$\omega_s \setminus T_s(\lambda) \setminus \overline{\bigcup_{(s,n)\in S(\lambda)} \omega_{s,n}}$ est un voisinage de a disjoint de $K(\lambda)$. Donc $K(\lambda) \cap X = \emptyset$ alors que $K(\lambda) \neq \emptyset$. Il en résulte que $K(\lambda) \notin FX$.

Donc $A = K^{-1}(FX)$. Et si FX était borélien dans KL, il en serait de même de A dans C, contrairement à l'hypothèse faite sur A, et ceci démontre le théorème.

COROLLAIRE 7. *Pour que la structure borélienne de FX soit standard, il faut et il suffit que X soit union d'un K_σ et d'un polonais.*

Ceci résulte immédiatement des théorèmes 1 et 6.

Bibliographie

[1] J. P. R. Christensen, *Topology and Borel structure.*
[2] W. Hurewicz, *Relativ perfekte Teile von Punktmengen und Mengen* (*A*), Fund. Math. 12 (1928), pp. 78–109.
[3] J. Saint Raymond, *Topologie sur l'ensemble des compacts...* Sém. Choquet 1969–70.

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On pure semi-simple Grothendieck categories I

by

Daniel Simson (Toruń)

Abstract. A Grothendieck category $\mathcal{A}$ is called *pure semi-simple* if it is locally finitely presented and each of its objects is pure-projective. It is shown that $\mathcal{A}$ is pure semi-simple if and only if a coproduct of any family of pure-injective objects in $\mathcal{A}$ is pure-injective. We study pure semi-simple categories with a finite number of non-isomorphic simple objects. Every such category is locally finite and, under some special assumptions, is equivalent to a module category over a ring. Applying Auslander's results ([2], [3]) we obtain connections between pure semi-simple categories and categories of finite representation type. For instance, the category of all left modules R-Mod over an artin algebra R is pure semi-simple if and only if R is of finite representation type.

Introduction. Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. $\mathcal{A}$ is called *pure semi-simple* if it has pure global dimension zero, which means that each of its objects is a direct summand of a coproduct of finitely presented objects (see [19] and [20]). A ring R will be called *left pure semi-simple* if the category R-Mod of all left R-modules is pure semi-simple.

A characterization of pure semi-simple categories is given in [18] and [20], where, among others things, it is proved that $\mathcal{A}$ is pure semi-simple if and only if $\mathcal{A}$ is pure-perfect, or equivalently, $\mathcal{A}$ satisfies a pure quasi-Frobenius property [see [18]), or equivalently, every object of $\mathcal{A}$ is a coproduct of indecomposable noetherian subobjects with left coperfect local endomorphism rings.

In the present paper we prove that $\mathcal{A}$ is pure semi-simple if and only if it satisfies the following pure noetherian property: a coproduct of any family of pure-injective objects in $\mathcal{A}$ is pure-injective. The idea of the proof is essentially due to Gruson and Jensen [10] and is the following. The category $\mathcal{A}$ is embedded in some locally coherent Grothendieck category $D(\mathcal{A})$ as its full subcategory consisting of all FP-injective objects (see [22]) and in such a way that the classes of pure-injective objects in $\mathcal{A}$ and injective objects in $D(\mathcal{A})$ coincide. Moreover, pure exact sequences in $\mathcal{A}$ are exact in $D(\mathcal{A})$. As a simple corollary we obtain a result, proved in [20], which asserts that any locally finitely presented Grothendieck category has enough pure-injective objects. These results are contained in Section 1.

In Section 2 we study pure semi-simple categories with a finite number of non-isomorphic simple objects. It is shown that such a category is always locally finite. The classification of these categories is reduced to the classification of left artinian rings Λ with the following two properties: (i) $\Lambda/J(\Lambda)$ is a finite product of division rings, (ii) the functor category ${}_\Lambda\mathrm{fp}$-Mod is perfect, where ${}_\Lambda\mathrm{fp}$ denotes the category of all finitely presented left Λ-modules. Then on the basis of Auslander's results in [2] and [3] we obtain the following connection between pure semi-simple categories and categories of finite representation type. Let $\mathscr{C}$ be a skeletally small abelian category (i.e. the isomorphism classes of objects of $\mathscr{C}$ form a set) with a finite number of non-isomorphic simple objects. Then the following statements are equivalent:

(i) Each object of $\mathscr{C}$ is both noetherian and artinian and $\mathscr{C}$ has only a finite number of non-isomorphic indecomposable objects.

(ii) $\mathrm{Lex}\,\mathscr{C}$ and $\mathrm{Lex}\,\mathscr{C}^{\mathrm{op}}$ are both pure semi-simple categories.

Furthermore, an artin algebra R is left pure semi-simple if and only if R is of finite representation type. This result may be considered as a "pure" version of the Wedderburn–Artin theorem for artin algebras.

In general, we follow the notation and conventions established in [20]. In particular, $\mathscr{A}$ denotes a locally finitely presented Grothendieck category and $\mathrm{fp}(\mathscr{A})$ its full subcategory consisting of all finitely presented objects. If $\mathscr{C}$ is an additive category, $\mathscr{C}$-Mod is the category of all covariant additive functors from $\mathscr{C}$ to the category of abelian groups $\mathscr{A}b$. When $\mathscr{C}$ has kernels, we denote by $\mathrm{Lex}\,\mathscr{C}$ the full subcategory of $\mathscr{C}$-Mod consisting of all left exact functors. Finally, R will denote a ring with an identity element and ${}_R\mathrm{fp}$ (resp. fp_R) is the category of all finitely presented left (resp. right) R-modules.

1. The category $D(\mathscr{A})$. Recall that in [20] the pure-projective dimension is investigated by using the full and faithful embedding

$$h.\colon \mathscr{A}\to L(\mathscr{A}),$$

where $L(\mathscr{A}) = \mathrm{fp}(\mathscr{A})^{\mathrm{op}}$-Mod and $h. = \mathrm{Hom}_{\mathscr{A}}(?,\cdot)$. It is observed that $h.$ establishes an equivalence between $\mathscr{A}$ and $\mathrm{Lex}\,\mathrm{fp}(\mathscr{A})^{\mathrm{op}} = \mathrm{fp}(\mathscr{A})^{\mathrm{op}}$-Fl. Furthermore, there are natural isomorphisms

$$\mathrm{Pext}^n_{\mathscr{A}}(M,N) = \mathrm{Ext}^n_{L(\mathscr{A})}(h_M, h_N), \quad n\geqslant 0.$$

The following proposition sheds light on the categories $L(\mathscr{A})$ and $\mathrm{fp}(\mathscr{A})$-Mod; we will need part of it subsequently.

PROPOSITION 1.1. (a) *The category* $\mathrm{fp}(\mathscr{A})$-Mod *is locally coherent.*

(b) $\mathrm{w.gl.dim}\,L(\mathscr{A})$ *is either* 0 *or* 2.

Proof. Statement (a) follows immediately since $\mathrm{fp}(\mathscr{A})$ has cokernels.

We now prove (b). First observe that since $h.$ commutes with inverse limits, the filtered inverse limit of flat objects in $L(\mathscr{A})$ is flat. Then, by Theorem 4.3 and Corollary 4.5 in [14], $\mathrm{w.gl.dim}\,L(\mathscr{A})\leqslant 2$. Suppose $\mathrm{w.gl.dim}\,L(\mathscr{A})\leqslant 1$ and let $I\subset h_X$, $X\in\mathrm{fp}(\mathscr{A})$, be a finitely generated ideal. Then there exists an exact sequence

$$0\to I\to h_X\xrightarrow{f} h_Y$$

with $Y\in\mathrm{fp}(\mathscr{A})$. By our assumption $\mathrm{Im}\,f$ is flat and finitely presented, and so it is projective and therefore the sequence splits. It follows that $\mathrm{w.gl.dim}\,L(\mathscr{A}) = 0$ and the proposition is proved.

Since $\mathrm{fp}(\mathscr{A})$-Mod is locally coherent, by [17] its full subcategory $\mathscr{B} = \mathrm{Coh}(\mathrm{fp}(\mathscr{A})\text{-Mod})$ consisting of all coherent objects is abelian, $\mathrm{fp}(\mathscr{A})$-Mod $= \mathrm{Lex}\,\mathscr{B}^{\mathrm{op}}$ and the embedding functor $\mathscr{B}\to\mathrm{fp}(\mathscr{A})$-Mod is exact.

Let $\tau\colon \mathrm{fp}(\mathscr{A})\to\mathscr{B}^{\mathrm{op}}$ be the functor given by $\tau(-) = \mathrm{Hom}_{\mathscr{A}}(-,?)$. Then τ^{op} induces a functor

$$G\colon \mathscr{B}\text{-Mod}\to L(\mathscr{A})$$

$G(-) = \mathrm{Hom}_{\mathscr{B}\text{-Mod}}(?,-)$, which by [1, § 3] admits a left adjoint functor

$$T\colon L(\mathscr{A})\to\mathscr{B}\text{-Mod}$$

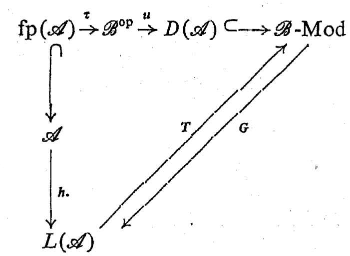
given by $T(-) = -\otimes_{\mathrm{fp}(\mathscr{A})}?$, where

$$\otimes_{\mathrm{fp}(\mathscr{A})}\colon \mathrm{fp}(\mathscr{A})^{\mathrm{op}}\text{-Mod}\times\mathrm{fp}(\mathscr{A})\text{-Mod}\to\mathscr{A}b$$

is the tensor product functor. We put $D(\mathscr{A}) = \mathrm{Lex}\,\mathscr{B}$. By Proposition 3.1 in [1] there is a natural isomorphism

$$(*)\qquad \mathrm{Hom}_{L(\mathscr{A})}(A,A') = \mathrm{Hom}_{\mathscr{B}\text{-Mod}}(T(A),T(A'))$$

and the following diagram commutes:



where $u(-) = \mathrm{Hom}_{\mathscr{B}}(-,?)$. Moreover, it follows from [17] that $D(\mathscr{A})$ is locally coherent, $\mathrm{Coh}\,D(\mathscr{A}) = \mathscr{B}^{\mathrm{op}}$ and u is an exact functor.

Now if $A\in\mathscr{A}$ then $A = \mathrm{colim}\,A_i$, $A_i\in\mathrm{fp}(\mathscr{A})$, and

$$\mathrm{Th}_A = \mathrm{colim}\,\mathrm{Th}_{A_i} = \mathrm{colim}\,u\tau(A_i)\in D(\mathscr{A}).$$

Then we have a unique factorization

$$t\colon \mathscr{A}\to D(\mathscr{A})$$

of the functor Th. On the other hand, if $F\in D(\mathscr{A})$ then $G(F)$ is left exact, and so it is flat. It follows that the functor t has a right adjoint functor $g\colon D(\mathscr{A})\to\mathscr{A}$ such that $G = h.g$. Furthermore, according to $(*)$ we have a natural isomorphism

$$(**)\qquad \mathrm{Hom}_{\mathscr{A}}(A,A') = \mathrm{Hom}_{D(\mathscr{A})}(t(A),t(A')).$$

It is clear that t commutes with filtered direct limits, coproducts and products, and carries over pure exact sequences into the exact ones.

Lemma 1.2. *If $F \in D(\mathscr{A})$ is an exact functor, then $F \cong t(A)$ for an $A \in \mathscr{A}$.*

Proof. Since g is right adjoint to t, there is a canonical natural transformation $\varphi\colon tg \to \mathrm{id}_{D(\mathscr{A})}$. The functors $h^Z = \mathrm{Hom}_{\mathscr{A}}(Z, -)$, $Z \in \mathrm{fp}(\mathscr{A})$, are finitely generated projective generators of $\mathrm{fp}(\mathscr{A})$-Mod; thus for any $B \in \mathscr{B}$ there exists an exact sequence in $\mathrm{fp}(\mathscr{A})$-Mod

$$h^Y \to h^X \to B \to 0$$

with $X, Y \in \mathrm{fp}(\mathscr{A})$, which is also exact in $\mathscr{B}$. For a given $A \in \mathscr{A}$ the functor $t(\mathscr{A})\colon \mathscr{B} \to \mathscr{A}b$ is right exact (as the restriction of the tensor product functor) and therefore it is exact because $t(A) \in D(\mathscr{A})$.

Now assume that $F \in D(\mathscr{A})$ is an exact functor. Then we derive a commutative diagram

$$\begin{array}{ccccccc} tg(F)h^Y & \longrightarrow & tg(F)h^X & \longrightarrow & tg(F)B & \longrightarrow & 0 \\ \downarrow{\scriptstyle \varphi(F)h^Y} & & \downarrow{\scriptstyle \varphi(F)h^X} & & \downarrow{\scriptstyle \varphi(F)B} & & \\ F(h^Y) & \longrightarrow & F(h^X) & \longrightarrow & F(B) & \longrightarrow & 0 \end{array}$$

with exact rows. To prove the lemma it is sufficient to show that the left two vertical maps are isomorphisms. But this follows from the fact that for any $Z \in \mathrm{fp}(\mathscr{A})$ we have isomorphisms

$$tg(F)h^Z = t(Fh^{\cdot})h^Z = (Fh^{\cdot}) \otimes_{fp(\mathscr{A})} h^Z = F(h^Z)$$

and the composed isomorphism is equal to $\varphi(F)h^Z$.

Proposition 1.3. *Let $A\colon 0 \to A' \xrightarrow{i} A \to A'' \to 0$ be an exact sequence in the category $\mathscr{A}$. The following statements are equivalent:*

(1) *A is pure.*

(2) *The sequence h_A in $L(\mathscr{A})$ is exact.*

(3) *h_A is pure exact.*

(4) *The sequence $t(A)$ is exact in $D(\mathscr{A})$.*

(5) *$t(A)$ is pure exact.*

Proof. The equivalence (1)$\leftrightarrow$(2) is immediate and (2) is equivalent to (3) because $h_{A''}$ is flat.

(1)$\leftrightarrow$(4). The monomorphism i is pure if and only if $h_i\colon h_{A'} \to h_A$ is a pure monomorphism in $L(\mathscr{A})$. But $\mathscr{B} = \mathrm{fp}(\mathrm{fp}(\mathscr{A})\text{-Mod})$; thus in view of Lemma 2.4(c) in [20] i is pure if and only if the abelian group homomorphism $t(i)M = h_i \otimes_{\mathrm{fp}(\mathscr{A})} M$ is a monomorphism for any coherent $\mathrm{fp}(\mathscr{A})$-module M, i.e. $t(i)$ is a monomorphism. Since the functor t is right exact, the required equivalence is proved.

To prove the equivalence of (1) and (5) it is sufficient to show that the sequence $t(A)$ is pure in $D(\mathscr{A})$ if it is exact in $\mathscr{B}$-Mod. Indeed, the purity of A implies that of h_A and therefore $t(A) = \mathrm{T}h_A$ is exact in $\mathscr{B}$-Mod because the tensor product functor carries over pure exact sequences into the exact ones. Now since $\mathrm{fp}(D(\mathscr{A})) = \mathrm{Coh}\, D(\mathscr{A}) = \mathscr{B}^{\mathrm{op}}$, we have a commutative diagram

$$\begin{array}{ccc} D(\mathscr{A}) & \xrightarrow{h.} & L(D(\mathscr{A})) \\ & \searrow & \wr\! \wr \\ & & \mathscr{B}\text{-Mod.} \end{array}$$

When the sequence $t(A)$ is exact in $\mathscr{B}$-Mod, $h_{t(A)}$ is exact in $L(D(\mathscr{A}))$ and from the implication (2)$\to$(1) applied to the category $D(\mathscr{A})$ we conclude that $t(A)$ is pure as required. This completes the proof.

We recall that an object Q is pure-injective if it is injective with respect to pure monomorphisms.

Proposition 1.4. *Let Q be an object of the category $\mathscr{A}$. Then Q is pure-injective if and only if $t(Q)$ is an injective object in $D(\mathscr{A})$.*

Proof. The proof presented here is due to L. Gruson and C. U. Jensen. Suppose Q is pure-injective and let $f\colon t(Q) \to I$ be a monomorphism in $D(\mathscr{A})$ with an injective object I. By Corollary 1, p. 354 in [8] I is an exact functor and therefore, by Lemma 1.2, $I \cong t(A)$ for a certain $A \in \mathscr{A}$. Since t is full and faithfull, there exists a monomorphism $i\colon Q \to A$ such that $f = t(i)$. By Proposition 1.3 i is pure. Hence i and f split, which shows that $t(Q)$ is injective.

Conversely, assume that $t(Q)$ is injective and consider a diagram in $\mathscr{A}$

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{i} & A' \\ & & \downarrow{\scriptstyle f} & & \\ & & Q & & \end{array}$$

with a pure monomorphism i. By Proposition 1.3, $t(i)$ is a monomorphism and hence $t(f) = pt(i)$ for a certain $p\colon t(A') \to t(Q)$. Since t is full and faithfull, $f = gi$ for some $g\colon A' \to Q$. Consequently Q is pure-injective and the proof is complete.

As a corollary we obtain the following result, proved in [20, § 4].

Corollary 1.5. *Every locally finitely presented category $\mathscr{A}$ has enough pure-injective objects.*

Proof. Let $A \in \mathscr{A}$ and let $f\colon t(A) \to I$ be a monomorphism in $D(\mathscr{A})$ with an injective object I. Then from Proposition 1.3 and its proof we conclude that there exists a pure monomorphism $i\colon A \to Q$ with a pure-injective object Q as desired.

Recall that a ring R is said to be *F-semiperfect* if idempotents can be lifted modulo the Jacobson radical $J(R)$ and $R/J(R)$ is regular in the sense of von Neumann (see [15]). Then in virtue of formula (**) from Theorem B in [15] and Proposition 1.4 above we immediately obtain the following result:

Corollary 1.6. *The endomorphism ring of any pure-injective object in a locally finitely presented Grothendieck category is F-semiperfect.*

Following Stenström [22], we call an object A of $\mathscr{A}$ FP-injective if it satisfies one of the following equivalent conditions:

PROPOSITION 1.7. *Let A be an object of a locally finitely presented category $\mathscr{A}$. The following conditions are equivalent:*

(a) $\mathrm{Ext}^1_{\mathscr{A}}(X, A) = 0$ *for every* $X \in \mathrm{fp}(\mathscr{A})$.

(b) *Every exact sequence* $0 \to A \to M \to N \to 0$ *is pure.*

(c) *There exists a pure exact sequence* $0 \to A \to Q \to N \to 0$ *with Q injective.*

The proof is left to the reader.

The FP-injective objects in the category $D(\mathscr{A})$ are characterized by the following proposition.

PROPOSITION 1.8. *Let $F \in D(\mathscr{A})$. The following conditions are equivalent:*

(1) *F is FP-injective.*

(2) *F: $\mathscr{B} \to \mathscr{A}b$ is an exact functor.*

(3) $F \in \mathrm{Im}(t\colon \mathscr{A} \to D(\mathscr{A}))$.

Proof. (1)→(2). It is sufficient to observe that F is a composition of the Yoneda functor $h^{\cdot}\colon \mathscr{B} \to D(\mathscr{A})$ and the functor $X \mapsto \mathrm{Hom}_{D(\mathscr{A})}(X, F)$. By [17] $h^{\cdot}$ is exact and by the assumption the second functor is exact on the image of $h^{\cdot}$.

The implication (2)→(3) follows from Lemma 1.2. To prove that (3) implies (1) suppose $F = t(A)$, $A \in \mathscr{A}$. According to Corollary 1.5 there exists a pure exact sequence

$$0 \to A \to Q \to K \to 0$$

with Q pure-injective. By Propositions 1.3 and 1.4 we derive a pure exact sequence in $D(\mathscr{A})$

$$0 \to t(A) \to t(Q) \to t(K) \to 0\,,$$

where $t(Q)$ is injective. Hence $t(A)$ is FP-injective and the proposition is proved.

It follows that $t\colon \mathscr{A} \to D(\mathscr{A})$ establishes an equivalence of $\mathscr{A}$ and the full subcategory of $D(\mathscr{A})$ consisting of all FP-injective objects. Pure-injective objects of $\mathscr{A}$ correspond to the injective ones in $D(\mathscr{A})$. Hence if $\mathbf{Q}$ is a pure-injective resolution of an object A in $\mathscr{A}$, then $t(\mathbf{Q})$ is an injective resolution of $t(A)$ in $D(\mathscr{A})$ and formula (**) yields

$$\mathrm{Pext}^n_{\mathscr{A}}(A, A') = \mathrm{Ext}^n_{D(\mathscr{A})}(t(A), t(A'))$$

for $n \geq 0$. Furthermore, $\mathrm{P.id}_{\mathscr{A}} A = \mathrm{inj.dim}_{D(\mathscr{A})} t(A)$.

We are now able to prove the following result:

THEOREM 1.9. *Let $\mathscr{A}$ be a locally finitely presented Grothendieck category. The following conditions are equivalent:*

(1) *$\mathscr{A}$ is pure semi-simple.*

(2) *$L(\mathscr{A})$ is perfect.*

(3) *$\mathrm{fp}(\mathscr{A})$-Mod is coperfect.*

(4) *$\mathrm{fp}(\mathscr{A})$-Mod is semiartinian (i.e. each $\mathrm{fp}(\mathscr{A})$-module has a non-zero simple submodule) and any finitely presented object in $\mathscr{A}$ is a finite coproduct of indecomposable subobjects.*

(5) *$D(\mathscr{A})$ is locally noetherian.*

(6) *A coproduct of any family of pure-injective objects in $\mathscr{A}$ is pure-injective.*

(7) *There exists an object D in $\mathscr{A}$ such that every object of $\mathscr{A}$ admits a pure embedding in a suitable coproduct of copies of D.*

Proof. The equivalence (1)↔(2) was proved in [20]. (3)→(4) is obvious and (4)→(2) may be proved by using the well-known arguments of Bass (see [16], p. 360).

By [17] the category $\mathrm{fp}(\mathscr{A})$-Mod is coperfect iff $\mathrm{Coh}(\mathrm{fp}(\mathscr{A})\text{-Mod}) = \mathscr{B}$ is artinian. Moreover, the category $D(\mathscr{A})$ is locally noetherian iff $\mathrm{Coh}\, D(\mathscr{A}) = \mathscr{B}^{\mathrm{op}}$ is noetherian. Consequently conditions (3) and (5) are equivalent. Finally, (1)→(6) and the equivalences (5)↔(6)↔(7) follow by [17] and the remark after Proposition 1.8. The theorem is proved.

The equivalence (1)↔(6) was proved by Gruson and Jensen for $\mathscr{A} = R$-Mod where R is a ring (see [10]).

COROLLARY 1.10. *If $\mathscr{A}$ is a pure semi-simple category, then any indecomposable injective object in $D(\mathscr{A})$ is noetherian.*

Proof. By Theorem 1.9 the category $D(\mathscr{A})$ is locally noetherian and hence $\mathscr{B}^{\mathrm{op}}$ is noetherian. If I is an indecomposable injective object in $D(\mathscr{A})$, then by Propositions 1.8 and 1.4 $I = t(Q)$ is an indecomposable pure-injective object in $\mathscr{A}$. But according to Theorem 6.3 in [20] Q is finitely presented and hence $I = t(Q) = u\tau(Q)$ is noetherian because the functor τ is exact (see the diagram beneath formula (*)).

2. Pure semi-simplicity and the finite representation type property. In this section we study the connections between the pure semi-simplicity and the finite representation type property [2]. We start with some preliminary results on endomorphism rings.

PROPOSITION 2.1. *Let M be a noetherian (resp. artinian) object of an abelian category $\mathscr{D}$. If P is a finitely generated projective object in $\mathscr{D}$, then the right $\mathrm{End}(P)$-module $\mathrm{Hom}_{\mathscr{D}}(P, M)$ is noetherian (resp. artinian). Dually, if Q is an injective object in $\mathscr{D}$, then the left $\mathrm{End}(Q)$-module $\mathrm{Hom}_{\mathscr{D}}(M, Q)$ is coperfect (resp. noetherian).*

Proof. Apply the arguments from the proof of Proposition 3.10 in [20].

Let us recall that an object is called finite if it is both noetherian and artinian.

COROLLARY 2.2. *If X and Y are finitely presented objects of a pure semi-simple category $\mathscr{A}$, then the ring $\mathrm{End}(Y)$ is left artinian and the left $\mathrm{End}(Y)$-module $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ is finite.*

Proof. By Theorem 1.9 $\mathscr{A}$ is pure semi-simple if and only if every $\mathrm{fp}(\mathscr{A})$-module $h^A = \mathrm{Hom}_{\mathscr{A}}(A, -)$, $A \in \mathrm{fp}(\mathscr{A})$, is artinian. Then in view of the isomorphism

$\operatorname{Hom}_{\mathscr{A}}(X, Y) = \operatorname{Hom}_{\mathrm{fp}(\mathscr{A})\text{-Mod}}(h^Y, h^X)$ the corollary is a consequence of Proposition 2.1.

As an immediate consequence of Corollary 2.2 and Corollary 6.5 in [20] we get

COROLLARY 2.3. *If R is a left pure semi-simple ring, then R is both left and right artinian.*

Finally, we note an easy consequence of Corollary 1.6.

COROLLARY 2.4. *Every object of a pure semi-simple category $\mathscr{A}$ has an F-semi-perfect endomorphism ring.*

When a category of all modules over a ring is pure semi-simple, then by Corollary 2.3 it is locally finite. The next theorem shows that the same result is true for any locally finitely presented category with a finite number of non-isomorphic simple objects.

THEOREM 2.5. *Let $\mathscr{A}$ be a pure semi-simple category such that for each of its finitely presented objects X there is only a finite number of non-isomorphic simple objects of the form X'/X'' where $X'' \subset X' \subset X$. Then $\mathscr{A}$ is locally finite.*

Proof. By Theorem 6.3 in [20] $\mathscr{A}$ is locally noetherian. Hence $\mathrm{fp}(\mathscr{A})$ is a subcategory of $\mathscr{A}$ consisting of all noetherian objects and to prove the theorem it is sufficient to show that every noetherian object in $\mathscr{A}$ is artinian.

Let X be a noetherian object in $\mathscr{A}$ and let $P_1, \ldots, P_n$ are all non-isomorphic simple objects of the form X'/X'', $X'' \subset X' \subset X$. Moreover, let us denote by Q the injective envelope of $P_1 \oplus \ldots \oplus P_n$. Since $\mathscr{A}$ is pure semi-simple, by Theorem 6.3 in [20] every indecomposable object in $\mathscr{A}$ is noetherian, and so Q is noetherian. Now let us consider a descending chain $X \supset X_1 \supset X_2 \supset \ldots$ Then the induced sequence of epimorphisms $X/X_1 \leftarrow X/X_2 \leftarrow X/X_3 \leftarrow \ldots$ derives an ascending chain

$$\operatorname{Hom}_{\mathscr{A}}(X/X_1, Q) \hookrightarrow \operatorname{Hom}_{\mathscr{A}}(X/X_2, Q) \hookrightarrow \operatorname{Hom}_{\mathscr{A}}(X/X_3, Q) \hookrightarrow \ldots$$

of submodules of the left $\operatorname{End}(Q)$-module $\operatorname{Hom}_{\mathscr{A}}(X, Q)$, which is noetherian by Corollary 2.2. Hence the Hom chain terminates for n greater than a certain n_0. Let $n > n_0$. Then the exact sequence

$$0 \to X_n/X_{n+1} \to X/X_{n+1} \to X/X_n \to 0$$

induces the exact sequence

$$0 \to \operatorname{Hom}_{\mathscr{A}}(X/X_n, Q) \xrightarrow{i} \operatorname{Hom}_{\mathscr{A}}(X/X_{n+1}, Q) \to \operatorname{Hom}_{\mathscr{A}}(X_n/X_{n+1}, Q) \to 0,$$

where i is an isomorphism. Hence $\operatorname{Hom}_{\mathscr{A}}(X_n/X_{n+1}, Q) = 0$ and it follows that $X_n/X_{n+1} = 0$. In fact, since X_n is noetherian, the assumption $X_n/X_{n+1} \neq 0$ allows one to choose a maximal subobject Y of X_n such that $X_{n+1} \subset Y$. This is a contradiction because $X_n/Y = P_j$ for a certain j and the composed map $X_n/X_{n+1} \to X_n/Y = P_j \subset Q$ must be zero. Consequently $X_n = X_{n+1}$ for $n > n_0$ and the proof is finished.

To observe that the assumption on the number of non-isomorphic simple object cannot be dropped in Theorem 2.5, consider the following example of a pure semi-simple category.

Let $\mathscr{H}$ be the category of all commutative and cocommutative, connected graded Hopf algebras over a perfect field of a finite characteristic $p > 2$ and let $\mathscr{PH}$ be the full subcategory of $\mathscr{H}$ consisting of all primitively generated Hopf algebras. It follows from [12] and [23] that $\mathscr{PH}$ is a locally noetherian Grothendieck category. Moreover, it is shown in [21] that $\mathscr{PH}$ is pure semi-simple. Since the polynomial algebra $k[x]$ with $\deg x = 2p$ is not an artinian object in $\mathscr{PH}$, $\mathscr{PH}$ is not locally finite.

A skeletally small category $\mathscr{C}$ is a length category if it is abelian and each of its objects is finite (see [9]).

On the basis of Auslander [2] and [3] we have the following

COROLLARY 2.6. *Let $\mathscr{C}$ be a skeletally small abelian category with a finite number of non-isomorphic simple objects. Then the following statements are equivalent:*

(a) $\operatorname{Lex} \mathscr{C}^{\mathrm{op}}$ *is pure semi-simple.*

(b) $\mathscr{C}$-Mod *is coperfect.*

(c) *$\mathscr{C}$ is a length category and for a given sequence*

$$C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \to \ldots \to C_n \xrightarrow{f_n} C_{n+1} \to \ldots$$

of monomorphisms between indecomposable objects in $\mathscr{C}$ there is an integer n such that f_i is an isomorphism for $i > n$.

Proof. By [17] the category $\mathscr{A} = \operatorname{Lex} \mathscr{C}^{\mathrm{op}}$ is locally coherent, $\mathrm{fp}(\mathscr{A}) = \mathscr{C}$, and the full embedding $\mathrm{fp}(\mathscr{A}) \subset \mathscr{A}$ is an exact functor. Then it follows from Theorem 1.9 that conditions (a) and (b) are equivalent.

Now suppose that (b) is satisfied. By Theorem 6.3 in [20] $\mathscr{A}$ is locally noetherian. Since each simple object is noetherian and $\mathrm{fp}(\mathscr{A}) = \mathscr{C}$, $\mathscr{A}$ has only a finite number of non-isomorphic simple objects. By Theorem 2.5 $\mathscr{A}$ is locally finite and therefore $\mathscr{C}$ is a length category. Moreover, by Theorem 6.3 in [20] the Jacobson radical of $\mathscr{C}$ is right T-nilpotent and hence the second statement in condition (c) follows. Since (c)→(b) is a part of the proof of Theorem 3.1 in [2] (see also [3]), the proof of the corollary is complete.

Recall that an additive category is of finite representation type if it has only a finite number of non-isomorphic indecomposable objects (see [2] and [9]).

COROLLARY 2.7. *Let $\mathscr{C}$ be an abelian category with a finite number of non-isomorphic simple objects. Then the following conditions are equivalent:*

(a) *The categories* $\operatorname{Lex}\mathscr{C}$ *and* $\operatorname{Lex}\mathscr{C}^{\mathrm{op}}$ *are pure semi-simple.*

(b) $\mathscr{C}^{\mathrm{op}}$-Mod *is locally finite.*

(c) *$\mathscr{C}$ is a length category of finite representation type.*

Proof. It follows from Theorem 3.1 in [2] that (b) implies (c) because by Corollary 2.6 $\mathscr{C}$ is a length category whenever (b) is satisfied. Since, in virtue of Corollary 2.6 (a) is an immediate consequence of (c), it remains to prove (a)→(b).

Assume (a). By Theorem 6.3 in [20] categories $L(\mathrm{Lex}\mathscr{C}) = \mathscr{C}^{\mathrm{op}}$-Mod and $L(\mathrm{Lex}\mathscr{C}^{\mathrm{op}}) = \mathscr{C}$-Mod are perfect. Then it follows from Theorem 5.4 in [20] that the Jacobson radical of $\mathscr{C}$ is both left and right T-nilpotent. Furthermore, by Corollary 2.6, $\mathscr{C}$ is a length category. Consequently, the statement (e) in Theorem 3.1 in [2] is satisfied and hence $\mathscr{C}^{\mathrm{op}}$-Mod is locally finite.

We recall that a ring R is of finite representation type if it is left artinian and the category ${}_R\mathrm{fp}$ of all finitely presented left R-modules is of finite representation type (see [2]). By Proposition 1.1 in [6] we know that such a ring is also right artinian and fp_R is of finite representation type. By an artin algebra we mean an artinian ring R having the property that the centre of R is an artinian ring and R is a finitely generated module over its centre (see [1], [2], [3]).

As a consequence of Theorem A in [3] and Theorem 6.3 in [20] we have

COROLLARY 2.8. *Let R be an artin algebra. Then the following conditions are equivalent*:

(a) $\mathrm{l.P.gl.dim}\, R = 0$.

(b) *R is of finite representation type.*

(c) *Every left R-module is a direct sum of indecomposable finitely generated submodules.*

It follows that for an artin algebra R $\mathrm{l.P.gl.dim}\, R = 0$ if and only if $\mathrm{r.P.gl.dim}\, R = 0$. Moreover, as an immediate consequence of [5] (see also [11]) we obtain

COROLLARY 2.9. *Let R be a local artin algebra such that the square of its unique maximal ideal is zero. Then* $\mathrm{l.P.gl.dim}\, R = 0$ *and only if R is a Gorenstein ring (i.e.* $\mathrm{l.inj.dim}_R R$ *is finite).*

We now return to our discussion of pure semi-simple categories with a finite number of non-isomorphic simple objects.

Let $\mathscr{A}$ be such a category and let $P_1, \ldots, P_n$ be a complete set of non-isomorphic simple objects in $\mathscr{A}$. We denote by $I_{\mathscr{A}}$ the injective envelope of $P_1 \oplus \ldots \oplus P_n$. By Theorem 2.5, $\mathscr{A}$ is locally finite and according to Gabriel's characterization ([8], Ch. IV) the correspondence $A \mapsto \mathrm{Hom}_{\mathscr{A}}(A, I_{\mathscr{A}})$ establishes an equivalence of the category $\mathscr{A}^{\mathrm{op}}$ and the category $\mathrm{PC}(\Lambda)$ of all left pseudocompact modules over the left pseudocompact ring

$$\Lambda = \Lambda_{\mathscr{A}} = \mathrm{End}(I_{\mathscr{A}}).$$

Since $\mathscr{A}$ is pure semi-simple, $I_{\mathscr{A}}$ is finitely presented and by Corollary 2.2 the ring $\Lambda = \Lambda_{\mathscr{A}}$ is left artinian. Hence the left linear topology on Λ is discrete. It follows that the category ${}_\Lambda\mathrm{fp}$ is equal to the full subcategory of $\mathrm{PC}(\Lambda)$ consisting of all finite objects, and hence we get

$$(***) \qquad \mathrm{fp}(\mathscr{A})^{\mathrm{op}} \cong {}_\Lambda\mathrm{fp},$$

since any duality preserves the finite length property. Observe that $\Lambda/J(\Lambda)$ is a finite direct product of division rings and by Theorem 1.9 ${}_\Lambda\mathrm{fp}$-Mod $= \mathrm{fp}(\mathscr{A})^{\mathrm{op}}$-Mod is a perfect category. Conversely, if Λ is a left artinian ring with the last two properties, then by [8], Ch. II, the category $\mathrm{Lex}\,{}_\Lambda\mathrm{fp}$ is locally finite, $\mathrm{fp}(\mathrm{Lex}\,{}_\Lambda\mathrm{fp})^{\mathrm{op}} = {}_\Lambda\mathrm{fp}$ and hence $\mathrm{Lex}\,{}_\Lambda\mathrm{fp}$ has only a finite number of non-isomorphic simple objects. Furthermore, since $L(\mathrm{Lex}\,{}_\Lambda\mathrm{fp}) = {}_\Lambda\mathrm{fp}$-Mod is perfect, by Theorem 1.9 $\mathrm{Lex}\,{}_\Lambda\mathrm{fp}$ is pure semi-simple. We have thus proved the following

THEOREM 2.10. *The map $\mathscr{A} \mapsto \Lambda_{\mathscr{A}}$ defines a one-to-one correspondence between equivalence classes of pure semi-simple categories with a finite number of non-isomorphic simple objects and isomorphy classes of left artinian rings Λ with the following two properties*:

(i) *$\Lambda/J(\Lambda)$ is a finite product of division rings.*

(ii) *${}_\Lambda\mathrm{fp}$-Mod is a perfect category.*

COROLLARY 2.11. *Let $\mathscr{A}$ be a pure semi-simple category with a finite number of non-isomorphic simple objects. Then $\mathscr{A}$ is equivalent to a module category R-*Mod *over a certain ring R if and only if the injective envelope of each simple left $\Lambda_{\mathscr{A}}$-module is finite.*

Proof. In view of the duality (***) the corollary is a consequence of [7], p. 106, F.

Let S be a commutative ring. We recall that an additive category $\mathscr{B}$ is an S-category if $\mathrm{Hom}_{\mathscr{B}}(X, Y)$ is an S-module for any pair of objects in $\mathscr{B}$ and the morphism composition is S-bilinear (see [13]).

We are now able to prove

COROLLARY 2.12. *Let S be a commutative artinian ring and let $\mathscr{A}$ be a pure semi-simple S-category with a finite number of non-isomorphic simple objects. If the S-module $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ is finitely generated for any pair X, Y of finitely presented objects in $\mathscr{A}$, then the associated ring $\Lambda_{\mathscr{A}}$ is of finite representation type and $\mathscr{A} \cong \mathrm{Mod}\text{-}\Lambda_{\mathscr{A}}$.*

Proof. By our assumption the ring $\Lambda = \Lambda_{\mathscr{A}}$ is an S-algebra finitely generated as an S-module. Hence there is a duality $\mathrm{fp}_\Lambda = {}_\Lambda\mathrm{fp}^{\mathrm{op}}$ (see [8]) and Λ is an artin algebra. Then in view of formula (***) it follows that $\mathrm{fp}(\mathscr{A}) = \mathrm{fp}_\Lambda$. Furthermore, by Theorem 2.5 the category $\mathscr{A}$ is locally finite, thus, according to Theorem 1 p. 356 in [8] we have

$$\mathscr{A} \cong \mathrm{Lex}\,\mathrm{fp}(\mathscr{A})^{\mathrm{op}} \cong \mathrm{Lex}\,\mathrm{fp}_\Lambda^{\mathrm{op}} \cong \mathrm{Mod}\text{-}\Lambda.$$

Finally, by Corollary 2.8 Λ is of finite representation type and the proof is complete.

References

[1] M. Auslander, *Representation theory of Artin algebras I*, Comm. in Algebra 1 (1974), pp. 177–268.

[2] — *Representation theory of Artin algebras II*, ibid. 1 (1974), pp. 269–310.

[3] M. Auslander, *Large modules over Artin algebras*, in: A Collection of Papers in Honour of Samuel Eilenberg. Academic Press, 1975.
[4] S. U. Chase, *Direct product of modules*, Trans. Amer. Math. Soc. 97 (1960), pp. 457–473.
[5] S. E. Dicson, *On algebras of finite representation type*, Trans. Amer. Math. Soc. 135 (1969), pp. 127–141.
[6] V. Dlab and C. M. Ringel, *Decomposition of modules over right uniserial rings*, Math. Z. 129 (1972), pp. 207–230.
[7] P. Freyd, *Abelian Categories*, Harper Row Publishers, New York–Evanston–London 1964.
[8] P. Gabriel, *Des catégories abeliennes*, Bull. Soc. Math. France 90 (1962), pp. 323–448.
[9] — *Indecomposable representations II*, Symposia Math. Ist. Naz. di Alta Mat. 11 (1973), pp. 81–104.
[10] L. Gruson and C. U. Jensen, *Modules algébraiquement compact et foncteurs* $\varprojlim^{(i)}$, C. R. Acad. Sci., Paris, Sér. A 276 (1973), pp. 1651–1653.
[11] M. S. Menzin, *Indecomposable modules over Artin local algebras*, J. Algebra 32 (1974), pp. 207–214.
[12] J. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. 81 (1965), pp. 211–264.
[13] B. Mitchell, *Rings with several objects*, Advances in Math. 8 (1972), pp. 1–161.
[14] U. Oberst and H. Rohrl, *Flat and coherent functors*, J. Algebra 14 (1970), pp. 91–105.
[15] — and H. J. Schneider, *Die Struktur von projektiven Moduln*, Inventiones Math. 13 (1971), pp. 295–304.
[16] N. Popescu, *Abelian Categories with Applications to Rings and Modules*, Academic Press, New York–London 1973.
[17] J. E. Roos, *Locally noetherian categories and generalized linearly compact rings. Applications*, Lecture Notes in Math., Springer-Verlag 92 (1969), pp. 197–277.
[18] D. Simson, *On colimits of injectives in Grothendieck categories*, Arkiv för Matematik 12 (1974), pp. 161–165.
[19] — *Functor categories in which every flat object is projective*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 22 (1974), pp. 375–380.
[20] — *On pure global dimension of locally finitely presented Grothendieck categories*, Fund. Math. 96 (1977), pp. 91–116.
[21] — and A. Skowroński, *On the category of commutative connected graded Hopf algebras over a perfect field*, Fund. Math. (to appear).
[22] B. Stenström, *Coherent rings and FP-injective modules*, J. London Math. Soc. 2 (1970), pp. 323–329.
[23] C. Schoeller, *Etude de la catégorie des algébres de Hopf commutatives connexes sur un corps*, Manuscripta Math. 3 (1970), pp. 133–155.

INSTITUTE OF MATHEMATICS, NICHOLAS COPERNICUS UNIVERSITY
Toruń

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On a problem of Sikorski

by

I. Juhász and **William Weiss** (Budapest)

Abstract. It is shown that the existence of an ω_1-metrizable Lindelöf space of cardinality bigger than ω_1 is equivalent to the existence of a Kurepa tree with no Aronszajn subtree. Thus the problem whether such spaces exist (asked by Sikorski in [5]) turns out to be both consistent with and independent of the usual axioms of set theory.

Let us first recall the definition of ω_μ-metric. Let μ be an ordinal and G an ordered abelian group such that $\{g_\xi\colon \xi<\mu\}$ is a strictly decreasing sequence converging to the unit element $0\in G$. Let X be a set and let $\varrho\colon X\times X\to\{g\in G\colon g\geqslant 0\}$ be a function such that

(i) $\varrho(x,y)=0 \leftrightarrow x=y$,

(ii) $\varrho(x,y)\leqslant\varrho(x,z)+\varrho(y,z)$,

(iii) $\varrho(x,y)=\varrho(y,x)$.

ϱ is called an *ω_μ-metric* on X. A topological space is called *ω_μ-metrizable* iff it has the topology generated by some ω_μ-metric. As is shown in [7], the ω_0-metrizable spaces are the usual metrizable spaces. The ω_1-metrizable spaces are the "metric" spaces for countable folks.

A topological space X is *$\varkappa$-compact* iff every open cover has a subcover of cardinality $<\varkappa$. In 1950 R. Sikorski asked if there were ω_μ-compact, ω_μ-metrizable spaces of cardinality $>\omega_\mu$. In case $\mu=0$ the answer is clearly yes since the unit interval is such a space. Let us concentrate on the case $\mu=1$ and try to find a "unit interval" for the countable folks, i.e., a Lindelöf, ω_1-metrizable space of cardinality $>\omega_1$.

A tree $\langle T, <_T\rangle$ is a partial order such that for each $x\in T$ the set

$$\hat{x}=\{t\in T\colon t<_T x\}$$

is well-ordered. If α is an ordinal, the αth level of $\langle T, <_T\rangle$ is $\{x\in T\colon \hat{x}$ is order isomorphic to the ordinal $\alpha\}$. If $\varkappa$ is an ordinal and λ is a cardinal $\langle T, <_T\rangle$ is a *$(\varkappa,\lambda)$-tree* iff $T=\bigcup\{T_\alpha\colon \alpha<\varkappa\}$ and for all $\alpha<\varkappa$, $0<|T_\alpha|<\lambda$.

A branch $b\subset T$ is a maximal chain of $\langle T, <_T\rangle$. A cofinal branch intersects each level. An *Aronszajn tree* is an (ω_1,ω_1)-tree with no cofinal branches. A *Kurepa tree* is an (ω_1,ω_1)-tree with $\geqslant\omega_2$ cofinal branches. For further basic results about trees, please consult [1] or [3].