Enlargements of Boolean algebras and Stone space

by

H. Gonshor (New Brunswick, N. J.)

Abstract. We study enlargements of structures which contain both a Boolean algebra and its Stone space. The main results are that duality is not preserved, but comes close to being preserved by taking enlargements.

Introduction. In [1] the author studied enlargements of Boolean algebras showing, for example, that such enlargements always contain completions as subquotients. In [2] the author proved an analogous result for projective covers of compact Hausdorff spaces. These results suggest that it may be of interest to study enlargements of structures which contain both a Boolean algebra and its Stone space. The natural question to ask is to what extent the duality is preserved. Since the power set is not absolute, i.e. external sets exist, the question is non-trivial. In a way which will be made precise in the paper, we shall see that duality is not preserved but it comes close to being preserved.

It also appears to be worthwhile to extend this study to other types of pairings, e.g. topological groups and their character groups. Some work along these lines appear in [3, Section 8] where the nonstandard hull of a normed space is compared with the nonstandard hull of its conjugate space.

We assume that the reader is familiar with the basic properties of Stone duality for Boolean algebras.

Section I. Elementary results. Let M be a structure containing an infinite Boolean algebra B, its Stone space X, and the integers N. We use the notation (x, b) for the usual pairing. It is well known (e.g. see [5]) that any enlargement M^* contains in a natural way an enlargement B^* of B, X^* of X, and N^* of N. (Although only B is really needed, it is convenient to have the structure as above.)

Many facts follow immediately from transfer. For example we have that B^* is a Boolean algebra. Furthermore we have:

THEOREM 1. *There is an internal one-one correspondence between the elements of X^* and internal homomorphisms of B^* into* $(0, 1)$. *Thus X^* is a subset of the Stone space of B^*.*

Note. Many other pairings which occur in mathematics have an infinite range, e.g. the set of complex numbers Z. In such a case the corresponding result must use an enlargement of the original range.

Let $S(B^*)$ be the Stone space of B^*. We want to compare X^* and $S(B^*)$. We shall use the *open subsets of X^* as a basis for the topology on X^*.

THEOREM 2. *The topology on X^* is generated by the *clopen sets.*

Proof. Every open set in X is a union of clopen sets. The result follows by transfer.

Note that although X is of course a subset of X^* it is not a subspace! In fact, the following result follows immediately from concurrency.

THEOREM 3. *If X is a T_1 space and if the *open subsets of X^* are taken as a basis for the topology on X^* then the induced subspace topology on X is discrete.*

This result should not be disturbing since duality transforms subobjects into quotient objects, in fact, we shall see that X is a quotient space of X^*.

THEOREM 4. *X^* is a subspace of $S(B^*)$.*

Proof. According to the embedding in Theorem 1 an element $x \in X^*$ corresponds to a map $b \xrightarrow{f} (x, b) \in S(B^*)$ where (x, b) is defined by transfer. Since the topology on $S(B^*)$ is generated by sets of the form $(f\colon f(b) = 1)$ where $b \in B^*$, the subspace topology on X^* is generated by sets of the form $[x\colon (x, b) = 1]$. By transfer this is precisely the collection of *clopen sets. Hence by Theorem 2 this induces the given topology on X^*.

The inclusion of B in B^* induces a continuous map from $S(B^*)$ to X which, in turn, induces by restriction a map T from X^* to X. By Stone duality the map may be defined as follows:

If $x \in X^*$ then Tx is the unique $x' \in X$ such that $(\forall b \in B)\ [(x, b) = (x', b)]$

THEOREM 5. *The monad of a point $x \in X$ is precisely the set of all $y \in X^*$ such that $(\forall b \in B)\ [(y, b) = (x, b)]$.*

Proof. The sets of the form $[y\colon (y, b) = (x, b)]$ for fixed b form a basis of open sets at b. The result follows immediately.

Now since X is compact Hausdorff, the monads of the points of X form a partition of X^*. Thus it follows from Theorem 5 that the above map T is precisely the map which takes X into its standard part.

Section 2. The relationship between X^* and $S(B^*)$.

THEOREM 6. *$X^* \neq S(B^*)$, i.e., B^* has external homomorphisms into $(0, 1)$.*

Proof. Since stone spaces are compact it is enough to show that X^* is not compact. Now for any integer N, X can be expressed as a union of n disjoint non-empty clopen sets. By transfer this is valid in X^* for an infinite integer w. Since unions are absolute this shows that X^* is a union of an infinite number of disjoint non-empty *clopen sets. Hence, by the way the topology is defined on X^*, X^* is not compact.

THEOREM 7. *X^* is dense in $S(B^*)$.*

Proof. Let $[f\colon f(b) = 1]$ where $b \in B^*$ and $b \neq 0$ be a basis set in $S(B^*)$. By applying transfer to the prime ideal theorem for Boolean algebras we see immediately that X^* intersects this basis set.

Section 3. The algebra of *clopen sets in X^*. We are now interested in the dual problem, i.e. that of comparing the algebra of *clopen sets in X^* to B^*. We have a result which is similar to Theorem 1 which also follows immediately from transfer.

THEOREM 8. *There is an internal one-one correspondence between the *clopen sets of X^* and the elements of B^*, given by $[x\colon (x, b) = 1)] \leftrightarrow b$, which preserves the Boolean operations.*

THEOREM 9. *The *clopen sets of X^* are the same as the internal clopen sets.*

Proof. *clopen sets are internal and open by the definition of the topology. By transfer the complement of a *clopen set is *clopen hence open. Thus a *clopen set is clopen in the topology. Conversely any internal clopen set A has the property that for every $p \in A$ there exists a *open set U such that $p \in U \subset A$. By transfer A is *open. Similarly A' is *open. Hence, again by transfer A is *clopen. (Note that complements are absolute in sets of lowest type.)

Note. In understanding the above proof it is essential note to confuse concepts such as open in the topology and *open which is obtained by transfer, although the two are closely related because of the way the topology is defined.

THEOREM 10. *All monads are clopen.*

Proof. Since the monads from a partition of X^* it suffices to prove that monads are open. Let $x \in X$ and let $p \in \mu(x)$ the monad of X. We must find a *clopen set V such that $p \in V \subset \mu(x)$. Now we know that for any finite set $b_1, b_2, \ldots, b_n$ of elements in a Boolean algebra and point f in its Stone space

$$\forall i[f, b_i) = 1] \rightarrow (f, \bigcap_{i=1}^{n} b_i) = 1 .$$

Hence

$$\exists b\{\forall i(b \leqslant b_i) \wedge \forall f[(\forall i[(f, b_i) = 1]) \rightarrow (f, b) = 1]\} .$$

By transfer the corresponding result is true with respect to B^* and X^* for *finite sets. Now although B is infinite it is well-known that there exists a *finite subset F of B^* containing B. By transfer any internal subset of a *finite set is *finite. Now $D = (b \in B^*\colon (p, b) = 1)$ is an internal set since it is definable. Hence $F \cap D$ is *finite. Note that $[p \in B\colon (p, b) = 1] \subset F \cap D$. By the above remark there exists a $d \in B^*$ such that

$$(\forall b \in F \cap D)(d \leqslant b) \wedge (\forall f \in X^*)[(\forall b \in F \cap D)(f, b) = 1) \rightarrow (f, d) = 1)] .$$

Let $V = (q \in X^*\colon (q, d) = 1)$. Then V is *clopen. Clearly by the way D was defined $(\forall b \in F \cap D)(p, b) = 1$. Hence $(p, d) = 1$ thus $p \in V$. Now let q be arbitrary in V.

Since $(\forall b \in F \cap D)(d \leqslant b)$ it follows that $(\forall b \in F \cap D)(q, b) = 1$. In particular, since $[p \in B\colon (p, b) = 1] \subset F \cap D$, then $(\forall b \in B)[(p, b) = 1 \rightarrow (q, b) = 1]$. Also $(p, b) = 0 \rightarrow (p, b') = 1 \rightarrow (q, b') = 1 \rightarrow (q, b) = 0$. Therefore $(\forall b \in B)[(p, b) = (q, b)]$. Since $p \in \mu(x)$ it follows from Theorem 5 that $q \in \mu(x)$. This proves that $V \subset \mu(x)$ and completes the proof of Theorem 10.

Note. The proof can be simplified if stronger saturation properties are assumed. In fact, it seems surprising that the result is valid even without these further saturation conditions.

It is amusing that the trite statement that a "subset of a finite set is finite" plays a crucial role (by transfer) in proving nontrivial theorems.

COROLLARY. *X^* contains external clopen sets.*

Proof. By an important theorem of W. A. J. Luxemburg, [4, Theorem 2.26] non-principal filters have external monads. Hence every point in X which is not discrete has an external monad in X^*. Since X is compact and infinite such points exist. By the theorem such monads are clopen. Theorem 10 gives another proof that X^* is hot compact. In contrast to the proof of Theorem 6, the partition used here involves external clopen sets by the corollary.

We have shown that the correspondence in Theorem 8 identifies B^* with a *proper* subalgebra of the Boolean algebra $B(X^*)$ of clopen subsets of X^*.

Since the *clopen sets from a basis for the topology on X^*, every clopen set contains a non-empty *clopen set. Hence B^* is dense in $B(X^*)$ in the sense made precise by the following theorem:

THEOREM 11. *For all $y > 0$ in $B(X^*)$ there exists an $x \in B^*$ such that $0 < x \leqslant y$.*

THEOREM 12. *B^* is not complete.*

Proof. This is a special case of a general result in the theory of Boolean algebras. If B is a proper subalgebra of B' and if $(\forall y \in B')[y > 0 \rightarrow (x \in B)(0 < x \leqslant y)]$ then B is not complete. In fact, it is easy to see that if $y \in B' - B$ then $(x \in B\colon x \leqslant y)$ does not have an l.u.b in B.

CONCLUSION. As mentioned in the introduction it would be interesting to extend this study to other kinds of pairings. We conjecture that the principal theorems in this paper, Theorems 6, 7, 10, and 11 have analogues in many situations. On the other hand [3, Theorem 8.7] runs counter to the spirit of this conjecture so that it is not clear what type of results are to be expected.

References

[1] H. Gonshor, *Enlargements contain various kinds of completions*, to appear in the proceedings of a symposium on nonstandard analysis held in Victoria, B. C. in May, 1972.

[2] — *Projective covers as subquotients of enlargements*, Israel J. Math. 14 (1973), pp. 257–261.

[3] C. W. Henson and L. C. Moore, Jr., *The nonstandard theory of topological vector spaces*, Trans. Amer. Math. Soc. 172 (1972), pp. 405–435.

[4] W. A. J. Luxemburg, *A general theory of monads* in W. A. J. Luxemburg, ed., *Applications of model to algebra, analysis, and probability theory*, New York 1969, pp. 18–86.

[5] A. Robinson, *Nonstandard Analysis*, Amsterdam 1966.

Accepté par la Rédaction le 29. 10. 1975