinear points, if q', r' are points between p and q and p and r, respectively, and if s is a point collinear with q and r but not between them such that*

$$(pr'/rr')(rs/sq)(qq'/pq') = 1\,,$$

then q', r' and s are collinear.

Thus we postulate the following version of the Theorem of Menelaus and its converse. We then show a complete, convex, convex, externally convex, metric space is a Banach space over the reals if and only if it has that property. We accomplish this by showing it is equivalent to the Young Postulate.

THE MENELAUS PROPERTY. If p, q, r are noncollinear points of a metric space, if q' and r' are points between p and q and p and r, respectively, and if s is collinear with q and r but not between them, then q', r', and s are collinear if and only if

$$(pr'/rr')(rs/sq)(qq'/pq') = 1\,.$$

2. The characterization. Throughout the remainder of this paper, M will denote a complete, convex, externally convex, metric space which satisfies the Menelaus Property.

LEMMA 1. *Each two points of M lie on a unique metric line.*

Proof. Since M is complete, convex, and externally convex each two distinct points of M lie on at least one metric line. Suppose M contains a pair of distinct points which do not lie on a unique line. Then distinct points p, r', q, r can be found such that r' is a midpoint of p and q and r' is a midpoint of p and r, for example, see [3]. Let q' be a midpoint of p and r' and let $r = s$. Clearly p, q, r and noncollinear points and the points, q' and r' are between p and q and p and r, respectively, q', r', and s are collinear, s is not between q and r, but

$$(pr'/rr')(rs/sq)(qq'/pq') = 0\,.$$

This contradiction completes the proof.

LEMMA 2. *If p, q, r are noncollinear points of M and if q', r' are points between p and q and p and r, respectively, with*

$$1/2 = pq'/pq < pr'/pr\,,$$

then the line joining q' and r' and the line joining q and r have a common point s and r' is between q' and s and r is between q and s.

Proof. Define a continuous real-valued function f on the half-line $H = \{x\colon r \text{ is between } q \text{ and } x \text{ or } x = r\}$ by

$$f(x) = (pr'/rr')(qq'/pq')(rx/qx)\,.$$

Since $pr'/pr > 1/2$, it follows that $pr'/rr' > 1$. Moreover, since r is between q and x, or $r = x$, $\lim_{rx\to+\infty} rx/qx = 1$. Consequently, f is continuous on the connected set H and $\lim_{rx\to+\infty} f(x) > 1$ and $f(r) = 0$. Therefore H contains a point s such that

$$(pr'/rr')(qq'/pq')(rs/qs) = 1\,.$$

Clearly q', r' and s are distinct, r is between q and s, and by the Menelaus Property, q', r' and s are collinear. We show r' is between q' and s by way of contradiction. Two cases must be considered.

Case 1. The point s is between q' and r'.

Now q', p, r' are noncollinear. Consider the real-valued function g defined by

$$g(x) = (r's/sq')(qq'/pq)(xp/r'x)-1$$

which is continuous on the segment joining p and r' except at r'. Since $g(p)<0$ and $g(x)>0$ for points close to r', there is a point t between r' and p such that

$$g(t) = (r's/sq')(qq'/pq)(tp/r't)-1 = 0\,.$$

By the Property of Menelaus, q, s and t are collinear. But this means the line joining q and s and the line joining p and r' have the distinct points t and r in common contrary to Lemma 1. Thus s is not between q' and r'.

Case 2. The point q' is between r' and s.

The points r', r, s are noncollinear. Similar to Case 1, we see the real valued function h defined by

$$h(x) = (sq'/q'r')(r'p/rp)(xr/xs)-1$$

is continuous on the segment joining r and s, except at s, $g(r)<0$, $g(x)>0$ for x close to s, so there is a point u between r and s such that

$$(sq'/q'r')(r'p/rp)(ur/us) = 1\,.$$

But now the line joining r and s and the line joining p and q' have the distinct points q and u in common contradicting Lemma 1. Therefore q' is not between r' and s.

Since one of the distinct collinear points q', r', s is between the other two, we may conclude r' is between q' and s.

THEOREM. *A complete, convex, externally convex, metric space is a rotund Banach space over the reals if and only if it satisfies the Menelaus Property.*

Proof. Since the Young Postulate implies a complete, convex, externally convex, metric space with the two-triple property is a Banach space over the reals [1] and from this work and the remarks in the introduction, a rotund Banach space over the reals satisfies the Menelaus Property.

Conversely, from Lemma 1, each two distinct points of M lie on a unique metric line and consequently M has the two-triple property [2, Theorem 21.3]. We show M satisfies the Young Postulate.

Let p, q, r be noncollinear points of M and let q', r' be the respective midpoints of p and q and p and r. Select a sequence $\{r_n\}$ of points between p and r with $\lim r_n = r'$ and $pr_n/pr > 1/2$. From Lemma 2, we obtain a sequence $\{s_n\}$ of points with r_n between q' and s_n and r between q and s_n, $n = 1, 2, \ldots$, and

$$(1)\qquad (s_n r_n/r_n q')(pq'/pq)(rq/rs_n) = 1 .$$

But $pq'/pq = 1/2$ and $\lim s_n r_n/rs_n = 1$ since

$$(rs_n - rr_n)/rs_n \leqslant s_n r_n/rs_n \leqslant (rs_n + rr_n)/rs_n$$

and the extreme sides this inequality have limit 1. It follows from this and (1) that $\lim rq/r_n q' = 2$. However, by the continuity of the metric, $\lim rq/r_n q' = rq/r'q' = 2$, which implies $q'r' = qr/2$. Thus M satisfies the Young Postulate, so by the result of Andalafte and Blumenthal M is a real Banach space.

References

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Properties of a function of E. Marczewski

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Abstract. Let $E \subset R$ be measurable set with the positive measure $|E|$. H. Steinhaus has proved that there exists an interval $(0, \delta)$ such that

$$\alpha \in (0, \delta) \Rightarrow E \cap (E+\alpha) \neq \emptyset$$

where $E+\alpha = \{x\colon x-\alpha \in E\}$. Let $(0, \delta_E)$ denote the maximal of those intervals. Then the function φ defined by the formula

$$\varphi(x) = \inf\{\delta_E\colon E \subset \langle 0, 1\rangle \wedge |E| = x\} \quad \text{for} \quad 0 \leqslant x \leqslant 1$$

can be expressed explicitly by the formula

$$\varphi(x) = \begin{cases} 0 & \text{for} \quad 0 \leqslant x \leqslant \tfrac{1}{2}, \\ \dfrac{x}{n} & \text{for} \quad \dfrac{n+1}{2n+1} < x \leqslant \dfrac{n}{2n-1}, \end{cases} \quad n \in N.$$

H. Steinhaus has proved the following

THEOREM. *Let $E \subset R$ be a measurable set such that $|E| > 0$. Then there is a number $\delta > 0$ such that $E \cap (E+\alpha) \neq \emptyset$ for all $\alpha \in (0, \delta)$.*

Put

$$(1)\qquad \delta(E) = \sup\{\delta\colon 0 < \alpha < \delta \Rightarrow E \cap (E+\alpha) \neq \emptyset\}$$

for any measurable set $E \subset R$. Unless otherwise stated all the sets under considerations will be Lebesgue measurable and $|E|$ will denote the Lebesgue measure of E. The following function was defined by E. Marczewski:

$$(2)\qquad \varphi(x) = \operatorname{int}\{\delta(E)\colon E \subset \langle 0, 1\rangle \wedge |E| = x\} \quad \text{for} \quad x \in \langle 0, 1\rangle .$$

In this paper we investigate the properties of function (2).

LEMMA 1. *If $E \subset (0, 1)$ is a measurable set and $E \cap (E+\alpha) = \emptyset$ then $|\alpha| > 2|E| - 1$.*