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On a problem of Sikorski

by

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Abstract. It is shown that the existence of an ω_1-metrizable Lindelöf space of cardinality bigger than ω_1 is equivalent to the existence of a Kurepa tree with no Aronszajn subtree. Thus the problem whether such spaces exist (asked by Sikorski in [5]) turns out to be both consistent with and independent of the usual axioms of set theory.

Let us first recall the definition of ω_μ-metric. Let μ be an ordinal and G an ordered abelian group such that $\{g_\xi\colon \xi<\mu\}$ is a strictly decreasing sequence converging to the unit element $0\in G$. Let X be a set and let $\varrho\colon X\times X\to\{g\in G\colon g\geqslant 0\}$ be a function such that

(i) $\varrho(x,y)=0 \leftrightarrow x=y$,

(ii) $\varrho(x,y)\leqslant\varrho(x,z)+\varrho(y,z)$,

(iii) $\varrho(x,y)=\varrho(y,x)$.

ϱ is called an *ω_μ-metric* on X. A topological space is called *ω_μ-metrizable* iff it has the topology generated by some ω_μ-metric. As is shown in [7], the ω_0-metrizable spaces are the usual metrizable spaces. The ω_1-metrizable spaces are the "metric" spaces for countable folks.

A topological space X is *$\varkappa$-compact* iff every open cover has a subcover of cardinality $<\varkappa$. In 1950 R. Sikorski asked if there were ω_μ-compact, ω_μ-metrizable spaces of cardinality $>\omega_\mu$. In case $\mu=0$ the answer is clearly yes since the unit interval is such a space. Let us concentrate on the case $\mu=1$ and try to find a "unit interval" for the countable folks, i.e., a Lindelöf, ω_1-metrizable space of cardinality $>\omega_1$.

A tree $\langle T,<_T\rangle$ is a partial order such that for each $x\in T$ the set

$$\hat{x}=\{t\in T\colon t<_T x\}$$

is well-ordered. If α is an ordinal, the αth level of $\langle T,<_T\rangle$ is $\{x\in T\colon \hat{x}$ is order isomorphic to the ordinal $\alpha\}$. If $\varkappa$ is an ordinal and λ is a cardinal $\langle T,<_T\rangle$ is a *$(\varkappa,\lambda)$-tree* iff $T=\bigcup\{T_\alpha\colon \alpha<\varkappa\}$ and for all $\alpha<\varkappa$, $0<|T_\alpha|<\lambda$.

A branch $b\subset T$ is a maximal chain of $\langle T,<_T\rangle$. A cofinal branch intersects each level. An *Aronszajn tree* is an (ω_1,ω_1)-tree with no cofinal branches. A *Kurepa tree* is an (ω_1,ω_1)-tree with $\geqslant\omega_2$ cofinal branches. For further basic results about trees, please consult [1] or [3].

In [4], a Suslin line is derived from a Suslin tree. In a similar manner, but using only cofinal branches, a linear order can be obtained from a Kurepa tree. We call such a line a *Kurepa line*. The process is as follows.

Assume $\langle K, <_K\rangle$ is a Kurepa tree. There exists another Kurepa tree $\langle T, <_T\rangle$ such that

(i) $|T_0| = 1$,

(ii) $(\forall\alpha<\omega_1)(\forall x\in T_\alpha)(|\{t\in T_{\alpha+1}:\ x<_T t\}| = \omega)$,

(iii) every $x\in T$ lies on ω_2 cofinal branches of T,

(iv) if β is a limit ordinal and $x, y\in T_\beta$ and $\hat{x} = \hat{y}$ then $x = y$.

The proof of this statement is not difficult. First, eliminate those elements of K which do not satisfy (iii). Add a "root" to the result to obtain a tree K' satisfying (i) and (iii). Now, as in [1], page 45 "squash" K' to obtain a subtree $K''\subset K'$ which satisfies (i), (ii) and (iii).

Let T be the "quotient tree" of K'' with respect to the following equivalence relation:

$$x\sim y \quad \text{iff } \exists \text{ limit ordinal } \beta \quad (x, y\in T_\beta \text{ and } \hat{x} = \hat{y})\,.$$

T satisfies (iv) and has the same number of cofinal branches as K'', since if two branches in K'' differed in K''_β, they now differ in $T_{\beta+1}$. Thus T satisfies (i)–(iv), and we call such a tree a *very normal Kurepa tree.*

The significance of such a tree is that, as in [4], a linear order $<_E$ can be imposed on the set $E = \{b\subset T:\ b \text{ is a cofinal branch of } T\}$ such that $\mathscr{U} = \{U_t:\ U_t = \{b: t\in b\}$ and $t\in T\}$ is a basis for the order topology on E induced by $<_E$. The order $<_E$ is obtained by first making $\{t\in T_{\alpha+1}:\ x<_T t\}$ order isomorphic to the integers for each $x\in T_\alpha$ and each $\alpha<\omega_1$, and then taking the lexicographic order on E.

LEMMA 1. *If there exists a Kurepa tree, then there exists a Kurepa line E such that $|E|\geqslant\omega_2$ and with respect to the order topology on E:*

(i) *there exists a base $\mathscr{U}$ for E of cardinality ω_1 such that every open cover $\mathscr{V}$ of E has a discrete refinement $\mathscr{V}'\subset\mathscr{U}$,*

(ii) *E is ω_1-additive, i.e., G_δ subsets are open.*

Proof. Consider a very normal Kurepa tree, T, and the Kurepa line, E, derived from the branches of T as above. $\mathscr{U} = \{U_t:\ t\in T\}$ is, as above, a basis for the order topology on E. Clearly $|\mathscr{U}| = |T| = \omega_1$.

In order to prove (i) let $\mathscr{V}$ be an open cover of E. We can assume $\mathscr{V}\subset\mathscr{U}$ and $\mathscr{V} = \{U_t:\ t\in I\}$, where $I\subset T$. Let $\mathscr{V}' = \{U_t:\ t\in J\}$, where $J = \{t\in I:$ for no $x\in I$ is $x<_T t\}$. $\mathscr{V}'$ is a subcover: if $b\in E$, since $\mathscr{V}$ is a cover, there exists $x\in I$ such that $x\in b$; let x' be the $<_T$-least such x, so that $b\in U_{x'}\in\mathscr{V}'$. $\mathscr{V}'$ is pairwise disjoint: if $b\in U_x\cap U_t$, then $x, t\in b$; thus $x<_T t$ or $t<_T x$ or $x = t$, but if $x, t\in J$ we must have $x = t$. Thus $\mathscr{V}'$ is a pairwise disjoint subcover of $\mathscr{V}$ and hence a discrete open refinement of $\mathscr{V}$.

In order to prove (ii) it suffices to show that $\bigcap\{U_t:\ t\in I\}$ is open, where I is some countable subset of T. If $b\in\bigcap\{U_t:\ t\in I\}$, there exists $x\in b$ such that for all $t\in I$, $t<_T x$. Thus $b\in U_x\subset\bigcap\{U_t:\ t\in I\}$.

It is known that the axiom of constructibility, $V = L$, implies there exists a Kurepa tree. In fact, in [2] it is shown that $V = L$ implies there exists a Kurepa tree with no Aronszajn subtree, and hence a very normal Kurepa tree with no Aronszajn subtree. The relevance of this lies in the following:

LEMMA 2. *Let T be a very normal Kurepa tree and let E be a Kurepa line associated with T. E is Lindelöf iff T has no Aronszajn subtree.*

Proof. In order to prove sufficiency, we suppose E is Lindelöf but T contains an Aronszajn subtree $A\subset T$; we derive a contradiction. For each $b\in E$, let $\alpha_b = \sup\{\alpha<\omega_1:\ b\cap T_\alpha\cap A\neq\emptyset\}$. Since A has no cofinal branches $\alpha_b<\omega_1$. For each $b\in E$, let $\beta_b = \sup\{\beta:\ \alpha_b<\beta<\omega_1$ and there exists x^b_β, the $<_T$-least member of A such that the element of $b\cap T_\beta$ is $<_T$-less than $x^b_\beta\}$. The set $\{x^b_\beta:\ \alpha_b<\beta\leqslant\beta_b\}$ is contained in a single level of A because the set of predecessors of each x^b_β in A is $b\cap A$; therefore we must have $\beta_b<\omega_1$. In other words, if we let $I = \{t\in T:$ there does not exist $x\in A$ such that $t\leqslant_T x\}$, each $b\in E$ contains an element of I. Thus $\mathscr{V} = \{U_t:\ t\in I\}$ is an open cover of E and hence has a countable subcover $\mathscr{V}'$. Since every $x\in T$ lies on a $b\in E$, we have $A\subset\{x\in T:$ there exists $t\in T$ such that $x<_T t$ and $U_t\in\mathscr{V}'\}$. Therefore A is countable and hence A has no Aronszajn subtrees.

Now suppose E is not Lindelöf and so there exists an open cover $\mathscr{W}$ with no countable subcover. Let $\mathscr{V}$ be a disjoint refinement of $\mathscr{W}$ as in Lemma 1; $\mathscr{V}$ must then be uncountable. Let $A = \{x\in T:$ there exists $U_t\in\mathscr{V}$ such that $x\leqslant_T t\}$. In order to show that A is an Aronszajn subtree, it suffices to show that A has no uncountable branches. To this end, suppose b is an uncountable branch of A; since A is closed under T-predecessors, we have $b\in E$. Therefore, there exists $U_t\in\mathscr{V}$ such that $b\in U_t$. Since $\mathscr{V}$ is pairwise disjoint, $U_x\notin\mathscr{V}$ for any $x>_T t$ such that $x\in b$.

Now pick $y\in b$ such that $t<_T y$; $y\in A$, hence there is $x\in T$ such that $y\leqslant_T x$ and $U_x\in\mathscr{V}$, which contradicts the previous conclusion.

By the results of [7] the Kurepa line E is ω_1-metrizable, hence the existence of a Kurepa tree with no Aronszajn subtree yields a solution to Sikorski's problem for the case $\mu = 1$. In fact, as follows from the next theorem, this strange-looking tree is just the essence of that very natural problem.

THEOREM 3. *The following are equivalent.*

(i) *There exists an ω_1-metrizable Lindelöf space of cardinality $>\omega_1$.*

(ii) *There exists an ω_1-metrizable Lindelöf space of cardinality $>\omega_1$ which has no isolated points.*

(iii) *There exists a Kurepa tree with no Aronszajn subtree.*

(iv) *There exists an ω_1-additive Lindelöf linearly ordered topological space with weight ω_1 and cardinality $>\omega_1$.*

Proof. (i)→(ii). Let X be a space as given in (i). We first claim that $w(X) = \omega_1$. Let $\{g_\xi: \xi<\omega_1\}$ be as in the definition of the ω_1-metric. For each $\xi<\omega_1$, let $\mathscr{W}_\xi = \{N_\xi(x): x \in X\}$, where $N_\xi(x) = \{y \in X: \varrho(x, y)<g_\xi\}$. Let $\mathscr{W}'_\xi$ be a countable subcover of $\mathscr{W}_\xi$. It is now straightforward to show that $\mathscr{W} = \bigcup\{\mathscr{W}'_\xi: \xi<\omega_1\}$ is a basis for X of cardinality ω_1. Now a Cantor–Bendixon type of argument will show that X has a closed subset X^* of cardinality $|X|$ and with no isolated points.

(ii)→(iii). Suppose X is a space given by (ii). Since X is ω_1-metrizable it is also zero-dimensional. We construct a Kurepa tree T from the clopen subsets of X with inclusion as $<_T$. Let $\{g_\xi: \xi<\omega_1\}$ be as in the definition of ω_1-metric. We first note that if U is a clopen subset of X, there exists a countable collection $\mathscr{C}(U, \xi)$ of pairwise disjoint clopen subsets of U of diameter $<g_\xi$ such that $\bigcup\mathscr{C}(U, \xi) = U$. To see this, cover U with clopen subsets of diameter $<g_\xi$, take a countable subcover and "disjointify".

Now let $T_0 = \{X\}$. If T_β has been defined and is a countable collection of clopen subsets of X, define $T_{\beta+1} = \bigcup\{\mathscr{C}(U, \beta+1): U \in T_\beta\}$. If T_β has been defined for all $\beta<\lambda$ and λ is a limit ordinal, let $\mathscr{V} = \{\bigcap B: B$ is a branch of $\bigcup\{T_\beta: \beta<\lambda\}\}$. Since X is ω_1-additive $\mathscr{V}$ is a clopen cover of X. $\mathscr{V}$ is disjoint, hence countable. Let $T = \mathscr{V}\setminus\{\emptyset\}$.

Let $T = \bigcup\{T_\beta: \beta<\omega_1\}$ and let $<_T$ be set inclusion. Suppose B is a branch of T; since X is Lindelöf $\bigcap B \neq \emptyset$. Since $\operatorname{diam}(B \cap T_{\beta+1}) \leqslant g_{\beta+1}$, $\bigcap B$ must be a single point. Thus the number of cofinal branches of $T = |X|$, which is $>\omega_1$ and so T is a Kurepa tree. Furthermore, it is now straightforward to show that topologically X is a Kurepa line associated with the Kurepa tree T. Thus, by Lemma 2, T can have no Aronszajn subtree.

(iii)→(iv) is clear from Lemmas 1 and 2.

(iv)→(i). In [5] it is shown that a regular ω_1-additive space with weight ω_1 is ω_1-metrizable.

The following corollary answers the question of Sikorski.

COROLLARY. *The existence of an ω_1-metrizable Lindelöf space of cardinality $>\omega_1$ is consistent with and independent of the usual axioms of set theory.*

Proof. As mentioned before, it is shown in [2] that V = L implies (iii) of the theorem. In fact, since V = L implies GCH, we can have an ω_1-metrizable Lindelöf space of cardinality 2^{ω_1}, which is of course the largest possible cardinality for such a space. However, in [6] it is shown that it is consistent with the usual axioms of set theory and with the assumption of a large cardinal to assume that there are no Kurepa trees.

Theorem 3 may be generalized. Note that if ω_μ is singular and X is ω_μ-metrizable, then the intersection of ω_μ open sets in ω_μ is open. Thus, if X is also ω_μ-compact, it is easy to show that X must be a discrete space of cardinality $<\omega_\mu$. We therefore generalize Theorem 3 for regular cardinals only.

THEOREM 4. *If ω_μ is regular and $\varkappa>0$, then the following are equivalent.*

(i) *There exists an ω_μ-metrizable, ω_μ-compact space of cardinality $\varkappa$.*

(ii) *There exists an (ω_μ, ω_μ)-tree T with $\varkappa$ cofinal branches such that any (ω_μ, ω_μ) subtree of T has a branch (of cardinality ω_μ).*

Proof is similar to Theorem 3.

We mention that the following known result is also an immediate consequence of this theorem.

COROLLARY. *If ω_μ is a weakly compact cardinal, then there exists an ω_μ-metrizable, ω_μ-compact space of cardinality 2^{ω_μ}.*

Proof. ω_μ is weakly compact iff ω_μ is inaccessible and every (ω_μ, ω_μ)-tree has a cofinal branch. Consider the tree $T = 2^{\omega_\mu} = \{s: s$ is a function from α into 2 for some $\alpha<\omega_\mu\}$ with the order $<_T$ as set inclusion. $\langle T, <_T\rangle$ satisfies (ii) of the theorem.

Let us now look back to see if we have constructed a unit interval for the countable folks. We have, assuming V = L, been able to construct an ω_1-metrizable, ω_1-compact linearly ordered zero-dimensional space of cardinality 2^{ω_1}. This is the Cantor set for the countable folks and this is the best we can do for them since every ω_1-metrizable space is zero-dimensional.

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