Homotopically fixed arcs and the contractibility of dendroids

by

J. J. Charatonik and **Z. Grabowski** (Wrocław)

Abstract. The concept of an R-arc in a dendroid is introduced and studied in the paper, to show a new class of non-contractible dendroids, namely of dendroids which contain an R-arc. Using this concept, hereditarily contractible fans are characterized as smooth ones, but the problem of finding a characterization of hereditarily contractible dendroids remains open.

Although the theory of contractible spaces is well-known (see e.g. [4] and [6]), general methods are not very useful for investigation of the contractibility of some special spaces, e.g. of some curves. In particular there are only a few papers which concern the contractibility of dendroids; no characterization of the class of contractible dendroids is known at present, and merely several conditions which are either necessary or sufficient have appeared in the literature. This paper is a contribution to the attempt to find further conditions concerning the contractibility of dendroids.

All spaces considered in this paper are assumed to be metric. A continuum means a compact connected space. A property of a continuum X is called to be *hereditary* if each subcontinuum of X has this property. A continuum X is said to be *arcwise connected* if for every two points a and b of X there exists an arc ab joining a with b and contained in X. A continuum X is called *unicoherent* if for each two subcontinua A and B of X such that $X = A \cup B$ the intersection $A \cap B$ is connected. A dendroid means an arcwise connected and hereditarily unicoherent continuum. A point p of an arcwise connected space X is called a *ramification point* of X provided there are three arcs pa, pb, pc such that p is the only common point of every two of them. A dendroid which has only one ramification point is called a *fan*. A point x of an arcwise connected space X is called an *end point* of X if it is an end point of every arc containing it and contained in X.

Given a space X with a metric d, let A be a subset of X and let ε be a positive number. We denote by $Q(A, \varepsilon)$ the spherical ε-neighbourhood of A, i.e., the set of all points x of X for which there exist points a of A with $d(x, a) < \varepsilon$. We denote by $[\alpha, \beta]$ the closed segment of reals from α to β, i.e. $[\alpha, \beta] = \{t\colon \alpha \leqslant t \leqslant \beta\}$, and we put I for $[0, 1]$. The closure of a set $A \subset X$ is denoted by $\bar{A}$, and we put $\mathrm{Fr}\,A = \bar{A} \cap \overline{X \setminus A}$ for the boundary of A. To describe some examples we use the symbol $\overline{xy}$ for the straight line segment joining the points x and y in the Euclidean space. We hope there will be no confusion with the closure of a set. Given a sequence of subsets A_n of X, we denote by $\operatorname{Lim}_{n\to\infty} A_n$ the topological limit of A_n in sense of [6], § 29, VI, p. 339.

A mapping means a continuous transformation. A mapping $H\colon X\times I\to Y$ is called a *homotopy*. If $X\subset Y$ and if $H(x,0)=x$ for each $x\in X$, then the homotopy H is said to be a *deformation* of X in Y. Furthermore, if for each $x\in X$ the point $H(x,1)$ is the same (i.e., if $H(\cdot,1)$ is a constant mapping), then the deformation H is called a *contraction* of X in Y. They are simply called deformations and contractions of X if $Y=X$. If a contraction of X (in Y) exists, then X is said to be *contractible* (in Y) (see [4], I, 8, p. 11 and 12; cf. [7], § 54, I, p. 360; IV, V and VI, p. 368–375).

PROPOSITION 1. *Let a space X contain two sets A and B such that*

(1) $$A\neq\emptyset,\quad B\neq X$$

and

(2) *for every deformation $H\colon X\times I\to X$ we have $H(A\times I)\subset B$.*

Then X is not contractible.

In fact, suppose there exists a contraction H with $H(A\times I)\subset B$ which contracts X to a point y, i.e., $H(x,1)=y$ for every $x\in X$. Thus X is arcwise connected ([7], § 54, VI, Theorem 1, p. 374) and hence we may take an arbitrary point of X as y. Taking y in $X\setminus B$ and x in A we get a contradiction.

DEFINITION 2. A non-empty subset A of a space X is said to be *homotopically fixed* if for every deformation $H\colon X\times I\to X$ we have $H(A\times I)\subset A$.

As an immediate consequence of the above definition and of Proposition 1 we get

PROPOSITION 3. *If a space X contains a proper subset A which is homotopically fixed, then X is not contractible.*

Let X be a dendroid. Consider an arbitrary arc ab contained in X and a point x of X. If, for some number $\varepsilon>0$ we have $x\in\overline{Q(ab,\varepsilon)}$ and the intersection $xa\cap\overline{Q(ab,\varepsilon)}$ is non-connected, then we denote by $x(\varepsilon)$ the first point of the arc xa ordered from x to a which lies in $\operatorname{Fr}Q(ab,\varepsilon)$, i.e., $x(\varepsilon)\in xa\cap\operatorname{Fr}Q(ab,\varepsilon)$ and $xx(\varepsilon)\cap\operatorname{Fr}Q(ab,\varepsilon)=\{x(\varepsilon)\}$.

DEFINITION 4. An arc ab contained in a dendroid X is called an *R-arc* if

1° there are two sequences $\{u_n\}$ and $\{v_n\}$ of end points of X such that

$$\lim_{n\to\infty} u_n=a\quad\text{and}\quad\lim_{n\to\infty} v_n=b;$$

2° there is a number $\varepsilon>0$ such that for almost all positive integers n the sets $u_nb\cap\overline{Q(ab,\varepsilon)}$ and $v_na\cap\overline{Q(ab,\varepsilon)}$ are non-connected (thus the points $u_n(\varepsilon)$ and $v_n(\varepsilon)$ are well-defined) and the sets $u_nu_n(\varepsilon)\setminus\{u_n(\varepsilon)\}$ and $v_nv_n(\varepsilon)\setminus\{v_n(\varepsilon)\}$ contain no ramification point of X;

3° $[\operatorname{Lim}_{n\to\infty} u_nu_n(\varepsilon)]\cap[\operatorname{Lim}_{n\to\infty} v_nv_n(\varepsilon)]=ab$.

The case of a degenerate R-arc (i.e. when $a=b$) is also acceptable.

THEOREM 5. *Every R-arc contained in a dendroid X is homotopically fixed.*

Proof. Let a dendroid X contain an R-arc ab. Suppose, on the contrary, that there exists a deformation $H\colon X\times I\to X$ for which $H(ab\times I)\setminus ab\neq\emptyset$, i.e., for which there exists a number $t'\in I$ with

(3) $$H(ab\times\{t'\})\setminus ab\neq\emptyset.$$

Let a number $\varepsilon>0$ satisfy condition 2° of Definition 4. For each point $p\in ab$ put

$$t_p=\sup\{t\in I\colon H(\{p\}\times[0,t])\subset Q(ab,\tfrac{1}{2}\varepsilon)\},$$

and let $t_0=\inf\{t_p\colon p\in ab\}$. Observe that $t_0>0$. In fact, suppose $t_0=0$. Then there is a sequence of points $p_n\in ab$ such that the sequence of corresponding numbers t_{p_n} tends to zero. It follows from the compactness of the arc ab that the sequence $\{p_n\}$ contains a subsequence $\{p_{n_k}\}$ which converges to a point $p_0\in ab$. We see by the definition of t_p that if $t_p<1$, then $H(p,t_p)\in\operatorname{Fr}Q(ab,\frac{1}{2}\varepsilon)$. Since the sequence $\{t_{p_{n_k}}\}$ tends to zero as a subsequence of $\{t_{p_n}\}$, we may assume $t_{p_{n_k}}<1$ for sufficiently large k, and therefore $H(p_{n_k},t_{p_{n_k}})\in\operatorname{Fr}Q(ab,\frac{1}{2}\varepsilon)$. The mapping H being continuous and the set $\operatorname{Fr}Q(ab,\frac{1}{2}\varepsilon)$ being closed, we conclude $H(p_0,0)\in\operatorname{Fr}Q(ab,\frac{1}{2}\varepsilon)$. But since H is a deformation we have $H(p_0,0)=p_0$, whence $p_0\in\operatorname{Fr}Q(ab,\frac{1}{2}\varepsilon)$, which contradicts to $p_0\in ab$. Thus the inequality $t_0>0$ is established.

It follows from the definition of t_0 that $H(ab\times[0,t_0])\subset\overline{Q(ab,\frac{1}{2}\varepsilon)}$. We claim that there is a number $t'\in[0,t_0]$ such that condition (3) is satisfied. In fact, if $t_0=1$, then the claim follows from the supposition done in the beginning of the proof. If $t_0<1$, then there exists a sequence of points $p_n\in ab$ such that $t_0\leqslant t_{p_n}$ and $t_0=\lim_{n\to\infty} t_{p_n}$. Let — as previously — $\{p_{n_k}\}$ be a subsequence of $\{p_n\}$ converging to a point $p_0\in ab$. Applying once more the same arguments as above we get $H(p_{n_k},t_{p_{n_k}})\in\operatorname{Fr}Q(ab,\frac{1}{2}\varepsilon)$, whence $H(p_0,t_0)\in\operatorname{Fr}Q(ab,\frac{1}{2}\varepsilon)$ and thus the point $H(p_0,t_0)$ is not in ab. Therefore we may take t_0 as t'.

Let $t'\in[0,t_0]$ be any number such that (3) holds and let $p_0\in ab$ be such a point that $H(p_0,t')$ is not in ab. Put $H(p_0,t')=q$ and define

$$t_0'=\inf\{t'\in[0,t_0]\colon H(p_0,t')=q\}.$$

Thus by the continuity of H we have $H(p_0,t_0')=q\in X\setminus ab$.

Now let us recall that ab is an R-arc and thus it satisfies condition 3° of Definition 4. Since $p_0\in ab\subset(\operatorname{Lim}_{n\to\infty} u_nu_n(\varepsilon))\cap(\operatorname{Lim}_{n\to\infty} v_nv_n(\varepsilon))$, there exist two sequences of points p_n' and p_n'' such that $p_n'\in u_nu_n(\varepsilon)$, $p_n''\in v_nv_n(\varepsilon)$ and $p_0=\lim_{n\to\infty} p_n'=\lim_{n\to\infty} p_n''$. Put $q_n'=H(p_n',t_0')$ and $q_n''=H(p_n'',t_0')$ and consider the following two conditions:

(4) $$q_n'\in u_nu_n(\varepsilon)\quad\text{for almost all } n,$$

(5) $$q_n''\in v_nv_n(\varepsilon)\quad\text{for almost all } n.$$

If (4) holds, then we have

$$q = H(p_0, t_0') = \lim_{n\to\infty} H(p_n', t_0') = \lim q_n' \in \operatorname*{Lim}_{n\to\infty} u_n u_n(\varepsilon).$$

Similarly if (5) holds, then $q \in \operatorname*{Lim}_{n\to\infty} v_n v_n(\varepsilon)$. Therefore if both (4) and (5) hold, we have $q \in (\operatorname*{Lim}_{n\to\infty} u_n u_n(\varepsilon)) \cap (\operatorname*{Lim}_{n\to\infty} v_n v_n(\varepsilon)) \subset ab$ which contradicts to $q \in X \setminus ab$.

So, either (4) or (5) does not hold. Assume (4) is not true. Thus there is a subsequence $\{q_{n_k}'\}$ of the sequence $\{q_n'\}$ with q_{n_k}' not in $u_{n_k}u_{n_k}(\varepsilon)$. Since the points p_{n_k}' are in the arcs $u_{n_k}u_{n_k}(\varepsilon)$ and q_{n_k}' are not, and since u_{n_k} are end points of the dendroid X, we conclude by 2° that $u_{n_k}(\varepsilon) \in H(\{p_{n_k}'\} \times [0, t_0'])$. Thus there are numbers $t_{n_k} \in [0, t_0']$ such that $H(p_{n_k}', t_{n_k}) = u_{n_k}(\varepsilon)$. By the compactness of $[0, t_0']$ we may assume that the sequence $\{t_{n_k}\}$ converges to a number $t'' \in [0, t_0']$. Hence it follows from the continuity of H that

$$\lim_{k\to\infty} H(p_{n_k}', t_{n_k}) = H(p_0, t'') \in H(ab \times [0, t_0']) \subset H(ab \times [0, t_0]) \subset \overline{Q(ab, \tfrac{1}{2}\varepsilon)}.$$

But $\lim_{k\to\infty} H(p_{n_k}', t_{n_k}) = \lim_{k\to\infty} u_{n_k}(\varepsilon)$ which is not in $\overline{Q(ab, \frac{1}{2}\varepsilon)}$ by the definition of the points $u_n(\varepsilon)$. This contradiction finishes the proof of the theorem.

Proposition 3 and Theorem 5 imply the following

COROLLARY 6. *If a dendroid contains an R-arc, then it is not contractible.*

We shall prove now that all hypotheses of Corollary 6 are essential, i.e., that all conditions of Definition 4 are necessary to show the non-contractibility of a dendroid which contains an R-arc.

EXAMPLE 7. Let a point p be the pole (i.e. the origin) of the polar coordinate system in the Euclidean plane. Put in the polar coordinates (ϱ, φ)

$$p_0 = (1, 0), \quad p_n = (1, 2^{1-n}), \quad q_n = (\tfrac{1}{2}, \tfrac{3}{4}\cdot 2^{1-n}) \quad \text{for} \quad n = 1, 2, \ldots$$

and let

$$X = \overline{pp_0} \cup \bigcup_{n=1}^{\infty} (\overline{pp_n} \cup \overline{p_n q_n})$$

(we recall that here $\overline{xy}$ stands for the straight line segment joining points x and y). So X is a fan with the top p and with end points p_0 and q_n for $n = 1, 2, \ldots$ Define $H\colon X \times I \to X$ as follows. $H(p, t) = p$ for every $t \in I$. If $x \neq p$, then $H(x, t)$ belongs to the same component of the set $X \setminus \{p\}$ as the point x for every $t \in I$. Further, if ϱ_1 and ϱ are the radii, i.e. the first polar coordinates, of the points $H(x, t)$ and x respectively, we put

$$\varrho_1 = \begin{cases} (1+2t)\varrho & \text{if} \quad \varrho < \frac{1}{2} \text{ and } 0 \leqslant t < \frac{1}{2}, \\ (1-2t)\varrho + 2t & \text{if } \frac{1}{2} \leqslant \varrho \leqslant 1 \text{ and } 0 \leqslant t < \frac{1}{2}, \\ 4(1-t)\varrho & \text{if} \quad \varrho < \frac{1}{2} \text{ and } \frac{1}{2} \leqslant t \leqslant 1, \\ 2(1-t) & \text{if } \frac{1}{2} \leqslant \varrho \leqslant 1 \text{ and } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

It is easy to verify that H contracts X to its top p (cf. [1], p. 31). Putting, for the fan X defined above, $u_n = q_n$, $v_n = (\frac{3}{4}, 2^{1-n})$, $a = (\frac{1}{2}, 0)$, $b = (\frac{3}{4}, 0)$ and $\varepsilon = \frac{1}{8}$ we see that all the conditions of Definition 4 are satisfied except for v_n are end points of X. Thus 1° is essential.

EXAMPLE 8. Putting, under the same notation as in Example 7, $r_n = (2^{1-n}, 2^{1-n})$ and

$$R = \overline{pp_0} \cup \bigcup_{n=1}^{\infty} (\overline{pp_{2n}} \cup \overline{p_{2n}q_{2n}}) \cup \bigcup_{n=1}^{\infty} \overline{pr_{2n-1}}$$

we see that R is a subfan of the fan X which can be described as the union of a fan homeomorphic to X and of a locally connected fan $\bigcup_{n=1}^{\infty} \overline{pr_{2n-1}}$, the top p of which is the only common point of the both fans. Hence R is contractible.

Further, putting $u_n = q_{2n}$, $v_n = r_{2n-1}$, $a = (\frac{1}{2}, 0)$ and $b = p$ we see that condition 1° of Definition 4 holds and that for every ε such that $0 < \varepsilon < \frac{1}{2}$ the sets $u_n b \cap \overline{Q(ab, \varepsilon)}$ are non-connected for almost all n, but $v_n a \cap \overline{Q(ab, \varepsilon)}$ are connected (and thus the points $v_n(\varepsilon)$ cannot be defined). Therefore the first part of condition 2° is essential.

To see that the second part is essential consider the following example (due to P. Minc).

EXAMPLE 9. Let X be the fan described in Example 7, let $x_n = (\frac{3}{4}, \frac{3}{4}\cdot 2^{2-n})$ and $y_n = (\frac{3}{4}, 2^{1-n})$ for $n = 1, 2, \ldots$ and define

$$M = X \cup \bigcup_{n=1}^{\infty} \overline{x_n y_n}.$$

It is easy to observe that M is a contractible dendroid. Taking $u_n = q_n$, $v_n = x_n$, $a = (\frac{1}{2}, 0)$, $b = (\frac{3}{4}, 0)$ and $\varepsilon = \frac{1}{8}$ we see that all conditions mentioned in Definition 4 are satisfied except that the sets $v_n v_n(\varepsilon) \setminus \{v_n(\varepsilon)\}$ contain ramification points y_n of M.

EXAMPLE 10. Putting, under the same notations as in Example 7, $s_n = (\frac{1}{4}, 2^{1-n})$ and

$$S = \overline{pp_0} \cup \bigcup_{n=1}^{\infty} (\overline{pp_{2n}} \cup \overline{p_{2n}q_{2n}}) \cup \bigcup_{n=1}^{\infty} \overline{ps_{2n-1}}$$

we see that S is a subfan of the fan X which can be described as the union of a fan homeomorphic to X and of a fan homeomorphic to the harmonic fan, both having the common top and such that the limit segment of the latter one is contained in the limit segment of the former. Define $F\colon S \times I \to S$ as follows. $F(p, t) = p$ for every $t \in I$. If $x \in S \setminus \{p\}$, then $F(x, t)$ belongs to the same component of the set $S \setminus \{p\}$ as the point x for every $t \in I$. Further if ϱ_2 and ϱ are the radii of the points $F(x, t)$ and x respectively, we put

$$\varrho_2 = \begin{cases} \varrho & \text{if } \quad \varrho \leqslant \frac{1}{4} \text{ and } 0 \leqslant t < \frac{1}{2}, \\ (4t+1)\varrho - t & \text{if } \frac{1}{4} < \varrho < \frac{1}{2} \text{ and } 0 \leqslant t < \frac{1}{2}, \\ (1-2t)\varrho + 2t & \text{if } \frac{1}{2} \leqslant \varrho \leqslant 1 \text{ and } 0 \leqslant t < \frac{1}{2}, \\ 2(1-t)\varrho & \text{if } \quad \varrho \leqslant \frac{1}{4} \text{ and } \frac{1}{2} \leqslant t \leqslant 1, \\ (1-t)(6\varrho - 1) & \text{if } \frac{1}{4} < \varrho < \frac{1}{2} \text{ and } \frac{1}{2} \leqslant t \leqslant 1, \\ 2(1-t) & \text{if } \frac{1}{2} \leqslant \varrho \leqslant 1 \text{ and } \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

It is easy to verify that F is a contraction of S to its top p. Putting $u_n = q_{2n}$, $v_n = s_{2n-1}$, $a = (\frac{1}{2}, 0)$, $b = (\frac{1}{4}, 0)$ and $\varepsilon = \frac{1}{8}$ we see that conditions 1° and 2° Definition 4 hold true. Further, we see that $\operatorname{Lim}_{n\to\infty} u_n u_n(\varepsilon)$ is the straight line segment joining the point a with $(\frac{5}{8}, 0)$ and similarly $\operatorname{Lim}_{n\to\infty} v_n v_n(\varepsilon)$ is the straight line segment joining $(\frac{1}{8}, 0)$ with the point b. Thus $\emptyset = (\operatorname{Lim}_{n\to\infty} u_n u_n(\varepsilon)) \cap (\operatorname{Lim}_{n\to\infty} v_n v_n(\varepsilon)) \subset ab$ and the opposite inclusion does not hold. So this inclusion is essential.

EXAMPLE 11. Let, under the same notations as in Example 7, the point w_n denote the center of the straight line segment $\overline{p_n q_n}$; thus $w_n = (\frac{3}{4}, \varphi_n)$, where $\frac{3}{4}\cdot 2^{1-n} < \varphi_n < 2^{1-n}$. Putting

$$W = \overline{pp_0} \cup \bigcup_{n=1}^{\infty} (\overline{pp_{2n}} \cup \overline{p_{2n} q_{2n}}) \cup \bigcup_{n=1}^{\infty} (\overline{pp_{2n-1}} \cup \overline{p_{2n-1} w_{2n-1}})$$

we see that W is a subfan of the fan X which can be described as the union of two fans, both homeomorphic to X, having the top p and the limit segment $\overline{pp_0}$ in common. Further, it is easy to verify that $H|W\times I\colon W\times I \to W$ (where H is defined in Example 7) is a contraction of W to its top p. Taking $u_n = q_{2n}$, $v_n = w_{2n-1}$, $a = (\frac{1}{2}, 0)$, $b = (\frac{3}{4}, 0)$ and $\varepsilon = \frac{3}{8}$ we see that conditions 1^0 and 2^0 of Definition 4 are satisfied. Further, we see that $\operatorname{Lim}_{n\to\infty} u_n u_n(\varepsilon)$ and $\operatorname{Lim}_{n\to\infty} v_n v_n(\varepsilon)$ both are the straight line segment joining $(\frac{1}{8}, 0)$ with the point p_0, whence the arc ab is a proper subset of the intersection of these limits: the inclusion $ab \subset (\operatorname{Lim}_{n\to\infty} u_n u_n(\varepsilon)) \cap (\operatorname{Lim}_{n\to\infty} v_n v_n(\varepsilon))$ does hold but the opposite one does not. So this inclusion is essential.

Examples 10 and 11 show that every inclusion of equality 3^0 is essential, so condition 3^0 is essential even in so strong sense.

PROPOSITION 12. *There exist dendroids X and Y such that*

(6) *X is a countable plane fan,*

(7) *X is contractible,*

(8) *Y is contained in X,*

(9) *Y is not contractible,*

(10) *Y is contractible in X.*

In fact, take the fan X described in Example 7 and observe that (6) follows from the construction and (7) is proved in that place. To define Y put — under the same notations — $y_n = (\frac{3}{4}, 2^{1-n})$ and take

$$Y = \overline{pp_0} \cup \bigcup_{n=1}^{\infty} (\overline{pp_{2n}} \cup \overline{p_{2n} q_{2n}}) \cup \bigcup_{n=1}^{\infty} \overline{py_{2n-1}}.$$

Thus we see that (8) holds, i.e., Y is a subfan of the fan X, whence it is a plane fan. Taking $u_n = q_{2n}$, $v_n = y_{2n-1}$, $a = (\frac{1}{2}, 0)$, $b = (\frac{3}{4}, 0)$ and $\varepsilon = \frac{1}{8}$ we see that all the conditions of Definition 4 are fulfilled with Y for X, thus ab is an R-arc in Y. Therefore (9) follows from Corollary 6. Further, (7) and (8) imply (10).

It follows from Proposition 12 that Theorem 1 of [3] is not true not only for dendroids, which was known (see e.g. [5]) but even for plane fans. Thus the following question seems to be natural:

QUESTION 13. Give an internal characterization of hereditarily contractible dendroids.

A contribution to the attempt to find such a characterization is the following.

PROPOSITION 14. *If a dendroid is smooth, then it is hereditarily contractible.*

Indeed, it is known that the smoothness of dendroids is a hereditary property, i.e., that every subdendroid of a smooth dendroid is also smooth (see [2], Corollary 6, p. 299). Further, every smooth dendroid is contractible (see [8], Theorem 1.16, p. 371; cf. [3], Corollary, p. 93). Thus, for dendroids, the smoothness implies the hereditary contractibility.

LEMMA 15. *If a contractible dendroid contains a point p and a convergent point sequence $\{a_n\}$ such that the sequence of arcs pa_n is convergent, then the limit continuum $\operatorname{Lim}_{n\to\infty} pa_n$ is hereditarily locally connected.*

Proof. Let X be a dendroid that satisfies the hypotheses of the lemma, and let $H\colon X\times I \to X$ be a contraction of X. Without loss of generality we may assume that $H(X\times\{1\}) = \{p\}$. Let $a = \lim_{n\to\infty} a_n$. We claim that the point a must pass through all points of the continuum $\operatorname{Lim}_{n\to\infty} pa_n$ during the contraction. In other words we claim that

(11) for every point $x \in \operatorname{Lim}_{n\to\infty} pa_n$ there exists a number $t \in I$ such that $H(a, t) = x$.

In fact, for each natural m consider the spherical neighbourhood $Q(x, 1/m)$. Since $x \in \operatorname{Lim}_{n\to\infty} pa_n$, there is a natural $n_0(m)$ such that for each $n > n_0(m)$ we have $pa_n \cap Q(x, 1/m) \neq \emptyset$. Let $x_{n,m} \in pa_n \cap Q(x, 1/m)$ and consider the dendrite $D_n = H(\{a_n\}\times I)$. Since $a_n = H(a_n, 0) \in D_n$ and $p = H(a_n, 1) \in D_n$, we have $ap_n \subset D_n$, whence $x_{n,m} \in D_n$. So there is a number $t_{n,m} \in I$ such that $H(a_n, t_{n,m}) = x_{n,m}$. Tending with m and n to infinity and considering convergent subsequences if necessary we get the limit t of $\{t_{n,m}\}$ with $H(a, t) = x$. So (11) is proved.

It follows from (11) that $\operatorname{Lim}_{n\to\infty} pa_n \subset H(\{a\}\times I)$. Since $H(\{a\}\times I)$ is a dendrite, thus a hereditarily locally connected continuum, therefore the continuum $\operatorname{Lim}_{n\to\infty} pa_n$ is also hereditarily locally connected.

THEOREM 16. *If a fan is hereditarily contractible, then it is smooth.*

Proof. Assume a fan X with the top p is not smooth. Thus there exists a convergent sequence of points $x_n \in X$ such that, putting $x = \lim_{n\to\infty} x_n$, we have $\operatorname{Lim}_{n\to\infty} px_n \setminus px \neq \emptyset$. If the continuum $\operatorname{Lim}_{n\to\infty} px_n$ is not locally connected, then X is not contractible by Lemma 15. So we can assume that $\operatorname{Lim}_{n\to\infty} px_n$ is locally connected. Since it is a subcontinuum of the fan X containing the top p, it is also a fan which can reduce to an arc in some particular case. In any case there is an end point y of $\operatorname{Lim} px_n$ which is not in the arc px. By the local connectedness of $\operatorname{Lim}_{n\to\infty} px_n$ there is a positive real number η such that

$$\eta < \tfrac{1}{2}\min\{\operatorname{dist}(y, px), \operatorname{dist}(y, (\operatorname{Lim}_{n\to\infty} px_n \setminus py) \cup \{p\})\}.$$

By the definition of an end point of an arc as a point of order 1 (see [7], § 51, I, p. 274) applied to the point y and to the arc py, there is an open set G such that $y \in G$, $\operatorname{diam} G < \frac{1}{2}\eta$ and $py \cap (\bar{G}\setminus G)$ is a one-point set. Denote by a the only point of $py \cap (\bar{G}\setminus G)$. Define (for $n = 1, 2, \ldots$) p_n as the first and q_n as the last point of the arc px_n ordered from p to x_n which lies in the boundary $\bar{G}\setminus G$ of G. Taking proper convergent subsequences if necessary we may assume without loss of generality that the sequences $\{p_n\}$ and $\{q_n\}$ are convergent and we see by construction that $\lim_{n\to\infty} p_n = a = \lim_{n\to\infty} q_n$ and $y \in \operatorname{Lim}_{n\to\infty} p_n q_n$. Putting $u_n = p_{2n}$, $v_n = q_{2n-1}$ and

$$Y' = \overline{\bigcup_{n=1}^{\infty} (pu_n \cup pv_n)}$$

we see that Y' is a subfan of X having u_n and v_n as its end points, and that the sequences $\{u_n\}$ and $\{v_n\}$ are convergent to the point a. Thus condition 1^0 of Definition 4 is satisfied with $a = b$ (the case when the R-arc is degenerate). We shall show that conditions 2^0 and 3^0 are satisfied too. Namely, let $0 < \varepsilon < \frac{1}{4}\min(\eta, d(y, a))$, where d denotes the metric on X. Since $p \in u_n a \setminus Q(a, \varepsilon)$ for almost all n, the sets $u_n a \cap Q(a, \varepsilon)$ are not connected. Thus $u_n(\varepsilon)$ are well defined and — taking convergent subsequences if necessary — we see that $\operatorname{Lim}_{n\to\infty} u_n u_n(\varepsilon) \subset ap$ by construction. Similarly, since

$$y \in \operatorname{Lim}_{n\to\infty} p_n q_n \setminus Q(a, \varepsilon) \subset \operatorname{Lim}_{n\to\infty} v_n a \setminus Q(a, \varepsilon)$$

we see that the sets $v_n a \cap Q(a, \varepsilon)$ are not connected. So $v_n(\varepsilon)$ are well defined and, using the same arguments as previously we conclude $\operatorname{Lim}_{n\to\infty} v_n v_n(\varepsilon) \subset ay$. Therefore

$[\operatorname{Lim}_{n\to\infty} u_n u_n(\varepsilon)] \cap [\operatorname{Lim}_{n\to\infty} v_n v_n(\varepsilon)] = \{a\}$. Observe further that the sets $u_n u_n(\varepsilon)$ and $v_n v_n(\varepsilon)$ are contained in $Q(a, \varepsilon)$ and therefore do not contain the only ramification point p of Y'. So conditions 2^0 and 3^0 are fulfilled for the degenerate R-arc $\{a\}$. Thus Y' is not contractible by Corollary 6 which finishes the proof.

Observe that the hypothesis that the dendroid under consideration is a fan is essential in Theorem 16. Namely, put in the rectangular Cartesian coordinate system in the plane $p = (0, 0)$, $q = (3, 0)$, $p_n = (1, 1/n)$ and $q_n = (2, 1/n)$ for $n = 1, 2, \ldots$ and define

$$D = \overline{pq} \cup \bigcup_{n=1}^{\infty} (\overline{pp_n} \cup \overline{qq_n}).$$

It is obvious that D is a non-smooth, hereditarily contractible dendroid having two ramification points p and q.

Proposition 14 and Theorem 16 imply

COROLLARY 17. *A fan is hereditarily contractible if and only if it is smooth.*

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INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY
INSTYTUT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO

Accepté par la Rédaction le 8. 4. 1976