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On the transformers of derivatives

by

M. Laczkovich and **G. Petruska** (Budapest)

Abstract. Let D be the class of derivatives defined on $[0, 1]$. Let T denote the class of transformers on D that is let $g \in T$ iff g is a homeomorphism of $[0, 1]$ onto itself, $g(0) = 0$, $g(1) = 1$ and $f \in D$ implies $f \circ g \in D$.

A result of R. J. Fleissner implies that if g is continuously differentiable and $1/g'(x)$ is of bounded variation then $g \in T$. On the other hand A. M. Bruckner has shown that $g \notin T$ can hold even if both g and g^{-1} (the inverse function of g) satisfy Lipschitz 1 condition.

In this paper a necessary and sufficient condition is given for g to be a transformer. The authors give some applications of this result as well. Some of them:

If $g(0) = 0$, $g(1) = 1$, there exist k and K such that

$$0<k\leqslant \underline{D}g(x)\leqslant \overline{D}g(x)<K<\infty$$

for every $x \in [0, 1]$ and $g'(x)$ is of bounded variation on the set of its existence then $g \in T$. ($\underline{D}g$ and $\overline{D}g$ denote the lower and upper derivative of g.)

If $g(0) = 0$, $g(1) = 1$ and g is convex then $g \in T$ if and only if

$$\limsup_{x\to 1} \frac{g(x)-1}{x-1}\cdot\frac{1}{\underline{D}g(x)}<\infty .$$

There exists a strictly increasing convex function g such that $g \in T$ and $g^{-1} \notin T$.

1. Introduction. Let $g(x)$ be a homeomorphism of $[0, 1]$ onto itself, $g(0) = 0$, $g(1) = 1$. If F is an arbitrary family of real functions defined on $[0, 1]$ then g is said to be a *transformer* on F if $f(g(x)) \in F$ holds for every $f \in F$. In this paper we are going to find the characteristic properties of the transformes on D where D denotes the class of (finite) derivatives defined on $[0, 1]$. T denotes the family of transformers on D.

A. M. Bruckner has shown in [2] that $g \notin T$ can hold even if both g and its inverse function satisfy Lipschitz 1 condition. On the other hand, it turns out from [3] that if $g(x)$ is continuously differentiable and $1/g'(x)$ is of bounded variation then $g \in T$. In fact, under the conditions mentioned above $1/g'(x)$ is a multiplier on D, i.e., $f \in D$ implies $f\cdot(1/g') \in D$. Now if F is a primitive of $f \in D$ then $F(g(x))' = f(g(x))g'(x) \in D$ and hence

$$f(g(x))g'(x)\frac{1}{g'(x)} = f(g(x)) \in D$$

holds, too.

As it will turn out the condition that $1/g'(x)$ is of bounded variation and continuous is not very far from being necessary as well.

2. We introduce the following notations. If $f(x)$ is an arbitrary (finite) real function then $D(f;x)$ denotes any one of the derived numbers of f in x. Of course, if $f'(x)$ exists then $D(f;x)=f'(x)$, otherwise $D(f;x)$ can be chosen in several ways. As usual, $f'_+(x)$, $f'_-(x)$ and $\underline{D}f(x)$, $\overline{D}f(x)$ denote the right hand side and left hand side derivatives (if they exist and are finite), and the lower and upper derivatives (possibly $\underline{D}=\pm\infty$, $\overline{D}=\pm\infty$). $V(f;E)$ denotes the total variation of $f(x)$ on the set E.

THEOREM 1. *If $g\in T$ and $a\in[0,1]$ then there exist $\delta>0$ and $K>0$ such that $\underline{D}g(x)>0$ and*

$$\left|\frac{g(x)-g(a)}{x-a}\cdot\frac{1}{\underline{D}g(x)}\right|<K$$

if $0<|x-a|<\delta$.

Proof. Suppose indirectly that there exists a sequence

$$x_0=1>x_1>x_2>\ldots,\quad x_n\to a\quad (a\in[0,1))$$

with

(1) $$\frac{g(x_n)-g(a)}{x_n-a}>n\underline{D}g(x_n)\quad (n=1,2,\ldots).$$

We can also suppose (after selecting a subsequence) that

$$\underline{D}g(x_n)=\lim_{t\to x_n+0}\frac{g(t)-g(x_n)}{t-x_n}.$$

(The proof runs analogously if $\underline{D}$ is produced by left hand side limit or x_n tends to a from the left.)

Let $\eta_n>0$ be so small that

(2) $$\frac{\eta_n}{x_n-a}\to 0\quad (n\to\infty),$$

denoting $\zeta_n=g(x_n+\eta_n)-g(x_n)$,

(3) $$\frac{g(x_n)-g(a)}{g(x_n)-\zeta_n-g(a)}<2\quad (n=1,2,\ldots),$$

(4) $$\zeta_n<\min[\tfrac{1}{2}(g(x_n)-g(x_{n+1}));\ \tfrac{1}{8}(g(x_{n-1})-g(x_n))]\quad (n=1,2,\ldots).$$

In addition to (2), (3) and (4) we can choose η_n such that

(5) $$\frac{1}{\eta_n}[g(x_n+\eta_n)-g(x_n)]<\frac{1}{n}\cdot\frac{g(x_n)-g(a)}{x_n-a}\quad (n=1,2,\ldots),$$

since by (1) we have

$$\underline{D}g(x_n)<\frac{1}{n}\cdot\frac{g(x_n)-g(a)}{x_n-a}.$$

Let (see Fig. 1)

$$f(x)=\begin{cases}0 & \text{if } x\leqslant g(a) \text{ or } x\geqslant g(x_1) \text{ or } g(x_n)+4\zeta_n\leqslant x\leqslant g(x_{n-1})-\zeta_{n-1}\\ \dfrac{1}{n}\cdot\dfrac{g(x_n)-g(a)}{g(x_n+\eta_n)-g(x_n)} & \text{if } g(x_n)\leqslant x\leqslant g(x_n+\eta_n),\\ -\dfrac{1}{n}\cdot\dfrac{g(x_n)-g(a)}{g(x_n+\eta_n)-g(x_n)} & \text{if } g(x_n)+2\zeta_n\leqslant x\leqslant g(x_n)+3\zeta_n.\end{cases}\quad (n=2,3,\ldots)$$

continuously linear on the remaining intervals

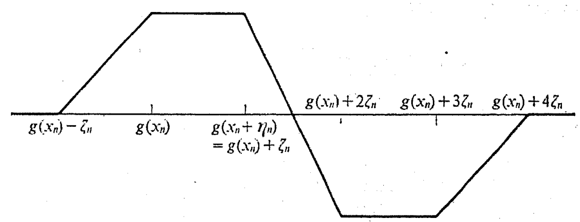

Fig. 1

We can easily check that

$$F(x)=\begin{cases}0 & \text{if } x\leqslant g(a),\\ -\int_x^1 f(t)\,dt & \text{if } x>g(a)\end{cases}$$

is a primitive of $f(x)$, i.e., $f\in D$. In fact, $F'(x)=f(x)$ is trivial for $x\neq g(a)$ and also for the left hand side limit in $g(a)$.

For $g(x_n)-\zeta_n\leqslant x<g(x_{n-1})-\zeta_{n-1}$ we have

$$\left|\frac{F(x)-F(g(a))}{x-g(a)}\right|=\frac{1}{x-g(a)}\left|\int_{g(x_n)-\zeta_n}^{x}f(t)\,dt\right|$$

since

$$\int_{g(x_n)-\zeta_n}^{g(x_n)+4\zeta_n}f(t)\,dt=0\quad (n=1,2,\ldots).$$

Hence

$$\left|\frac{F(x)-F(g(a))}{x-g(a)}\right| \leqslant \frac{2\zeta_n\cdot\frac{1}{n}\cdot\frac{g(x_n)-g(a)}{g(x_n+\zeta_n)-g(x_n)}}{g(x_n)-\zeta_n-g(a)} \leqslant \frac{4}{n}\to 0 \quad \text{for } n\to\infty$$

by (3), i.e., $F'(g(a))=0$.

Thus $f\in D$ and hence $f(g(x))\in D$. Let $G(x)$ be a primitive of $f(g(x))$, then

$$\frac{G(t)-G(a)}{t-a}\to f(g(a))=0 \quad \text{if} \quad t\to a+0\,.$$

Hence

$$\frac{G(x_n)-G(a)}{x_n-a}\to 0 \quad (n\to\infty)$$

and by $x_n-a\sim x_n+\eta_n-a$ (see (2))

$$\frac{G(x_n+\eta_n)-G(a)}{x_n-a}\to 0 \quad (n\to\infty)\,.$$

Thus by subtraction

$$\frac{G(x_n+\eta_n)-G(x_n)}{x_n-a}\to 0 \quad (n\to\infty)\,. \tag{6}$$

We have, however

$$G(x_n+\eta_n)-G(x_n)=\int_{x_n}^{x_n+\eta_n} f(g(x))\,dx=\eta_n\frac{1}{n}\cdot\frac{g(x_n)-g(a)}{g(x_n+\eta_n)-g(x_n)}>x_n-a$$

by (5). Hence

$$\frac{G(x_n+\eta_n)-G(x_n)}{x_n-a}>1$$

which contradicts (6). This contradiction proves the theorem.

Corollary 1. *If $g\in T$ then $\underline{D}g(x)>0$ holds for all but a finite number of $x\in[0,1]$.*

Proof. If a is a limit point of the set $\{x;\ \underline{D}g(x)=0\}$ then Theorem 1 fails to hold in a.

Lemma 1. *Let $f(x)$ be a monotone increasing function on $[a,b]$. The following properties are equivalent to each other.*

(i) *$f'(x)$ is of bounded variation on the set $E=\{x;\ f'(x)$ exists and finite$\}$ and $\overline{D}f(x)<\infty$ at every point of $[a,b]$.*

(ii) *$f(x)=r(x)-s(x)$ where both $r(x)$ and $s(x)$ are convex functions on $[a,b]$ and they have finite derived numbers everywhere. Hence $f'_+(x)$ and $f'_-(x)$ exist for $x\in[a,b)$ and $x\in(a,b]$, respectively; moreover*

$$\begin{aligned} f'_+(x)&=\lim_{t\to x+0}D(f;t)=\lim_{\substack{t\to x+0\\ t\in E}}f'(t) \quad (x\in[a,b))\,,\\ f'_-(x)&=\lim_{t\to x-0}D(f;t)=\lim_{\substack{t\to x-0\\ t\in E}}f'(t) \quad (x\in(a,b]) \end{aligned} \tag{7}$$

hold for any choice of $D(f;t)$.

(iii) *$D(f;x)$ is of bounded variation on $[a,b]$ for any choice of $D(f;x)$.*

Proof. (i)⇒(ii). By (i) there exist bounded and increasing functions $m_1(x)$ and $m_2(x)$ defined on E such that

$$f'(x)=m_1(x)-m_2(x) \quad (x\in E)\,.$$

Let

$$r(x)=\int_a^x m_1(t)\,dt+f(a) \quad (a\leqslant x\leqslant b)\,,$$

$$s(x)=\int_a^x m_2(t)\,dt \quad (a\leqslant x\leqslant b)\,.$$

Since f is monotone, $\lambda(E)=b-a$, $m_1(t)$ and $m_2(t)$ are defined almost everywhere on $[a,b]$ thus the integrals above make sense. r and s are obviously convex functions and since they are integrals of bounded functions, r and s can have finite derived numbers only. It remains to prove $r(x)-s(x)=f(x)$. Consider the function $\Delta(x)=f(x)-(r(x)-s(x))$. Since $\Delta'(x)=0$ holds a.e. on $[a,b]$ and any derived number of Δ is finite ($0\leqslant\underline{D}f\leqslant\overline{D}f<\infty$ by our assumption) referring to [1], 7.2.2.1 (p. 222). Δ is constant and hence $\Delta(x)\equiv\Delta(a)=0$.

(ii)⇒(iii). (ii) obviously implies that for any choice of $D(f;x)$, $D(f;x)=f'_+(x)$ or $f'_-(x)$. Thus if $D(f;x)$ is given then $D(r;x)$ and $D(s;x)$ can be chosen such that $D(f;x)=D(r;x)-D(s;x)$. By (ii) $D(r;x)$ and $D(s;x)$ are finite and increasing functions.

(iii)⇒(i). Trivial.

Lemma 2. *Let the monotone function $f(x)$ satisfy any of* (i), (ii), (iii) *in Lemma* 1 *and let E denote the set $\{x\in[a,b];\ f'(x)$ esixts$\}$. Then*

a) $V(D(f;x);[\alpha,\beta])=V(f';E\cap[\alpha,\beta])+|D(f;\alpha)-f'_+(\alpha)|+|D(f;\beta)-f'_-(\beta)|$ *for any choice of $D(f;x)$ and $[\alpha,\beta]\subset[a,b]$;*

b) *for any continuous function $h(x)$ the integral $\int_\xi^\eta h(t)\,dD(f;t)$ is independent of the choice of $D(f;t)$ whenever $\xi,\eta\in E$, furthermore*

$$\left|\int_\xi^\eta h(t)\,dD(f;t)\right|\leqslant\int_\xi^\eta |h(t)|\,dV(f';E\cap[\xi,t]);$$

c) *for any continuous function* $h(x)$

$$\int_c^d h(t)\,dD(f;t) = \lim_{\substack{\xi\to c+0\\ \eta\to d-0\\ \xi,\eta\in E}} \int_\xi^\eta h(t)\,dD(f;t) - h(c)[D(f,c)-f'_+(c)] + h(d)[D(f,d)-f'_-(d)]$$

$$([c,d]\subset[a,b]);$$

d) $\int_a^b D(f;t)\,dt = f(b)-f(a)$.

Proof. a) Let F_k: $\alpha\leqslant x_1^{(k)}<\ldots<x_{n_k}^{(k)}\leqslant\beta$ be a sequence of subdivisions of $[\alpha,\beta]$ such that $x_i^{(k)}\in E$ $(i=1,2,\ldots,n_k;\ k=1,2,\ldots)$, $x_1^{(k)}\to\alpha$, $x_{n_k}^{(k)}\to\beta$ and

$$\sum_{i=2}^{n_k} |f'(x_i^{(k)})-f'(x_{i-1}^{(k)})| \to V(f';E\cap[\alpha,\beta])$$

if $k\to\infty$. Thus

$$V(D(f;x);[\alpha,\beta])$$
$$\geqslant \lim_{k\to\infty}\Big[|D(f,\alpha)-f'(x_1^{(k)})| + \sum_{i=2}^{n_k}|f'(x_i^{(k)})-f'(x_{i-1}^{(k)})| + |f'(x_{n_k}^{(k)})-D(f,\beta)|\Big]$$
$$= V(f';E\cap[\alpha,\beta]) + |D(f;\alpha)-f'_+(\alpha)| + |D(f;\beta)-f'_-(\beta)|\,.$$

Now let $\alpha=x_0<x_1<\ldots<x_n=\beta$ be an arbitrary subdivision of $[\alpha,\beta]$. According to Lemma 1(ii), (7) there exist sequences $\{t_i^{(k)}\}_{k=1}^\infty$ such that $t_i^{(k)}\in E$ $(k=1,2,\ldots;\ i=0,1,\ldots,n)$, $\lim_{k\to\infty} t_i^{(k)}=x_i$ $(i=0,1,\ldots,n)$ and $\lim_{k\to\infty} f'(t_i^{(k)}) = D(f;x_i)$ $(i=1,2,\ldots,n-1)$. Thus

$$\sum_{i=1}^{n} |D(f,x_i)-D(f,x_{i-1})|$$
$$\leqslant \lim_{k\to\infty}\Big[|D(f,\alpha)-f'(t_0^{(k)})| + \sum_{i=1}^{n}|f'(t_i^{(k)})-f'(t_{i-1}^{(k)})| + |f'(t_n^{(k)})-D(f;\beta)|\Big]$$
$$\leqslant V(f';E\cap[\alpha,\beta]) + |D(f;\alpha)-f'_+(\alpha)| + |D(f;\beta)-f'_-(\beta)|\,.$$

Since the subdivision $\{x_i\}$ was arbitrary we have

$$V(D(f;x);[\alpha,\beta]) \leqslant V(f';E\cap[\alpha,\beta]) + |D(f,\alpha)-f'_+(\alpha)| + |D(f,\beta)-f'_-(\beta)|$$

and the assertion a) is proved.

b) Since $h(x)$ is continuous the integral $\int_\xi^\eta h(t)\,dD(f;t)$ is the limit of sums $\sum_{i=1}^{n} h(x_i)\big(D(f;x_i)-D(f;x_{i-1})\big)$ where we can assume $x_i\in E$ $(i=0,1,\ldots,n)$.

In such cases $D(f;x_i)=f'(x_i)$ which shows that the integral is independent of the choice of $D(f,t)$. Furthermore we have

$$\Big|\sum_{i=1}^{n} h(x_i)\big(D(f;x_i)-D(f;x_{i-1})\big)\Big| \leqslant \sum_{i=1}^{n} |h(x_i)|\cdot V(D(f;x);[x_{i-1},x_i])$$
$$= \sum_{i=1}^{n} |h(x_i)|\cdot V(f';E\cap[x_{i-1},x_i])$$
$$\to \int_\xi^\eta |h(t)|\,dV(f';E\cap[\xi,t])\,.$$

c) Obviously

$$\lim_{\substack{\xi\to c+0\\ \eta\to d-0\\ \xi,\eta\in E}} \int_\xi^\eta h(t)\,dD(f;t) = \int_c^d h(t)\,du(t)$$

where

$$u(t) = \begin{cases} D(f;t) & \text{if } c<t<d\,,\\ f'_+(c) & \text{if } \quad t=c\,,\\ f'_-(d) & \text{if } \quad t=d\,.\end{cases}$$

This trivially implies the assertion.

d) (7) trivially implies that $\overline{D}f(x)$ is bounded on $[a,b]$. In particular f is a Lipschitz 1 function and *a fortiori* absolutely continuous. Thus

$$\int_a^b D(f;t)\,dt = \int_a^b f'(t)\,dt = f(b)-f(a)\,.$$

THEOREM 2. *Let* $g\in T$, $x_0\in[0,1)$. *We denote* $\gamma=g^{-1}$ *and*

$$E=\{x\in[0,1];\ \gamma'(x) \text{ exists}\}\,.$$

Suppose $x_1>x_2>\ldots$, $x_n\to x_0$, $x_n\in E$ $(n=1,2,\ldots)$ *and*

$$\lim_{n\to\infty}\frac{x_n-x_0}{x_{n+1}-x_0}=1\,.$$

Then there exists $K>0$ *such that*

$$\sum_{n=N}^{\infty}(x_n-x_0)\,V(\gamma';E\cap[x_{n+1},x_n]) \leqslant K\big(\gamma(x_N)-\gamma(x_0)\big)$$

if N *is sufficiently large.*

Proof. Suppose indirectly that K can not be chosen according to Theorem 2. Then for $K=1$ there exists N_1 with

$$\sum_{n=N_1}^{\infty}(x_n-x_0)\,V_n > 1\cdot\big(\gamma(x_{N_1})-\gamma(x_0)\big)$$

and hence $M_1 > N_1$ with

$$\sum_{n=N_1}^{M_1} (x_n - x_0) V_n > 1 \cdot (\gamma(x_{N_1}) - \gamma(x_0)),$$

where we put briefly $V_n = V(\gamma'; E \cap [x_{n+1}, x_n])$. If $N_1 < M_1 < N_2 < M_2 < \ldots < N_k < M_k$ has been selected then we can find $N_{k+1} > M_k$ such that

$$\sum_{n=N_{k+1}}^{\infty} (x_n - x_0) V_n > (k+1)(\gamma(x_{N_{k+1}}) - \gamma(x_0))$$

and hence M_{k+1} with

$$\sum_{n=N_{k+1}}^{M_{k+1}} (x_n - x_0) V_n > (k+1)(\gamma(x_{N_{k+1}}) - \gamma(x_0)).$$

Hence by $(x_n - x_0) V_n \geqslant 0$

$$\sum_{n=N_k}^{N_{k+1}-1} (x_n - x_0) V_n > k(\gamma(x_{N_k}) - \gamma(x_0)) \qquad (k = 1, 2, \ldots).$$

For every fixed n we can find $y_0^{(n)} = x_{n+1} < y_1^{(n)} < \ldots < y_{p_n}^{(n)} = x_n$ such that $y_j^{(n)} \in E$ $(j = 0, 1, \ldots, p_n)$ and

$$\sum_{n=N_k}^{N_{k+1}-1} (x_n - x_0) \sum_{j=1}^{p_n} |\gamma'(y_j^{(n)}) - \gamma'(y_{j-1}^{(n)})| > k(\gamma(x_{N_k}) - \gamma(x_0)) \tag{8}$$

still holds for $k = 1, 2, \ldots$

We put

$$\alpha_n = \frac{x_n - x_0}{k} \quad \text{for} \quad N_k \leqslant n < N_{k+1}.$$

By (8)

$$\sum_{n=N_k}^{\infty} \alpha_n \sum_{j=1}^{p_n} |\gamma'(y_j^{(n)}) - \gamma'(y_{j-1}^{(n)})| > \gamma(x_{N_k}) - \gamma(x_0). \tag{9}$$

Further, we fix a sequence $\Delta_n > 0$ with

$$\sum_{n=N}^{\infty} \Delta_n = o(\gamma(x_N) - \gamma(x_0)) \tag{10}$$

($\Delta_n = [\gamma(x_n) - \gamma(x_0)]^2 - [\gamma(x_{n+1}) - \gamma(x_0)]^2$ applies).

Now, making use of our indirect assumption we are going to construct a derivative $f(x)$ such that $f(g(x)) \notin D$.

We choose $0 < \eta_n < h_n$ so small that

$$2h_n + 4\eta_n < \min_{1 \leqslant j \leqslant p_n} (y_j^{(n)} - y_{j-1}^{(n)}).$$

Let

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant x_0, \text{ or } x \geqslant x_1, \text{ or } x = y_j^{(n)} \ (0 \leqslant j \leqslant p_n,\ n = 1, 2, \ldots), \\ & \text{or } y_{j-1}^{(n)} + h_n + 2\eta_n \leqslant x \leqslant y_j^{(n)} - (h_n + 2\eta_n) \ (n = 1, 2, \ldots); \\ \operatorname{sgn}[\gamma'(y_j^{(n)}) - \gamma'(y_{j-1}^{(n)})] \dfrac{\alpha_n}{h_n} & \text{if } y_{j-1}^{(n)} + \eta_n \leqslant x \leqslant y_{j-1}^{(n)} + \eta_n + h_n; \\ -\operatorname{sgn}[\gamma'(y_j^{(n)}) - \gamma'(y_{j-1}^{(n)})] \dfrac{\alpha_n}{h_n} & \text{if } y_j^{(n)} - (\eta_n + h_n) \leqslant x \leqslant y_j^{(n)} - \eta_n \ (n = 1, 2, \ldots; \\ & 1 \leqslant j \leqslant p_n); \end{cases}$$

and let $f(x)$ be continuously linear on the remaining intervals $[y_{j-1}^{(n)}, y_{j-1}^{(n)} + \eta_n]$, $[y_{j-1}^{(n)} + h_n + \eta_n, y_{j-1}^{(n)} + h_n + 2\eta_n]$, $[y_j^{(n)} - h_n - 2\eta_n, y_j^{(n)} - h_n - \eta_n]$ and $[y_j^{(n)} - \eta_n, y_j^{(n)}]$ (see Figure 2).

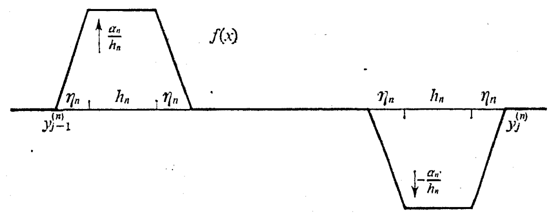

Fig. 2

We prove $f(x) \in D$. Let

$$F(x) = \begin{cases} -\int_x^1 f(t)\,dt & \text{if } x > x_0; \\ 0 & \text{if } x \leqslant x_0. \end{cases}$$

Since $f(x)$ is continuous for $x \neq x_0$, $F'(x) = f(x)$ $(x \neq x_0)$ and $F'_-(x_0) = 0 = f(x_0)$ trivially hold. If $y_{j-1}^{(n)} \leqslant x \leqslant y_j^{(n)}$ then

$$\frac{F(x) - F(x_0)}{x - x_0} = -\frac{1}{x - x_0} \int_x^1 f(t)\,dt = \frac{1}{x - x_0} \int_{y_{j-1}^{(n)}}^x f(t)\,dt$$

since

$$\int_{y_{j-1}^{(n)}}^{y_j^{(n)}} f(t)\,dt = \int_{y_j^{(n)}}^1 f(t)\,dt = 0.$$

$$\left| \int_{y_{j-1}^{(n)}}^x f(t)\,dt \right| \leqslant \int_{y_{j-1}^{(n)}}^{y_j^{(n)}} |f(t)|\,dt < 4\alpha_n = \frac{4}{k}(x_n - x_0)$$

if $N_k \leqslant n < N_{k+1}$, therefore

$$\left|\frac{F(x)-F(x_0)}{x-x_0}\right| \leqslant \frac{4}{k}\cdot\frac{x_n-x_0}{x_{n+1}-x_0} \quad \text{for every } x_0<x<x_{N_k}$$

which implies

$$\lim_{x\to x_0+0}\frac{F(x)-F(x_0)}{x-x_0} = 0 = f(x_0)\,.$$

Now we prove that for suitably chosen sequences $\{\eta_n\}$ and $\{h_n\}$ the function $f(g(x))$ is not a derivative.

Since $y_j^{(n)} \in E$ we can fix h_n so small that

$$\left|\frac{\gamma(y_{j-1}^{(n)}+h_n)-\gamma(y_{j-1}^{(n)})}{h_n}-\gamma'(y_{j-1}^{(n)})\right| < \frac{\Delta_n}{2\alpha_n p_n}$$

and

$$\left|\frac{\gamma(y_j^{(n)})-\gamma(y_j^{(n)}-h_n)}{h_n}-\gamma'(y_j^{(n)})\right| < \frac{\Delta_n}{2\alpha_n p_n}$$

hold for $j = 1, 2, \ldots, p_n$.

Let n be fixed and suppose $\eta_n \to 0$. Since

$$f(g(x)) = \operatorname{sgn}[\gamma'(y_j^{(n)})-\gamma'(y_{j-1}^{(n)})]\frac{\alpha_n}{h_n}$$

if

$$\gamma(y_{j-1}^{(n)}+\eta_n) \leqslant x \leqslant \gamma(y_{j-1}^{(n)}+\eta_n+h_n)$$

and

$$f(g(x)) = -\operatorname{sgn}[\gamma'(y_j^{(n)})-\gamma'(y_{j-1}^{(n)})]\frac{\alpha_n}{h_n}$$

if

$$\gamma(y_j^{(n)}-(\eta_n+h_n)) \leqslant x \leqslant \gamma(y_j^{n}-\eta_n)\,,$$

it is easy to see that

$$\int_{\gamma(x_{n+1})}^{\gamma(x_n)} f(g(t))\,dt$$

$$\to \alpha_n \sum_{j=1}^{p_n} \operatorname{sgn}[\gamma'(y_j^{(n)})-\gamma'(y_{j-1}^{(n)})]\left[\frac{\gamma(y_{j-1}^{(n)}+h_n)-\gamma(y_{j-1}^{(n)})}{h_n}-\frac{\gamma(y_j^{(n)})-\gamma(y_j^{(n)}-h_n)}{h_n}\right]$$

$$> \alpha_n \sum_{j=1}^{p_n} |\gamma'(y_j^{(n)})-\gamma'(y_{j-1}^{(n)})| - \Delta_n\,.$$

Therefore we can choose η_n such that

$$\int_{\gamma(x_{n+1})}^{\gamma(x_n)} f(g(t))\,dt > \alpha_n \sum_{j=1}^{p_n} |\gamma'(y_j^{(n)})-\gamma'(y_{j-1}^{(n)})| - \Delta_n\,.$$

Suppose $f(g(x)) \in D$ and let $G(x)$ be a primitive of $f(g(x))$. Then

$$\text{(11)} \qquad \frac{G(\gamma(x_{N_k}))-G(\gamma(x_0))}{\gamma(x_{N_k})-\gamma(x_0)} \to f(g(\gamma(x_0))) = f(x_0) = 0\,.$$

On the other hand

$$G(\gamma(x_{N_k}))-G(\gamma(x_0)) = \sum_{n=N_k}^{\infty}[G(\gamma(x_n))-G(\gamma(x_{n+1}))]$$

$$= \sum_{n=N_k}^{\infty}\int_{\gamma(x_{n+1})}^{\gamma(x_n)} f(g(t))\,dt$$

$$> \sum_{n=N_k}^{\infty}\alpha_n \sum_{j=1}^{p_n}|\gamma'(y_j^{(n)})-\gamma'(y_{j-1}^{(n)})| - \sum_{n=N_k}^{\infty}\Delta_n$$

$$> \gamma(x_{N_k})-\gamma(x_0) - \sum_{n=N_k}^{\infty}\Delta_n \quad \text{by (9).}$$

Hence, by (10)

$$\liminf_{k\to\infty}\frac{G(\gamma(x_{N_k}))-G(\gamma(x_0))}{\gamma(x_{N_k})-\gamma(x_0)} \geqslant 1$$

which contradicts (11). This proves the theorem.

COROLLARY 2. *If $g \in T$, $\gamma = g^{-1}$, then there exists a finite set $U \subset [0, 1]$ such that for $[a, b] \subset [0, 1] \setminus U$, γ possesses properties* (i)-(iii) *formulated in Lemma* 1.

Proof. By Corollary 1, $\overline{D}\gamma < \infty$ holds apart from a finite set $U_1 \subset [0, 1]$. By Theorem 2, for every $x_0 \in [0, 1]$ a neighbourhood $(x_0-\delta, x_0+\delta)$ can be given such that for $[a, b] \subset (x_0-\delta, x_0+\delta) \setminus \{x_0\}$, $V(\gamma'; E \cap [a, b]) < \infty$ where $E = \{x; \gamma'(x)$ exists and finite$\}$. Thus there exists a finite set $U_2 \subset [0, 1]$ such that $V(\gamma'; E \cap [a, b]) < \infty$ for $[a, b] \subset [0, 1] \setminus U_2$. Putting $U = U_1 \cup U_2$ and $[a, b] \subset [0, 1] \setminus U$, condition (i) in Lemma 1 is satisfied by γ. Hence (ii) and (iii) hold, too.

COROLLARY 3. *If $g \in T$, $\gamma = g^{-1}$ then for every $x_0 \in [0, 1]$ there exist K and $\delta > 0$ such that*

$$\left|\int_0^x t\,dV(t)\right| \leqslant K[\gamma(x_0+x)-\gamma(x_0)]$$

holds for every $|x| < \delta$ where

$$\text{(12)} \qquad V(t) = \begin{cases} V(\gamma'; E \cap [x_0+t, x_0+\delta]), & \text{if } 0 < t \leqslant \delta\,,\\ V(\gamma'; E \cap [x_0-\delta, x_0+t]), & \text{if } -\delta \leqslant t < 0\,,\end{cases}$$

$E=\{x;\gamma'(x)<\infty$ exists$\}$ and the integral above makes sense as an improper Riemann–Stieltjes integral:

$$\int_0^x t\,dV(t)=\lim_{\varepsilon\to+0}\int_{(\operatorname{sgn}x)\varepsilon}^{x} t\,dV(t).$$

Proof. Given $x_0\in[0,1)$ we choose $\delta_1>0$ with $[x_0,x_0+\delta_1)\cap(U\setminus\{x_0\})=\emptyset$. Let the sequence $y_1>y_2>\dots>y_n>\dots$ be the union of the sequences $\{x_0+1/n\}$ and $\{g(\gamma(x_0)+1/n)\}$. Since E is everywhere dense in $[0,1]$ we can choose an $x_n\in(y_{n+1},y_n)\cap E$ for every n. Then the sequence x_n has the following properties: $x_n\in E$, $x_1>x_2>\dots$, $x_n\to x_0+0$,

$$\lim_{n\to\infty}\frac{x_n-x_0}{x_{n+1}-x_0}=1\quad\text{and}\quad\frac{\gamma(x_n)-\gamma(x_0)}{\gamma(x_{n+1})-\gamma(x_0)}\leqslant 2,$$

if $n\geqslant n_0$ (there is at most one k with $x_{n+1}<g(\gamma(x_0)+1/k)<x_n$ and hence

$$\frac{\gamma(x_n)-\gamma(x_0)}{\gamma(x_{n+1})-\gamma(x_0)}<\frac{\dfrac{1}{k-1}}{\dfrac{1}{k+1}}\leqslant 2$$

if $x_{n+1}<g(\gamma(x_0)+\frac{1}{3})$).

Referring to Theorem 2 there exist $K>0$ and $N_0>n_0$ such that

$$\sum_{n=N}^{\infty}(x_n-x_0)V(\gamma';E\cap[x_{n+1},x_n])\leqslant K[\gamma(x_N)-\gamma(x_0)]$$

for every $N\geqslant N_0$. Let $\delta=\min(\delta_1,x_{N_0}-x_0)$. Then

$$\Big|\int_{\varepsilon}^{x}t\,dV(t)\Big|\leqslant\sum_{n=N}^{M}(x_n-x_0)V(\gamma';E\cap[x_{n+1},x_n])$$
$$\leqslant K(\gamma(x_N)-\gamma(x_0))\leqslant 2K[\gamma(x)-\gamma(x_0)]$$

where N is the maximal and M is the minimal index satisfying $x_N-x_0\geqslant x$ and $x_M-x_0\leqslant\varepsilon$, respectively. Since $\int_{\varepsilon}^{x}t\,dV(t)$ is a monotone function of ε the proof is complete.

Now we turn to prove our main result which gives a complete description of transformers.

THEOREM 3. *Let g be a continuous and strictly increasing function on* $[0,1]$ $g(0)=0$, $g(1)=1$. $g\in T$ *holds if and only if*

α) *there exists a finite set $U\subset[0,1]$ such that for $[a,b]\subset[0,1]\setminus U$, γ possesses properties* (i), (ii) *and* (iii) *formulated in Lemma* 1.

In addition at every $x_0\in[0,1]$ g satisfies

β) $$\limsup_{x\to x_0}\left|\frac{g(x)-g(x_0)}{x-x_0}\cdot\frac{1}{\underline{D}g(x)}\right|<+\infty$$

and

γ) $$\limsup_{h\to0}\frac{1}{\gamma(x_0+h)-\gamma(x_0)}\left|\int_0^h t\,dV(t)\right|<+\infty$$

where γ denotes the inverse function g^{-1} and $V(t)$ is defined under (12) *in Corollary* 3.

Proof. Conditions α), β) and γ) are necessary by Corollary 2, Theorem 1 and Corollary 3.

Suppose α), β), γ) and let $f(x)\in D$ be arbitrary. We are going to find a primitive for $f(g(x))$. It is obvious that, if $t_0=0<t_1<\dots<t_n=1$ is a finite decomposition and a primitive of $f(g(x))$ has been found on each $[t_{i-1},t_i]$ $(i=1,2,\dots,n)$ then $f(g(x))\in D$. Hence, without loss of generality we can suppose $U=\{0\}$. Let $F(x)$ denote a primitive of $f(x)$, $F(0)=0$.

$U=\{0\}$ means that on $[x,1]$ γ satisfies (i), (ii) and (iii) formulated in Lemma 1 if $0<x\leqslant1$. For $x>0$ we define

$$G(x)=F(g(x))D(\gamma;g(x))+\int_{g(x)}^{1}F(t)\,dD(\gamma;t).\tag{13}$$

We ought to verify that this definition is independent of the choice of $D(\gamma;x)$. This can be done easily applying Lemma 2. Our proof below gives $G'(x)=f(g(x))$ which of course implies the uniqueness. We prove that

$$G(0)\overset{\text{def}}{=}\int_0^1F(t)\,dD(\gamma;t)$$

is a continuous extension of (13). In fact,

$$|F(g(x))D(\gamma;g(x))|\leqslant x\left|\frac{F(g(x))}{g(x)}\right|\frac{g(x)}{x}(\bar{D}\gamma)(g(x))$$
$$=x\left|\frac{F(g(x))}{g(x)}\right|\frac{g(x)}{x\underline{D}g(x)}.$$

If $x\to0$ then $\dfrac{F(g(x))}{g(x)}\to f(0)$ and by β) $\dfrac{g(x)}{x\underline{D}g(x)}$ is bounded thus the first term in (13) tends to zero. Since $|F(t)|\leqslant Kt$ $(0\leqslant t\leqslant1)$, by Lemma 2b) and c) we have

$$\left|\int_x^y F(t)\,dD(\gamma,t)\right|\leqslant\left|\int_x^y|F(t)|\,dV(t)\right|+|F(y)||D(\gamma;y)-\gamma'_-(y)|+|F(x)||D(\gamma;x)-\gamma'_+(x)|$$
$$\leqslant K\left|\int_0^y t\,dV(t)\right|+2Ky\bar{D}\gamma(y)+2Kx\bar{D}\gamma(x).$$

Now by β) we have

β′) $\dfrac{x-x_0}{\gamma(x)-\gamma(x_0)}\,\overline{D}\gamma(x)$ is locally bounded at every $x_0\in[0,1]$.

Hence

$$\Big|\int_x^y F(t)\,dD(\gamma;t)\Big|\le K_1\gamma(y)+K_2\gamma(y)+K_2\gamma(x)\to 0 \quad \text{if } x,y\to 0\,.$$

Therefore by Cauchy's principle the improper integral $\int_0^1 F(t)\,dD(\gamma;t)$ is convergent.

We prove first $G'(0)=f(g(0))=f(0)$.

$$\frac{G(x)-G(0)}{x}=\frac{F(g(x))}{g(x)}\cdot\frac{g(x)}{x}D(\gamma;g(x))-\frac{1}{x}\int_0^{g(x)}[f(0)t+\varepsilon(t)t]\,dD(\gamma;t)$$

where $\lim_{t\to 0}\varepsilon(t)=0$. Now

$$\frac{1}{x}\int_0^{g(x)}[f(0)t+\varepsilon(t)t]\,dD(\gamma;t)=\frac{f(0)}{x}\lim_{\eta\to 0}\int_\eta^{g(x)} t\,dD(\gamma;t)+\frac{1}{x}\lim_{\eta\to 0}\int_\eta^{g(x)}\varepsilon(t)\cdot t\,dD(\gamma;t)\,.$$

As for the first term we have

$$\int_\eta^{g(x)} t\,dD(\gamma;t)=g(x)D(\gamma;g(x))-\eta D(\gamma;\eta)-\int_\eta^{g(x)} D(\gamma;t)\,dt$$
$$=g(x)D(\gamma;g(x))-\eta D(\gamma;\eta)-(x-\gamma(\eta))$$

by Lemma 2 d). Referring to β′)

$$|\eta D(\gamma;\eta)|\le\eta\overline{D}\gamma(\eta)\le K_3\gamma(\eta)\to 0 \quad \text{if } \eta\to 0$$

thus

$$\lim_{\eta\to 0}\int_\eta^{g(x)} t\,dD(\gamma;t)=g(x)D(\gamma;g(x))-x\,.$$

As for the second term we have by Lemma 2 b) and c)

$$\Big|\int_\eta^{g(x)}\varepsilon(t)\cdot t\,dD(\gamma;t)\Big|$$
$$\le\Big|\int_\eta^{g(x)}|\varepsilon(t)|\,t\,dV(t)\Big|+\varepsilon(g(x))g(x)\big|D(\gamma;g(x))-\gamma'_-(g(x))\big|+\varepsilon(\eta)\eta\cdot|D(\gamma;\eta)-\gamma'_+(\eta)|$$
$$\le\max_{0\le t\le g(x)}|\varepsilon(t)|\Big|\int_0^{g(x)}t\,dV(t)\Big|+2\varepsilon(g(x))g(x)(\overline{D}\gamma)(g(x))+2\varepsilon(\eta)\eta\overline{D}\gamma(\eta).$$

By β′) $\eta\overline{D}\gamma(\eta)\to 0$ and hence by γ) and β) we obtain

$$\Big|\lim_{\eta\to 0}\int_\eta^{g(x)}\varepsilon(t)\,t\,dD(\gamma;t)\Big|\le\max_{0\le t\le g(x)}|\varepsilon(t)|K_1x+2\varepsilon(g(x))K_3x=o(x) \quad \text{if } x\to 0.$$

Combining the two parts

$$\frac{1}{x}\int_0^{g(x)}[f(0)t+\varepsilon(t)t]\,dD(\gamma;t)=\frac{f(0)}{x}g(x)D(\gamma;g(x))-f(0)+o(1)$$

that is

$$\frac{G(x)-G(0)}{x}=\left[\frac{F(g(x))}{g(x)}-f(0)\right]\frac{g(x)}{x}D(\gamma;g(x))+f(0)+o(1).$$

Referring to β) again and observing

$$\lim_{x\to 0}\left[\frac{F(g(x))}{g(x)}-f(0)\right]=0\,.$$

we have

$$\lim_{x\to 0}\frac{G(x)-G(0)}{x}=f(0).$$

We turn to the case $G'(a)=f(g(a))$ for $a>0$.

$$\frac{G(x)-G(a)}{x-a}=\frac{F(g(x))-F(g(a))}{x-a}D(\gamma;g(x))+$$
$$+\frac{F(g(a))}{x-a}[D(\gamma;g(x))-D(\gamma;g(a))]-$$
$$-\frac{1}{x-a}\int_{g(a)}^{g(x)}F(t)\,dD(\gamma;t)$$
$$=\frac{F(g(x))-F(g(a))}{g(x)-g(a)}\cdot\frac{g(x)-g(a)}{x-a}D(\gamma;g(x))-$$
$$-\frac{1}{x-a}\int_{g(a)}^{g(x)}[F(t)-F(g(a))]\,dD(\gamma;t).$$

Making use of $F(t)-F(g(a))=f(g(a))(t-g(a))+\varepsilon(t)(t-g(a))$ $\big(\lim_{t\to g(a)}\varepsilon(t)=0\big)$ we have

$$\frac{1}{x-a}\int_{g(a)}^{g(x)}[F(t)-F(g(a))]\,dD(\gamma;t)$$
$$=\frac{f(g(a))}{x-a}\int_{g(a)}^{g(x)}(t-g(a))\,dD(\gamma;t)+\frac{1}{x-a}\int_{g(a)}^{g(x)}\varepsilon(t)(t-g(a))\,dD(\gamma;t)$$
$$\overset{\text{def}}{=}I_1+I_2.$$

By partial integration

$$I_1 = \frac{f(g(a))}{x-a}\left[(g(x)-g(a))D(\gamma; g(x)) - \int_{g(a)}^{g(x)} D(\gamma; t)\,dt\right]$$

$$= f(g(a))\frac{g(x)-g(a)}{x-a}D(\gamma; g(x)) - f(g(a))$$

where we have applied Lemma 2 d). Referring to Lemma 2 b) and c)

$$|I_2| \leqslant \frac{1}{|x-a|}\int_{g(a)}^{g(x)} |\varepsilon(t)|(t-g(a))\,dV(t) +$$

$$+\frac{1}{|x-a|}\varepsilon(g(x))|g(x)-g(a)|\,|D(\gamma; g(x)) - \gamma'_-(g(x))|$$

$$\leqslant \frac{1}{|x-a|}\max_{g(a)\leqslant t\leqslant g(x)} |\varepsilon(t)|K_4|x-a| +$$

$$+2\varepsilon(g(x))\left|\frac{g(x)-g(a)}{x-a}\,\frac{1}{\underline{D}g(x)}\right| = o(1) \quad (x\to a)$$

by properties β) and γ).

Summing up our results we obtain

$$\frac{G(x)-G(a)}{x-a} = \left[\frac{F(g(x))-F(g(a))}{g(x)-g(a)} - f(g(a))\right]\frac{g(x)-g(a)}{x-a}D(\gamma; g(x)) + f(g(a)) + o(1)$$

and hence the theorem is proved, referring to property β) again.

THEOREM 4. (*) *Let g be a continuous and strictly increasing function in* $[0, 1]$, $g(0) = 0$, $g(1) = 1$. *If there exist $\varkappa$ and K such that* $0<\varkappa\leqslant \underline{D}g(x)\leqslant \overline{D}g(x)\leqslant K<\infty$ $(0\leqslant x\leqslant 1)$ *and $g'(x)$ is of bounded variation on the set of its existence E then $g\in T$.*

(**) *On the other hand if $g\in T$ then there exists a nowhere dense closed set of measure zero $H\subset[0, 1]$ such that for every $[a, b]\subset[0, 1]\setminus H$*

$$0< \inf_{x\in[a,b]} \underline{D}g(x) \leqslant \sup_{x\in[a,b]} \overline{D}g(x) < \infty$$

and $g'(x)$ is of bounded variation on the set $E\cap[a, b]$.

Proof. Let g have the properties described in (*). We are going to verify conditions α), β) and γ) of Theorem 3. α) is trivial. As for β) we have $1/\underline{D}g(x)\leqslant 1/\varkappa$ and

$$\limsup_{x\to a}\frac{g(x)-g(a)}{x-a}\leqslant K.$$

Turning to γ) we remark

$$\frac{1}{K}\leqslant \underline{D}\gamma(x)\leqslant \overline{D}\gamma(x)\leqslant\frac{1}{\varkappa} \quad (0\leqslant x\leqslant 1)$$

and

$$V(\gamma'; g(E)\cap[a, b])\leqslant\frac{1}{\varkappa^2}V(g'; E\cap[\gamma(a), \gamma(b)])$$

$$\leqslant\frac{1}{\varkappa^2}V(g'; E) = C.$$

Hence $\left|\int_0^h t\,dV(t)\right|\leqslant hC$ and

$$\limsup_{h\to 0}\frac{1}{\gamma(a+h)-\gamma(a)}\left|\int_0^h t\,dV(t)\right| \leqslant \limsup_{h\to 0} C\frac{h}{\gamma(a+h)-\gamma(a)}$$

$$= C\frac{1}{\underline{D}\gamma(a)}\leqslant KC.$$

(**) Let $g\in T$ and denote $\gamma = g^{-1}$. Let $H = \{x;\ \overline{D}g(x) = \infty\}\cup\gamma(U)$ where U is the finite set defined in Corollary 2.

H is closed: let $x_n\in H$ and $\lim_{n\to\infty} x_n = a\notin\gamma(U)$. Thus $g(x_n)\to g(a)\notin U$ and according to Corollary 2, γ possesses the properties (i), (ii) and (iii) in Lemma 1 in a neighbourhood of $g(a)$. Hence $(\underline{D}\gamma)(g(a))\leqslant\lim_{n\to\infty}(\underline{D}\gamma)(g(x_n)) = 0$ that is $\overline{D}g(a) = \infty$, $a\in H$.

$\lambda(H) = 0$ since g is differentiable almost everywhere in $[0, 1]$.

Let $[a, b)\subset[0, 1]\setminus H$, then $\underline{D}\gamma(x)>0$ if $x\in[g(a), g(b)]$ and by $[g(a), g(b)]\cap U=\varnothing$ we also have $\overline{D}\gamma(x)<\infty$ $(x\in[g(a), g(b)])$. Lemma 1 (ii) easily implies that $\underline{D}\gamma$ and $\overline{D}\gamma$ are lower semi-continuous and upper semi-continuous, respectively. Thus we can choose $\varkappa$ and K with

$$0<\frac{1}{K}\leqslant\underline{D}\gamma(x)\leqslant\overline{D}\gamma(x)\leqslant\frac{1}{\varkappa}<\infty \quad (x\in[g(a), g(b)]).$$

These inequalities and the application of Lemma 1 for γ in $[g(a), g(b)]$ gives the result.

EXAMPLE 1. In this section we show by an example that $H = \{x;\ g'(x) = \infty\}$ can be a perfect set for a suitably chosen $g\in T$. It means to find a perfect set H and a function γ such that $H = \{x;\ \gamma'(x) = 0\}$ and $\gamma^{-1} = g\in T$.

Let the sequence $r_1, r_2, \ldots$ run over the rational numbers of $(0, 1)$ and put

$$f_n(x) = \begin{cases} 0 & \text{if } 0\leqslant x\leqslant r_n,\\ \dfrac{1}{2^n} & \text{if } r_n<x\leqslant 1,\end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} f_n(x).$$

(x) is a strictly increasing jump function, $f(0) = 0$, $f(1) = 1$. Let

$$a_n = \lim_{x\to r_n - 0} f(x), \quad b_n = \lim_{x \to r_n + 0} f(x) \quad (n = 1, 2, \ldots),$$

$$H = [0, 1] \setminus \bigcup_{n=1}^{\infty} (a_n, b_n).$$

H is obviously is a nowhere dense perfect set, $0, 1 \in H$ and the intervals contiguous to H can be ordered in a sequence $I_n = (a_n, b_n)$ such that $\lambda(I_n) = b_n - a_n = 1/2^n$. In particular $\lambda(H) = 0$.

We define $\gamma'(x)$ by the formula

$$\gamma'(x) = \begin{cases} 0 & \text{if} \quad x \in H, \\ 12(x - a_n) & \text{if} \quad a_n \leqslant x \leqslant \frac{1}{2}(a_n + b_n), \\ 12(b_n - x) & \text{if} \quad \frac{1}{2}(a_n + b_n) \leqslant x \leqslant b_n \quad (n = 1, 2, \ldots). \end{cases}$$

γ' is continuous, let $\gamma(x) = \int_0^x \gamma'(t)\,dt$. γ is obviously a strictly increasing and continuously differentiable function, $\gamma(0) = 0$,

$$\gamma(1) = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} \gamma'(t)\,dt = \sum_{n=1}^{\infty} 3(b_n - a_n)^2 = 1.$$

We have to prove $\gamma^{-1} = g \in T$. For this reason we apply our Theorem 3, i.e., we are going to verify properties α), β) and γ) on g.

It is obvious that for any interval $[a, b] \subset [0, 1]$, $\gamma'(x)$ is of bounded variation on $[a, b]$ and

$$(14) \qquad V(\gamma'; [a, b]) = 12(b - a).$$

Thus α) holds. In order to check β) we prove first that

$$(15) \qquad \gamma(b) - \gamma(a) \geqslant \tfrac{3}{16}(b - a)^2$$

for every $0 \leqslant a \leqslant b \leqslant 1$.

In fact, let (α, β) be a component of $(a, b) \cap ([0, 1] \setminus H)$ having maximal length. If $a_n \leqslant a \leqslant b_n$, $a_m \leqslant b \leqslant b_m$ then

$$b - a = (b_n - a) + \sum_{I_k \subset [b_n, a_m]} \lambda(I_k) + (b - a_m) \leqslant 4(\beta - \alpha)$$

(we may suppose $(a, b) \cap H \neq \emptyset$, otherwise $(\alpha, \beta) = (a, b)$) thus we obtain

$$\gamma(b) - \gamma(a) \geqslant \gamma(\beta) - \gamma(\alpha) = \int_{\alpha}^{\beta} \gamma'(t)\,dt \geqslant 12 \cdot \tfrac{1}{4}(\beta - \alpha)^2 \geqslant \tfrac{3}{16}(b - a)^2$$

and (15) is proved.

Property β) can be written in the form

$$\limsup_{x \to a} \frac{x - a}{\gamma(x) - \gamma(a)} \overline{D}\gamma(x) < \infty.$$

If $a \notin H$ then $\gamma'(a) \neq 0$ and by the continuity of $\gamma'(x)$ we obtain

$$\limsup_{x \to a} \frac{x - a}{\gamma(x) - \gamma(a)} \overline{D}\gamma(x) = \lim_{x \to a} \frac{x - a}{\gamma(x) - \gamma(a)} \gamma'(x) = 1.$$

If $a \in H$ and $x \in H$ then

$$\frac{x - a}{\gamma(x) - \gamma(a)} \gamma'(x) = 0.$$

If $a < x$, $x \in (a_n, b_n)$ then by (15) we have

$$\frac{x - a}{\gamma(x) - \gamma(a)} \gamma'(x) \leqslant \frac{x - a}{\frac{3}{16}(x - a)^2} \cdot 12(x - a_n) = 64 \frac{x - a_n}{x - a} < 64.$$

The same argument applies also if $x < a$ and thus in any case

$$\frac{x - a}{\gamma(x) - \gamma(a)} \gamma'(x) < 64$$

which proves β).

Let $a \in [0, 1]$ be arbitrary, then by (14)

$$V(t) = \begin{cases} V(\gamma'; [a + t, a + \delta]) = \delta - t & \text{if} \quad 0 < t < \delta, \\ V(\gamma'; [a - \delta, a + t]) = \delta + t & \text{if} \quad -\delta < t < 0 \end{cases}$$

and hence

$$\left| \int_0^h t\,dV(t) \right| = \left| \int_0^h t\,dt \right| = \tfrac{1}{2}h^2 \leqslant \tfrac{1}{2} \cdot \tfrac{16}{3} |\gamma(a + h) - \gamma(a)| \quad \text{by (15)}.$$

That is, property γ) is verified, too.

We remark that the set $\{x; g'(x) = 0\}$ can not be infinite for any $g \in T$ because of Corollary 1. Thus our function $\gamma(x)$ constructed above is not a transformer since $\{x; \gamma'(x) = 0\}$ is infinite. Therefore $g \in T$ does not imply $g^{-1} \in T$.

We give another application our Theorem 3 by proving

THEOREM 5. *Let $g(x)$ be a strictly increasing continuous function on* $[0, 1]$, $g(0) = 0$, $g(1) = 1$. *If $g(x)$ is convex then $g \in T$ if and only if*

$$(16) \qquad \limsup_{x \to 1} \frac{g(x) - 1}{x - 1} \cdot \frac{1}{\underline{D}g(x)} < \infty.$$

In particular, if $g'_-(1) < \infty$ then $g \in T$.

Analogous assertion holds for concave functions.

Proof. We prove first that (16) and convexity imply properties α), β) and γ) at every point $x \in [0, 1]$. α) is trivial for convex functions. Since for $0 < a \leqslant x \leqslant b < 1$ we have

$$0 < g'_+(a) \leqslant \underline{D}g(x) \leqslant \overline{D}g(x) \leqslant g'_-(b) < \infty$$

we can apply the same argument used in the proof of part (*) in Theorem 4 which yields properties β) and γ) for $0<x<1$. For $a = 0$ we have

$$\frac{g(x)}{x\underline{D}g(x)} = \frac{\int_0^x g'(t)\,dt}{x\underline{D}g(x)} \leqslant 1.$$

Since $V(t) = -\gamma'_+(t)+c$ we obtain

$$\Big|\int_0^x t\,dV(t)\Big| = -\int_0^x t\,d\gamma'_+(t) = -\lim_{\varepsilon\to+0}\int_\varepsilon^x t\,d\gamma'_+(t)$$
$$= -\lim_{\varepsilon\to+0}\Big(x\gamma'_+(x)-\varepsilon\gamma'_+(\varepsilon)-\int_\varepsilon^x \gamma'_+(t)\,dt\Big)$$
$$= -\lim_{\varepsilon\to+0}\Big(x\gamma'_+(x)-\varepsilon\gamma'_+(\varepsilon)-\big(\gamma(x)-\gamma(\varepsilon)\big)\Big).$$

Making use of

$$\frac{g(x)}{x\underline{D}g(x)} \leqslant 1$$

we have

$$\varepsilon\gamma'_+(\varepsilon)\leqslant \varepsilon\overline{D}\gamma(\varepsilon)\leqslant\gamma(\varepsilon)$$

and hence

$$\Big|\int_0^x t\,dV(t)\Big| = -x\gamma'_+(x)+\gamma(x)\leqslant\gamma(x),$$

i.e., γ) is verified.

Consider now $a = 1$. Property β) is assumed in (16) thus there exists $K>0$ such that

$$\frac{g(1)-g(x)}{1-x}\cdot\frac{1}{\underline{D}g(x)}<K \quad \text{for} \quad \tfrac{1}{2}\leqslant x<1.$$

$$\Big|\int_0^h t\,dV(t)\Big| = \int_0^h t\,d\gamma'_-(1-t) = h\gamma'_-(1-h)-[-\gamma(1-h)+\gamma(1)]<h\gamma'_-(1-h)$$
$$= \big(1-(1-h)\big)\gamma'_-(1-h) = \big(g(1)-g(x)\big)\frac{1}{g'_-(x)}\Big|_{x=\gamma(1-h)}$$
$$<K\big(1-(1-h)\big)$$

thus γ) holds at $a = 1$, too.

If $g'_-(1)<\infty$ then

$$\lim_{x\to1}\frac{g(1)-g(x)}{1-x}\,\frac{1}{\underline{D}g(x)} = 1$$

and the previous result applies.

EXAMPLE 2. We show that there exist convex and concave functions not belonging to T. Since $g(x)\in T$ trivially implies $1-g(1-x)\in T$, it is enough to give a concave function $\gamma(x)\notin T$. Let

$$g(x) = \begin{cases} 0 & \text{if } x = 0,\\ e^{-\frac{2}{x}+2} & \text{if } 0<x\leqslant1.\end{cases}$$

It is easy to verify that $g(x)$ is a strictly increasing continuous and convex function on $[0, 1]$, $g(1) = 1$. Since $g'(1) = 2$, by Theorem 5 we have $g\in T$.

We prove that for the concave function $\gamma(x) = g^{-1}(x)$, $\gamma\notin T$. In fact, for $0<x\leqslant1$

$$\frac{\gamma(x)}{x}\cdot\frac{1}{\gamma'(x)} = \frac{tg'(t)}{g(t)}\Big|_{t=\gamma(x)} = \frac{te^{-\frac{2}{t}+2}\frac{2}{t^2}}{e^{-\frac{2}{t}+2}}\Bigg|_{t=\gamma(x)} = \frac{2}{\gamma(x)}\to\infty \quad \text{if } x\to0$$

and hence $\gamma(x)$ does not satisfy condition β) at $x = 0$.

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