Concerning decompositions of continua

by

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Abstract. The first purpose of this paper is to characterize two decompositions of a Hausdorff hereditarily unicoherent continuum. One of them is a unique minimal with respect to being upper semi-continuous, monotone and having a λ-dendroid as the quotient space and the other is a unique minimal with respect to being upper semi-continuous, monotone and having a dendroid as the quotient space. For definition of a λ-dendroid and of a dendroid see below. The second purpose pertains Hausdorff continua irreducible about a finite subset. It is proved that each such continuum has a unique minimal decomposition with respect to being upper semi-continuous, monotone and having a tree as the quotient space.

A *continuum* is a compact connected Hausdorff space. A *decomposition* (a *monotone decomposition*) of a continuum X is a family of mutually disjoint non-empty closed subsets (non-empty subcontinua) of X filling up X. If $\mathscr{D}$ and $\mathscr{E}$ are both decompositions of a continuum X, then "$\mathscr{D}$ *refines* $\mathscr{E}$" means each element of $\mathscr{D}$ is contained in some element of $\mathscr{E}$. Let X be a continuum and let P be a certain property of decompositions of X. We say that a decomposition $\mathscr{D}$ of X is *minimal with respect to* P if $\mathscr{D}$ possesses P and refines each decomposition of X possessing P. A *mapping* is a continuous function. A mapping $f(X) = Y$ is called *monotone* if the inverse image $f^{-1}(C)$ of each connected subset C of Y is connected.

The following is a consequence of a more general results (see [8], Propositions 3 and 4, p. 1090).

PROPOSITION 1. *For any continuum X and for any class $\mathscr{A}$ of connected subsets of X there exists a unique monotone decomposition $\mathscr{D}$ of X which is minimal with respect to the property: "$\mathscr{D}$ is upper semi-continuous, and each element of $\mathscr{A}$ is contained in some element of $\mathscr{D}$".*

A λ-dendroid is a hereditarily unicoherent hereditarily decomposable continuum (not necessarily metrizable).

THEOREM 2. *Let X be a hereditarily unicoherent continuum. There exists a unique decomposition $\mathscr{D}$ of X such that*

(1) *$\mathscr{D}$ is a unique minimal decomposition of X with respect to the property: "$\mathscr{D}$ is upper semi-continuous and each indecomposable subcontinuum of X is contained in some element of $\mathscr{D}$",*

(2) *$\mathcal{D}$ is a unique minimal decomposition of X with respect to the property: "$\mathcal{D}$ is upper semi-continuous, monotone and the quotient space $X/\mathcal{D}$ is hereditarily decomposable"*,

(3) *$\mathcal{D}$ is a unique minimal decomposition of X with respect to the property: "$\mathcal{D}$ is upper semi-continuous, monotone and the quotient space $X/\mathcal{D}$ is a λ-dendroid".*

Moreover, $\mathcal{D}$ is monotone.

To prove that we first need two lemmas.

LEMMA 2.1. *Let X be a continuum and let $\mathcal{D}$ be an upper semi-continuous monotone decomposition of X. If each indecomposable subcontinuum of X is contained in some element of $\mathcal{D}$, then the quotient space $X/\mathcal{D}$ is hereditarily decomposable.*

Proof. This result follows from [7], § 48, V, Theorem 4, p. 208 (this theorem is stated for metric continua, however, it is a consequence of [7], § 47, II, Theorem 7, p. 171 that is proved for continua).

LEMMA 2.2. *Let X be a hereditarily unicoherent continuum and let $\mathcal{D}$ be an upper semi-continuous monotone decomposition of X having a hereditarily decomposable quotient space. Then each indecomposable subcontinuum of X is contained in some element of $\mathcal{D}$.*

Proof. Let f denote the quotient mapping of $\mathcal{D}$. If we assume that K is a subcontinuum of X with non-degenerate image $f(K)$, then (since $X/\mathcal{D}$ is hereditarily decomposable) $f(K) = A \cup B$ where A and B are both proper subcontinua of $f(K)$. It follows $K = (K \cap f^{-1}(A)) \cup (K \cap f^{-1}(B))$. Since X is hereditarily unicoherent the sets $K \cap f^{-1}(A)$ and $K \cap f^{-1}(B)$ are continua; they are both proper subcontinua of K. In fact, if $K \cap f^{-1}(A) = K$, i.e., $K \subset f^{-1}(A)$, then

$$A \cup B = f(K) \subset f(f^{-1}(A)) = A ,$$

a contradiction. Therefore K is decomposable.

Proof of Theorem 2. By Proposition 1, taking the class of indecomposable subcontinua of X as $\mathcal{A}$, there exists a monotone decomposition $\mathcal{D}$ of X satisfying condition (1). The quotient space $X/\mathcal{D}$ is hereditarily decomposable according to Lemma 2.1. Consider an upper semi-continuous monotone decomposition $\mathcal{D}_1$ of X having a hereditarily decomposable quotient space. By Lemma 2.2 each indecomposable subcontinuum of X is contained in some element of $\mathcal{D}_1$. Therefore $\mathcal{D}$ refines $\mathcal{D}_1$. It follows that $\mathcal{D}$ satisfies condition (2). Since the hereditarily unicoherence of continua is an invariant under monotone mappings the quotient space $X/\mathcal{D}$ is hereditarily unicoherent, so it is a λ-dendroid. Since each λ-dendroid is hereditarily decomposable and $\mathcal{D}$ satisfies (2), it satisfies condition (3).

An *arc* is a continuum with precisely two non-separating points. A continuum X is called *irreducible about a set A* if X contains A and no proper subcontinuum of X contains A. A continuum X irreducible about a set of two its points (such continua will be called shortly *irreducible*) has a unique minimal decomposition with respect to the property of being upper semi-continuous, monotone and having an arc or a point as the quotient space (see [5], Theorem 2.4, p. 649). The elements of this

decomposition are called *layers* of X. Vought [11] was described the structure of the layers of X having non-empty interior in the case of metric X. The idea of his proof, after a slight modification, is applicable to the non-metric case. Namely, let J be a layer of X with non-empty interior J^0 and assume without loss of generality that $X-J = A \cup B$, a separation of X, where $\overline{J^0}$ is irreducible from each point of $\overline{J^0} \cap \bar{A}$ to each point of $\overline{J^0} \cap \bar{B}$ (for a general discussion of irreducible continua see [5]). Suppose that every subcontinuum of $\overline{J^0}$ with non-empty interior be decomposable. By [5], Theorem 2.7, p. 650 $\overline{J^0}$ has an upper semi-continuous monotone decomposition with an arc as the quotient space. But it involves a contradiction since J is a layer (compare the first part of the proof of Theorem 2.7, ibidem). Therefore $\overline{J^0}$ contains an indecomposable subcontinuum with non-empty interior. Denote all of them by I_i, where $i \in M$. We shall define now, by the transfinite induction, the continua C_α^i for each $i \in M$. Let $C_0^i = I_i$. Suppose for an ordinal α that C_β^i has been defined for each $\beta < \alpha$ and for each $i \in M$. Put

$$C_\alpha^i = \begin{cases} \overline{\bigcup \{C_\beta^j :\ C_\beta^j \cap C_\beta^i \neq \emptyset\}}, & \text{if } \alpha = \beta+1 , \\ \overline{\bigcup \{C_\beta^i :\ \beta < \alpha\}}, & \text{if } \alpha = \lim \beta . \end{cases}$$

It follows by the transfinite induction that for each ordinal α and for each $i \in M$ the set C_α^i is well defined. From the construction C_α^i is a continuum. Similarly as in [3], the proof of Theorem 4.3, p. 40 one can show that there exists a first ordinal γ such that $C_\gamma^i = C_{\gamma+1}^i$ for each $i \in M$, and for each $i, j \in M$ we have either $C_\gamma^i \cap C_\gamma^j = \emptyset$ or $C_\gamma^i = C_\gamma^j$. If we proceed as in [10], writing C_γ^i instead of $\mathrm{Ch}_\gamma(I_i)$ we obtain $C_\gamma^i = C_\gamma^j$ for each $i, j \in M$ and C^i intersects both $\bar{A}$ and $\bar{B}$. It implies $C_\gamma^i = \overline{J^0}$ in view of the irreducibility of $\overline{J^0}$.

LEMMA 3.1. *Let a continuum X be irreducible and let $\mathcal{D}$ be a decomposition of X such that each indecomposable subcontinuum of X is contained in some element of $\mathcal{D}$. If J is a layer of X with non-empty interior, then $\overline{J^0}$ is contained in some element of $\mathcal{D}$.*

Proof. The proof involves the transfinite induction. Let f denote the quotient mapping of $\mathcal{D}$. By assumption $f(C_0^i) = f(I_i) = \{z_i\}$ for some $z_i \in f(X)$. Now, suppose that for each $\beta < \alpha$ and for each $i \in M$, the set $f(C_\beta^i)$ be degenerate. Since $I_i \subset C_\beta^i$ we obtain $f(C_\beta^i) = \{z_i\}$. If α is a limit ordinal, then obviously $f(C_\alpha^i) = \{z_i\}$. If α is a non-limit ordinal, then conditions $y \in C_\beta^j$ (where $\alpha = \beta+1$) and $C_\beta^j \cap C_\beta^i \neq \emptyset$ imply $f(y) = \{z_i\}$. It follows $f(C_\alpha^i) = \{z_i\}$. Finally, for all i's, $f(\overline{J^0}) = f(C_\gamma^i) = \{z_i\}$.

LEMMA 3.2. *Let a decomposition $\mathcal{D}$ of a continuum X has the property that for each indecomposable or irreducible continuum M of X, each subcontinuum of M with empty interior relative to M is contained in some element of $\mathcal{D}$. Then*

(a) *each indecomposable subcontinuum of X is contained in some element of $\mathcal{D}$, and*

(b) *for each irreducible subcontinuum I of X, each layer of I is contained in some element of $\mathcal{D}$.*

Proof. Let f denote the quotient mapping of $\mathcal{D}$. For an indecomposable subcontinuum M of X consider a composant C of M containing some point y. If x is an arbitrary point of C, then there exists a proper subcontinuum K of M such that $x, y \in K$. Since K has empty interior (otherwise M is decomposable) $f(x) = f(y)$, hence $f(C) = f(y)$. Thus $f(M) = f(\overline{C}) \subset \overline{f(C)} = \{f(y)\}$, therefore condition (a) holds. Now, let I be an irreducible subcontinuum of X and let J be a layer of I. By condition (a), each indecomposable subcontinuum of I is mapped onto a point under f, hence by Lemma 3.1 the set $f(\overline{J^0})$ is degenerate provided J^0 is non-empty (the case of empty J^0 is trivial). We can assume that $I - J = A \cup B$, a separation of I. The sets $\overline{A} - A$ and $\overline{B} - B$ obviously have both empty interior relative to I, and by [5], Theorem 2.3, p. 649 they are subcontinua of I. Since $J \cap \overline{A} \subset \overline{A} - A$ and $J \cap \overline{B} \subset \overline{B} - B$ the images $f(J \cap \overline{A})$ and $f(J \cap \overline{B})$ are both degenerate. Obviously

$$J = J \cap \overline{A} \cup J \cap \overline{B} \cup J^0,$$

thus $f(J)$ is degenerate.

A continuum X is called *hereditarily arcwise connected* if for each subcontinuum Y of X, each pair of points of Y can be joined by an arc lying in Y.

THEOREM 3. *For any continuum X there exists a unique monotone decomposition $\mathcal{D}$ of X such that*

(1) *$\mathcal{D}$ is upper semi-continuous and for each indecomposable or irreducible subcontinuum M of X, each subcontinuum of M having empty interior (relative to M) is contained in some element of $\mathcal{D}$,*

(2) *$\mathcal{D}$ is a unique minimal decomposition of X with respect to property* (1),

(3) *$\mathcal{D}$ is upper semi-continuous,*

(a) *each indecomposable subcontinuum of X is contained in some element of $\mathcal{D}$ and*

(b) *for each irreducible subcontinuum I of X each layer of I is contained in some element of $\mathcal{D}$,*

(4) *$\mathcal{D}$ is a unique minimal decomposition of X with respect to property* (3),

(5) *the quotient space $X/\mathcal{D}$ is hereditarily arcwise connected and hereditarily decomposable.*

Proof. It follows from Proposition 1 that there exists a monotone decomposition $\mathcal{D}_2$ (resp. $\mathcal{D}_4$) of X satisfying condition (2) (resp. (4)). By Lemma 3.2, $\mathcal{D}_2$ satisfies condition (3), hence $\mathcal{D}_4$ refines $\mathcal{D}_2$. On the other hand, it follows from [5], Theorem 2.3, p. 649 that for each irreducible subcontinuum I of X, each subcontinuum of I having empty interior is contained in a layer of I. It follows that $\mathcal{D}_2$ refines $\mathcal{D}_4$, hence $\mathcal{D}_2 = \mathcal{D}_4$. Therefore $\mathcal{D} = \mathcal{D}_2$ is a required decomposition satisfying conditions (1), (2), (3), and (4). That $\mathcal{D}$ satisfies condition (5) follows from [11], Theorem 3 and from Lemma 2.1.

A *dendroid* is a hereditarily unicoherent, hereditarily arcwise connected and hereditarily decomposable continuum (not necessarily metrizable) (compare [6], p. 62).

COROLLARY 3.3. *For any hereditarily unicoherent continuum X there exists a unique minimal decomposition $\mathcal{D}$ of X with respect to the property: "$\mathcal{D}$ is upper semi-continuous, monotone and the quotient space is a dendroid".*

Proof. Such is above decomposition $\mathcal{D}$ (see Theorem 3). Since the hereditarily unicoherence of continua is an invariant under monotone mappings, the quotient space $X/\mathcal{D}$ is a dendroid. Now, let $\mathcal{D}_1$ be an upper semi-continuous monotone decomposition of X with a dendroid as the quotient space. Similarly as in [1], the proof of Theorem 5, p. 26 one can show that for each irreducible subcontinuum I of X, each layer of I is contained in some element of $\mathcal{D}_1$. By Lemma 2.2 each indecomposable subcontinuum of X is contained in some element of $\mathcal{D}_1$. It follows that $\mathcal{D}_1$ satisfies condition (3) of Theorem 3, hence $\mathcal{D}$ refines $\mathcal{D}_1$. This completes the proof.

For a metric continuum X, Charatonik [2] has defined a decomposition of X to be *admissible* if it is monotone, upper semi-continuous and the layers of irreducible subcontinua of X are contained in the elements of the decomposition. He has proved that the quotient space of an admissible decomposition is hereditarily arcwise connected and that X has a unique minimal admissible decomposition, say $\mathcal{D}$. If X is hereditarily unicoherent, then $\mathcal{D}$ is a unique minimal decomposition of X with respect to being upper semi-continuous, monotone and having a dendroid as the quotient space. Vought [10] has extended Charatonik's results to (Hausdorff) continua. The statement of our Theorem 3 and Corollary 3.3 is another extension to continua of mentioned Charatonik's results.

A continuum X is called *discoherent* if for any pair of its proper subcontinua A and B such that $X = A \cup B$ the intersection $A \cap B$ is not connected. By a *simple closed curve* we mean a non-degenerate continuum which is separated by each pair of its points.

THEOREM 4. *For any continuum X there exists a unique monotone decomposition $\mathcal{D}$ of X which is minimal with respect to the property: "$\mathcal{D}$ is upper semi-continuous and each discoherent subcontinuum of X is contained in some element of $\mathcal{D}$". Furthermore, the quotient space $X/\mathcal{D}$ is a λ-dendroid.*

To establish this we first need the following

LEMMA 4.1. *Assume that $\mathcal{D}$ is a monotone upper semi-continuous decomposition of a continuum X such that each discoherent subcontinuum of X is contained in some element of $\mathcal{D}$. Then the quotient space $X/\mathcal{D}$ is a λ-dendroid.*

Proof. Since each indecomposable continuum is discoherent, the quotient space $X/\mathcal{D}$ is hereditarily decomposable by Lemma 2.1. Let f denote the quotient mapping of $\mathcal{D}$ and suppose that $f(X)$ is not hereditarily unicoherent. By [5], Theorem 3.3, p. 652 there exists a continuum $N \subset f(X)$ and a monotone mapping of N onto a simple closed curve S. Put $h = gf \mid f^{-1}(N)$. It follows from [7], § 42, IV, Theorems 1 and 2, p. 54 that there exists a continuum $M\subset f^{-1}(N)$ which is irreducible with respect to the property $h(M) = S$. Consider a decomposition, $M = A \cup B$, of M onto its proper subcontinua A and B. Obviously, $h(A) \neq S$ and $h(B) \neq S$. But $h(A) \cup h(B) = S$, hence there exist disjoint closed and non-empty sets E and F such that $h(A) \cap h(B) = E \cup F$. Sets $A \cap B \cap h^{-1}(E)$ and $A \cap B \cap h^{-1}(F)$ are both closed, non-empty and disjoint. Furthermore,

$$A \cap B \subset h^{-1}(h(A)) \cap h^{-1}(h(B)) = h^{-1}(h(A) \cap h(B)) = h^{-1}(E \cup F)$$
$$= h^{-1}(E) \cup h^{-1}(F),$$

hence

$$A \cap B \cap h^{-1}(E) \cup A \cap B \cap h^{-1}(F) = A \cap B \cap (h^{-1}(E) \cup h^{-1}(F)) = A \cap B.$$

This implies that the intersection $A \cap B$ is not connected. So we have proved that M is discoherent. By assumption, $f(M)$ is degenerate, so $h(M) = S$ is. But it involves a contradiction. Therefore $f(X)$ is unicoherent.

Proof of Theorem 4. It follows from Proposition 1 that there exists a required decomposition $\mathscr{D}$. By Lemma 4.1 the quotient space $X/\mathscr{D}$ is a λ-dendroid.

Let I denote the interval $[0, 1]$. The square I^2 is an example of a continuum for which the above decomposition $\mathscr{D}$ is not minimal with respect to the property of having a λ-dendroid as the quotient space even in the class of monotone upper semi-continuous decompositions.

A *tree* is a continuum for which every pair of points is separated by some third point. It is well known that a continuum is a tree if and only if it is hereditarily unicoherent and locally connected (see [13], Theorem 9, p. 803).

THEOREM 5. *For any continuum X irreducible about a finite subset there exists a unique monotone decomposition $\mathscr{D}$ of X such that*

(1) *$\mathscr{D}$ is a unique minimal decomposition of X with respect to the property: "$\mathscr{D}$ is upper semi-continuous and each subcontinuum of X with empty interior is contained in some element of $\mathscr{D}$",*

(2) *$\mathscr{D}$ is a unique minimal decomposition of X with respect to the property: $\mathscr{D}$ is upper semi-continuous and for each irreducible subcontinuum I of X, each layer of I is contained in some element of $\mathscr{D}$",*

(3) *$\mathscr{D}$ is a unique minimal decomposition of X with respect to the property: "$\mathscr{D}$ is upper semi-continuous, monotone and the quotient space $X/\mathscr{D}$ is hereditarily arcwise connected",*

(4) *$\mathscr{D}$ is a unique minimal decomposition of X with respect to the property: "$\mathscr{D}$ is upper semi-continuous, monotone and the quotient space $X/\mathscr{D}$ is a tree".*

For the proof we need four lemmas.

The following is well known (compare [4]).

LEMMA 5.1. *If a continuum X is not locally connected at a point p, then there exists a continuum C with empty interior such that $p \in C$ and X is not locally connected at each point of C.*

In Lemmas 5.2, 5.3 and 5.4 the continuum X is supposed to be irreducible about a set of n, but no fewer of its points, say $a_1, a_2, \ldots, a_n$, where $n \geqslant 2$.

LEMMA 5.2. *Let $\mathscr{D}$ be an upper semi-continuous monotone decomposition of the continuum X with a hereditarily arcwise connected quotient space. Suppose that K is a subcontinuum of X with empty interior. Then X is contained in some element of $\mathscr{D}$.*

Proof. The proof of Lemma 1 of [9], p. 160 generalizes easily to the non-metric case.

LEMMA 5.3. *If the continuum X is hereditarily arcwise connected, then it is locally connected.*

Proof. By Lemma 5.2 each subcontinuum of X with empty interior is degenerate. Therefore X is locally connected according to Lemma 5.1.

LEMMA 5.4. *If the continuum X is locally connected, then it is a tree. Consequently, if X is hereditarily arcwise connected, then it is a tree.*

Proof. The first part of the lemma is established in [11], the proof of Theorem 1. Therefore the second one follows from Lemma 5.3.

Proof of Theorem 5. By Proposition 1 there exists a monotone decomposition $\mathscr{D}_1$ (resp. $\mathscr{D}_2$) of X satisfying condition (1) (resp. (2)). Let I be an irreducible subcontinuum of X. By Lemma 3.2 each layer of I is contained in some element of $\mathscr{D}_1$. It implies that $\mathscr{D}_2$ refines $\mathscr{D}_1$. The quotient space $X/\mathscr{D}_1$ as well as $X/\mathscr{D}_2$ is hereditarily arcwise connected according to [10], Theorem 3. Therefore they are both trees by Lemma 5.4. Thus Lemma 5.2 implies that each subcontinuum of X with empty interior is contained in some element of $\mathscr{D}_2$. It follows that $\mathscr{D}_1$ refines $\mathscr{D}_2$, so $\mathscr{D}_1 = \mathscr{D}_2$. Observe that we have proved, by the way, that $\mathscr{D}_1$ satisfies condition (3). Let now, $\mathscr{D}_4$ be an upper semi-continuous monotone decomposition of X with a tree as the quotient space. By [5], Theorem 4.1, p. 655, the quotient space $X/\mathscr{D}_4$ is hereditarily arcwise connected, so by Lemma 5.2 each subcontinuum of X with empty interior is contained in some element of $\mathscr{D}_4$. Therefore $\mathscr{D}_1$ refines $\mathscr{D}_4$, so $\mathscr{D}_1$ satisfies condition (4). Putting $\mathscr{D} = \mathscr{D}_1$ we have proved that $\mathscr{D}$ is a required decomposition.

Hausdorff continua irreducible about a finite subset were investigated recently by Vought [11]. He has proved that such a continuum has a minimal decomposition with respect to being upper semi-continuous, monotone and having a tree as the quotient space and elements with empty interior if and only if the continuum contains no indecomposable subcontinua with non-empty interior. Therefore our Theorem 5 seems to be a completion to his work.

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