Stability of algebraic inverse systems,
I: Stability, weak stability and the weakly-stable socle

by

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Abstract. In work in shape theory, one often finds that insufficient detail is given in the corresponding results in algebra. This paper is the first of a series written with the aim of removing this difficulty. By looking at some well known classical results on inverse limits from a slightly different viewpoint, we have managed to obtain more detail in these results and have been able to generalize them in such a way as hopefully to be of interest algebraically and not just for their shape theoretic applications.

1. Introduction. The concepts of stability and weak stability of inverse systems have been introduced, in the topological context, in [12] and [13]. The first of these properties, stability, was studied, under a different name, by Verdier [17] and Duskin [4]. Here a start is made on studying the weakened form, the intention being to provide more information on the class of inverse systems, M, satisfying the condition: $\mathbf{lim}^{(i)}M = 0$ for $i>0$.

Although many of the results are easily generalizable to pro-objects in a suitably structured abelian category, A, we have chosen to restrict ourselves to the case of modules over a ring, $\mathfrak{A}$. The results obtained in Sections 5 and 6 are used in [13].

We assume that the ring, $\mathfrak{A}$, is associative has a $1 \neq 0$ and we denote by $\mathcal{M}$od-$\mathfrak{A}$ the category of unitary right $\mathfrak{A}$-modules. A projective or inverse system of right $\mathfrak{A}$-modules in a functor

$$M\colon \mathcal{I} \to \mathcal{M}\text{od-}\mathfrak{A}$$

where $\mathcal{I}$ is a small cofiltering category. A morphism from $M\colon \mathcal{I} \to \mathcal{M}\text{od-}\mathfrak{A}$ to $N\colon \mathcal{J} \to \mathcal{M}\text{od-}\mathfrak{A}$ is an element of the set

$$\lim_{\mathcal{J}}\big(\operatorname*{colim}_{\mathcal{I}} \mathrm{Hom}_{\mathcal{M}\text{od-}\mathfrak{A}}(M(i), N(j))\big)\,.$$

pro($\mathcal{M}$od-$\mathfrak{A}$) denotes the category thus formed and we shall refer to the objects of pro($\mathcal{M}$od-$\mathfrak{A}$) as pro $\mathfrak{A}$-modules or $\mathfrak{A}$-systems for short, if no confusion will arise from this use. More details of the structure of pro-categories, such as pro($\mathcal{M}$od-$\mathfrak{A}$), will be found in Grothendieck [5], Duskin [4] and Artin and

Mazur [1], Appendix. Of special note is that pro($\mathscr{M}$od-$\mathfrak{A}$) is abelian with enough projectives but generally without enough injectives.

There is a full embedding

$$c\colon \mathscr{M}\text{od-}\mathfrak{A} \to \text{pro}(\mathscr{M}\text{od-}\mathfrak{A})$$

and $\mathbf{lim}$: pro($\mathscr{M}$od-$\mathfrak{A}$)$\to\mathscr{M}$od-$\mathfrak{A}$ is right adjoint to c. A pro $\mathfrak{A}$-module is called *stable* if it is isomorphic to some system $c(\mathfrak{N})$ for $\mathfrak{N}$ in $\mathscr{M}$od-$\mathfrak{A}$. M is called *essentially epimorphic* if it is isomorphic to a system in which all the structure maps (i.e. the maps $p_i^j\colon M(j)\to M(i)$ corresponding to $j\to i$ in $\mathscr{I}$) are epimorphisms and essentially monomorphic if the p_i^j's are monomorphisms in some isomorphic pro $\mathfrak{A}$-module.

It is well known that M is essentially epimorphic if and only if it satisfies the Mittag–Leffler condition:

Suppose $M\colon \mathscr{I}\to\mathscr{M}$od-$\mathfrak{A}$ then we say M satisfies (ML) if, given i in $\mathscr{I}$, there is some $f(i)\geqslant i$ such that for any $j\geqslant f(i)$, the natural morphism

$$\text{Im}\big(M(j)\to M(i)\big)\to\text{Im}\big(M(f(i))\to M(i)\big)$$

is an isomorphism in $\mathscr{M}$od-$\mathfrak{A}$. (We use $j\geqslant i$ to mean $\text{Hom}_{\mathscr{I}}(j,i)\neq\emptyset$.)

The proof of the equivalence between (ML) and "essentially epimorphic" is well known and can be found, for instance, in Laudal [10] or [11], and in Duskin [4].

Verdier, in [16] p. 4951, gives an analogous condition, (EM), which is equivalent to M being essentially monomorphic. Explicitly:

$M\colon \mathscr{I}\to\mathscr{M}$od-$\mathfrak{A}$ is essentially monomorphic if any only if it satisfies the condition: (EM) there is a i_0 in $\mathscr{I}$ such that given $j\geqslant i_0$ there is some $k\geqslant j$ so that the natural map

$$\text{Ker}\big(M(k)\to M(j)\big)\to\text{Ker}\big(M(k)\to M(i_0)\big)$$

is an isomorphism.

The proof is very similar to that of the previous result, see Duskin [4].

As well as these results, Verdier's note, [17], mentions that M is stable if and only if it satisfies both (EM) and (ML), and hence if and only if it is essentially both monomorphic and epimorphic.

Remark. Perhaps at this stage a note on a difficulty in terminology is required. The word "stable" is being used here for the situation called "essentially constant" by Verdier. Our change of terminology is motivated by its use in an algebraic topological context in [12] and [13] and also because Verdier's term does not decline nicely, e.g. stability, stabilizes, and so on. If this were the only terminological difficulty it would not matter that much, however both Laudal, [11], and Jensen [9], in quoting Laudal's work, have used "stable" to mean "essentially epimorphic". We therefore warn the reader against any possible confusion, but will continue to use stable until a better term is suggested.

It is well known that the limit functor

$$\mathbf{lim}\colon \text{pro}(\mathscr{M}\text{od-}\mathfrak{A})\to\mathscr{M}\text{od-}\mathfrak{A}$$

is left exact, but not right exact, and considerable study has been made of the right derived functors, $\mathbf{lim}^{(p)}$, of lim; see, for example, Jensen's lecture notes, [8].

The results we use depend heavily on the functionality of the derived limits,

$$\mathbf{lim}^{(p)}\colon \text{pro}(\mathscr{M}\text{od-}\mathfrak{A})\to\mathscr{M}\text{od-}\mathfrak{A}.$$

The proof that these functors are definable on pro($\mathscr{M}$od-$\mathfrak{A}$) is not easy as pro($\mathscr{M}$od-$\mathfrak{A}$) does not, in general, have enough injectives. The detailed proof is beyond the scope of this short paper. Proofs do exist in the literature, for instance in Duskin [4] and the author has an alternative method of proof which depends on the work of Bousfield and Kan on homotopy limits. Here, however, we will merely sketch the proof by mentioning some of the intermediate results. The central result is:

(i) Every M in pro($\mathscr{M}$od-$\mathfrak{A}$) has a resolution $C^*(M)$ in pro($\mathscr{M}$od-$\mathfrak{A}$) by lim-acyclic promodules for which $\mathbf{lim}^{(q)}M\cong H^q(\underleftarrow{\lim}\, C^*(M))$ for $q\geqslant 0$ and in fact if $M\colon\mathscr{I}\to\mathscr{M}$od-$\mathfrak{A}$ is a pro-module then for all $n\geqslant 0$, $\mathbf{lim}^{(n)}M\cong\underset{\mathscr{I}}{\mathbf{lim}}^{(n)}M$ (i.e. the derivation depends only on the pro-object and not on the particular representation chosen).

To prove this one proves

(ii) If M is essentially zero (and hence isomorphic to zero) in pro($\mathscr{M}$od-$\mathfrak{A}$) then

$$\mathbf{lim}^{(n)}M=0\quad\text{for all } n\geqslant 0.$$

Finally, to prove (ii) one needs

(iii) If $\varphi\colon\mathscr{J}\to\mathscr{I}$ is cofinal and $M\colon\mathscr{I}\to\mathscr{M}$od-$\mathfrak{A}$ then

$$\underset{\mathscr{I}}{\mathbf{lim}}^{(n)}M\cong\underset{\mathscr{J}}{\mathbf{lim}}^{(n)}M\varphi\quad\text{for all } n\geqslant 0.$$

The hard part of the proof is the derivation of (i) from (ii) as this involves some deep results on derived categories — again due to Verdier. (The author's alternative proof avoids this section but only by replacing it with another equally hard).

It is convenient here to recall two facts and some related problems.

(i) If $\mathfrak{A}$ is a field and $\dim M(i)$ is finite for each i in $\mathscr{I}$, then

$$\mathbf{lim}^{(p)}M=0\quad\text{for all } p>0.$$

(ii) If $\mathscr{I}$ is countable and M is essentially epimorphic, then

$$\mathbf{lim}^{(p)}M=0\quad\text{for all } p>0.$$

A quick check shows that, in situation (i), M always satisfies (ML), but $\mathbf{lim}^{(p)}M=0$, $p>0$ irrespective of whether $\mathscr{I}$ is countable or not. M does not in general satisfy (EM) in this situation and so is not stable; hence there is a possibility that such systems have some additional property which forces $\mathbf{lim}^{(p)}M$ to be zero. In fact, it is easy enough to show that, if $\dim M(i)$ is bounded, then M is stable and hence that for M an arbitrary inverse system of finite dimensional vector spaces, M is a direct limit of its stable subsystems.

This fact raises the possibility of recursively generating new systems satisfying $\lim^{(p)} M = 0$, given the class of stable systems, and it is this generation process we study in this paper.

In Section 2, we show that the class $\mathscr{L}$, of pro $\mathfrak{A}$-modules, defined by

$$\mathscr{L} = \{M \in \mathrm{ob}(\mathrm{pro}(\mathscr{M}\mathrm{od}\text{-}\mathfrak{A}))|\ \lim^{(p)} M = 0 \quad \text{for all } \underline{p}>0\},$$

is closed under direct limits, extensions and a restricted class of inverse limits. It thus almost forms a torsion class in the abelian category $\mathrm{pro}(\mathscr{M}\mathrm{od}\text{-}\mathfrak{A})$; however, since $\mathscr{L}$ is not closed under quotients, no associated torsion theory gives information relevant to our enquiry.

Since the class, $\mathscr{S}$, of stable objects is a sub-class of $\mathscr{L}$, we can generate a larger subclass, $\mathscr{W}\mathscr{S}$, of $\mathscr{L}$ using direct limits and extensions; the restrictions on the use of inverse limits makes their use at this stage impractical. The systems in $\mathscr{W}\mathscr{S}$ will be called *weakly stable* systems; thus any inverse system of finite dimensional vector spaces is weakly stable.

In Section 3, we examine some of the properties of the class, $\mathscr{W}\mathscr{S}$, especially under a change of rings situation,

$$\varphi\colon \mathfrak{A} \to \mathfrak{B},$$

particularly nice information being available if φ is a flat epimorphism.

In Section 4, we study the possibility of using $\mathscr{W}\mathscr{S}$ to form a torsion class, but again $\mathscr{W}\mathscr{S}$ is not closed under quotients. We must therefore restrict to an even smaller class. If we denote by $\mathscr{S}\mathscr{S}$ the class of simple stable pro $\mathfrak{A}$-modules, then we can define weakly stable "semi-simple" systems by the extension and colimit processes and study the resulting class, $\mathscr{W}\mathscr{S}\mathscr{S}$. More precisely, we define, in the terminology of [15], an associated idempotent preradical on $\mathrm{pro}(\mathscr{M}\mathrm{od}\text{-}\mathfrak{A})$, form the corresponding radical and this process will give us a maximal $\mathscr{W}\mathscr{S}\mathscr{S}$-subobject of any pro $\mathfrak{A}$-module, M. Using this machinery we produce several results which improve, to some degree, on the corresponding results, obtained by homological methods, by Jensen and others. We investigate the interpretation of some fairly standard torsion theoretic results in this context in a sequel to this note.

2. $\mathscr{L}$ is closed under direct limits, extensions, et cetera. If $M\colon \mathscr{I} \to \mathscr{M}\mathrm{od}\text{-}\mathfrak{A}$ is an inverse system of $\mathfrak{A}$-module, we can construct from M a cochain complex whose (co)-homology will give the $\mathfrak{A}$-modules, $\lim^{(i)} M$. Define

$$\prod^k M = \prod_{i_0 \leqslant \ldots \leqslant i_k} M(i_0, \ldots, i_k),$$

where $M(i_0, \ldots, i_k) = M(i_0)$, and

$$\delta^n\colon \prod^n M \to \prod^{n+1} M$$

by

$$\delta^n(\underline{m})(i_0, \ldots, i_{n+1}) = p_{i_0}^{i_1}(\underline{m}(i_1, \ldots, i_{n+1})) + \sum_{j=1}^{n+1} (-1)^j \underline{m}(i_0, \ldots, \hat{i}_j, \ldots, i_{n+1})$$

for $\underline{m} = (\underline{m}(i_0, \ldots, i_n)) \in \prod^n M$.

It is easily checked that $\delta^{n+1} \circ \delta^n = 0$ and that the (co)-homology groups satisfy

$$H^p(\prod^* M) \cong \lim^{(p)} M$$

for $p \geqslant 0$. Dually, if $M\colon \mathscr{I}^{\mathrm{op}} \to \mathscr{M}\mathrm{od}\text{-}\mathfrak{A}$ is an injective system, then we define

$$\sum^k M = \sum_{i_0 \leqslant \ldots \leqslant i_k} \oplus M(i_0, \ldots, i_k),$$

where $M(i_0, \ldots, i_k) = M(i_0)$, and

$$\partial^n(j(i_0, \ldots, i_n)\underline{m}) = j(i_1, \ldots, i_n) p_{i_0}^{i_1} \underline{m} + \sum_{i=1}^{1+n} (-1)^i j(i_0, \ldots, \hat{i}_i, \ldots, i_n)\underline{m}$$

for $\underline{m} \in M(i_0)$, where $j(i_0, \ldots, i_n)$ is the natural monomorphism from $M(i_0, \ldots, i_n)$ into $\sum^n M$ as the $(i_0, \ldots, i_n)$-th summand.

Again $\partial^{n-1} \circ \partial^n = 0$ and $H_0(\sum^* M) = \mathrm{colim}\, M$. Since colim is exact, $H_k(\sum^* M) = 0$ for $k>0$.

Now assume that M is in $\mathrm{pro}(\mathscr{M}\mathrm{od}\text{-}\mathfrak{A})$ and that there is a family of subobjects of M, $\{M_\alpha\}_{\alpha \in \Lambda}$, satisfying the following $M\colon \mathscr{I} \to \mathscr{M}\mathrm{od}\text{-}\mathfrak{A}$, $M_\alpha\colon \mathscr{I}_\alpha \to \mathscr{M}\mathrm{od}\text{-}\mathfrak{A}$,

(i) Λ is directed,

(ii) $M = \underset{\alpha \in \Lambda}{\mathrm{colim}}\, M_\alpha$ in $\mathrm{pro}(\mathscr{M}\mathrm{od}\text{-}\mathfrak{A})$, with $\mathscr{I}_\alpha = \mathscr{I}$ for all $\alpha \in \Lambda$,

(iii) for each $\alpha \in \Lambda$, and each $p>0$, $\lim^{(p)} M_\alpha = 0$.

We shall say, in this case, that M is the special direct limit of $\{M_\alpha\}$.

THEOREM 2.1. *L is closed under special (filtered) direct limits.*

Proof. Let $M = \underset{\alpha \in \Lambda}{\mathrm{colim}}\, M_\alpha$ in $\mathrm{pro}(\mathscr{M}\mathrm{od}\text{-}\mathfrak{A})$ where Λ is directed.

Combining the two constructions, $\prod^*$ and $\sum^*$, both of which are functorial, we get a double complex

$$C_{**} = \prod^* \sum^* \{M_\alpha(i)\},$$

where

$$C_{p,q} = \begin{cases} \prod^{-p} \sum^q \{M_\alpha(i)\} & \text{for } p \leqslant 0,\ q \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

As usual C_{**} gives rise to two spectral sequences. Using the notation of Hilton and Stammbach, [6], we get, for $B = \mathrm{Tot}\, C$:

$${}_1E_1^{p,q} = H_p(H_{q-p}(B, \partial''), \partial') = \lim_{\mathscr{I}}{}^{(-p)} \mathrm{colim}_{\Lambda}{}^{(q+p)} \{M_\alpha(i)\}$$

and

$${}_2E_1^{p,q} = H_p(H_{q-p}(B, \partial'), \partial'') = \mathrm{colim}_{\Lambda}{}^{(-p)} \lim_{\mathscr{I}}{}^{(q+p)} \{M_\alpha(i)\}.$$

Using the facts that, for each α, and $p>0$, $\mathbf{lim}^{(p)} M_\alpha = 0$ and that **colim** is exact, these two spectral sequences degenerate to give

$$ {}_1E_1^{p,q} = \begin{cases} \mathbf{lim}^{(-p)}_{\mathscr{I}} M & \text{if } q=-p, \\ 0 & \text{otherwise} \end{cases} $$

and

$$ {}_2E_1^{p,q} = \begin{cases} \mathbf{colim}\,\mathbf{lim}_{\mathscr{I}}\{M_\alpha(i)\} & \text{if } q=p=0, \\ 0 & \text{otherwise}. \end{cases} $$

Both spectral sequences converge to $H(B)$ and the second one gives that, for $q \neq 0$, $H^q(B) = 0$. Feeding this information back into the first spectral sequence then gives, again for $q \neq 0$,

$$ {}_1E_1^{p,q} = {}_1E_\infty^{p,q} = 0 $$

and so $\mathbf{lim}^{(p)} M = 0$ for all $p>0$.

The corresponding result for inverse limits of quotient objects is also true with some restrictions. In fact, the argument given below is based on that of Jensen, [8] p. 153, who used it to argue from systems of Artinian modules to systems of strictly linearly compact modules over a Noetherian ring.

Let $M\colon \mathscr{I} \to \mathscr{M}od\text{-}\mathfrak{A}$ be a pro-$\mathfrak{A}$-module and suppose there is a directed family of subobjects, $\{M_\alpha\}_{\alpha\in A}$, of M satisfying the following conditions:

(i) $M = \mathbf{lim}_{\alpha}\{M/M_\alpha\}$,

(ii) for each α and $p>0$, $\mathbf{lim}^{(p)}_{\mathscr{I}}\{M/M_\alpha\} = 0$,

(iii) for $p>0$, $\mathbf{lim}^{(p)}_{\mathscr{I}\times A^{\mathrm{op}}}\{M(i)/M_\alpha(i)\} = 0$.

THEOREM 2.2. *With M, $\{M_\alpha\}$ as above, $\mathbf{lim}^{(p)} M = 0$ for $p>0$.*

Since all that we have done is to axiomatise to cover Jensen's argument on p. 153 of [8], a proof would seem slightly superfluous. It basically follows the second half of the proof of 2.1 in form, but uses ${}_1E_1^{p,q} = \mathbf{lim}^{(p)}_{\mathscr{I}}(\mathbf{lim}^{(q)}_{A}\{M(i)/M_\alpha(i)\})$ which is the first spectral sequence associated with double complex

$$ \bar{C}_{**} = \prod_{\mathscr{I}}{}^{*} \prod_{A}{}^{*} \{M(i)/M_\alpha(i)\}. $$

Condition (iii) above is necessary because the second spectral sequence of $\mathbf{Tot}\,\bar{C}$ does not give enough information to claim $H^q(\mathbf{Tot}\,\bar{C}) = 0$ for $q \neq 0$. It is the use of this extra condition that is implied in the statement: "$\mathscr{L}$ is closed under restricted inverse limits" in the introduction.

The other construction that will be needed is that of extension:

A class, $\mathscr{C}$, of objects in an abelian category is said to be *closed under extensions* if, whenever, in a short exact sequence

$$ 0 \to X \to Y \to Z \to 0, $$

X and Z are in $\mathscr{C}$, then Y is in $\mathscr{C}$.

EXAMPLE. Let $\mathscr{S}$ denote the class of stable systems in pro$(\mathscr{M}od\text{-}\mathfrak{A})$, then $\mathscr{S}$ is closed under extensions by a result of Verdier [17] p. 4951, Proposition 3(2).

PROPOSITION 2.3. *$\mathscr{L}$ is closed under extensions.*

Proof. Let

$$ 0 \to \mathbf{X} \to \mathbf{Y} \to \mathbf{Z} \to 0 $$

be a short exact sequence then the associated long exact sequence of derived functors gives

$$ 0 \to \mathbf{lim}\,\mathbf{X} \to \mathbf{lim}\,\mathbf{Y} \to \mathbf{lim}\,\mathbf{Z} \to \mathbf{lim}^{(1)}\mathbf{X} \to \mathbf{lim}^{(1)}\mathbf{Y} \to \mathbf{lim}^{(1)}\mathbf{Z} \to \ldots $$

If $\mathbf{X}$ and $\mathbf{Z}$ are in $\mathscr{L}$, $\mathbf{lim}^{(p)}\mathbf{X} = 0$ and $\mathbf{lim}^{(p)}\mathbf{Z} = 0$ for $p>0$, hence $\mathbf{lim}^{(p)}\mathbf{Y} = 0$.

Since $\mathscr{S}$ is a subclass of $\mathscr{L}$ and $\mathscr{L}$ is closed under special direct limits and extensions, we can recursively form new pro $\mathfrak{A}$-modules from the stable systems by repeated application of the two processes. The class so formed will be denoted by $\mathscr{W}\mathscr{S}$ and will be called the class of weakly stable pro $\mathfrak{A}$-modules. Thus $\mathscr{W}\mathscr{S}$ is the smallest subclass of $\mathscr{L}$ closed under special direct limits and extensions, and which contains $\mathscr{S}$. There are several questions which pose themselves more or less naturally:

1. When is $\mathscr{W}\mathscr{S} = \mathscr{L}$?

2. If $\mathscr{W}\mathscr{S} \neq \mathscr{L}$, can we describe the $\mathscr{L}$-objects which are not in $\mathscr{W}\mathscr{S}$ by some explicit internal property?

and on a slightly more technical level,

3. Is either of $\mathscr{W}\mathscr{S}$ or $\mathscr{L}$ a torsion class in pro$(\mathscr{M}od\text{-}\mathfrak{A})$?

The reason one might hope for $\mathscr{W}\mathscr{S}$ or $\mathscr{L}$ to be a torsion class in that the resulting torsion theory would help greatly in attacking the first two questions. (For information on torsion theories, the reader is referred to Stenström, [15], and in particular to the extensive bibliography contain therein.)

Partial answers to some of these questions exist.

If $\mathfrak{A}$ is a field and we restrict attention to finite dimensional systems then, as mentioned in the introduction, all $\mathscr{L}$-objects are weakly stable. In fact a more general result than this will be proved later and will include a partial answer to 2. To question 3 the answer is simple: in general, no.

EXAMPLE. Let $\mathbf{X}$ be the "Jensen monster" constructed in Section 6 of [9], i.e. let $\mathfrak{A}$ be commutative, then $\mathbf{X}$ is an inverse system indexed by a well ordered set, $\mathscr{I}$, such that $\mathbf{lim}^{(n)}\mathbf{X} \neq 0$ for all $n>0$. Again, by [9] 1.1, let $\mathbf{X}$ be embedded in a pro $\mathfrak{A}$-module $\mathbf{Y}$ which is flasque and hence $\mathbf{lim}^{(n)}\mathbf{Y} = 0$ for all $n>0$. $\mathbf{Y}$ is thus in $\mathscr{L}$, but $\mathbf{Y}/\mathbf{X}$ is not since, using the long exact sequence of derived functors, it is easily shown that $\mathbf{lim}^{(n)}\mathbf{Y}/\mathbf{X} \cong \mathbf{lim}^{(n+1)}\mathbf{X}$. Thus although $\mathscr{L}$ is closed under extensions and direct limits, it is not closed under quotients and hence cannot be a torsion class.

Similarly, although quotients of weakly stable systems will be often essentially epimorphic, it is well known that epimorphic systems can be arbitrarily bad as above. So $\mathscr{W}\mathscr{S}$ cannot be expected to give a torsion theory either, in general.

Again we shall see later that, in fact, a subclass of $\mathscr{WS}$ does give a torsion theory and that this torsion theory provides a tool for examining weak stability.

3. Weak stability and change of rings. So far we have restricted attention to a single ring, $\mathfrak{A}$, and the resulting category, $\mathrm{pro}(\mathscr{M}od\text{-}\mathfrak{A})$, with classes $\mathscr{S}$, $\mathscr{WS}$ and $\mathscr{L}$, et cetera. Since, in this section, we will be dealing with "change of rings" induced by a ring homomorphism,

$$\varphi\colon \mathfrak{A}\to\mathfrak{B},$$

we shall need to modify the notation to talk of the categories, $\mathrm{pro}(\mathscr{M}od\text{-}\mathfrak{A})$ and $\mathrm{pro}(\mathscr{M}od\text{-}\mathfrak{B})$, and classes, $\mathscr{S}(\mathfrak{A})$, $\mathscr{WS}(\mathfrak{A})$ and $\mathscr{L}(\mathfrak{A})$ in $\mathrm{pro}(\mathscr{M}od\text{-}\mathfrak{A})$, $\mathscr{S}(\mathfrak{B})$, $\mathscr{WS}(\mathfrak{B})$ and $\mathscr{L}(\mathfrak{B})$ in $\mathrm{pro}(\mathscr{M}od\text{-}\mathfrak{B})$.

φ induces two functors

$$U^{\varphi}\colon \mathscr{M}od\text{-}\mathfrak{B}\to\mathscr{M}od\text{-}\mathfrak{A}$$

and

$$F^{\varphi}\colon \mathscr{M}od\text{-}\mathfrak{A}\to\mathscr{M}od\text{-}\mathfrak{B},$$

F^{φ} being a left adjoint for U^{φ}. Explicitly, if $\mathfrak{M}$ is a $\mathfrak{B}$-module, then $U^{\varphi}(\mathfrak{M})$ has the same underlying Abelian group structure as $\mathfrak{M}$ and has a right $\mathfrak{A}$-module structure given by

$$m\cdot a = m\cdot\varphi(a),$$

where the action on the right is that "native to" $\mathfrak{M}$.

F^{φ} is given by

$$F^{\varphi}(\mathfrak{N}) = \mathfrak{N}\otimes_{\mathfrak{A}}\mathfrak{B} \quad \text{for } \mathfrak{N} \text{ in } \mathscr{M}od\text{-}\mathfrak{A},$$

where $\mathfrak{B}$ is given an $\mathfrak{A}$-algebra structure by the above method.

These extend to give a pair of functors

$$\mathrm{pro}(U^{\varphi})\colon \mathrm{pro}(\mathscr{M}od\text{-}\mathfrak{B})\to\mathrm{pro}(\mathscr{M}od\text{-}\mathfrak{A})$$

and

$$\mathrm{pro}(F^{\varphi})\colon \mathrm{pro}(\mathscr{M}od\text{-}\mathfrak{A})\to\mathrm{pro}(\mathscr{M}od\text{-}\mathfrak{B}).$$

In our context here, the question arises as to whether the classes $\mathscr{L}$, $\mathscr{WS}$ and $\mathscr{S}$ are, in some way, carried over by these functorial transformations. One result is obvious; since $\mathrm{pro}(U^{\varphi})$ is essentially a forgetful functor, we immediately get:

PROPOSITION 3.1. $\mathrm{pro}(U^{\varphi})$ *sends each of* $\mathscr{L}(\mathfrak{B})$, $\mathscr{WS}(\mathfrak{B})$ *and* $\mathscr{S}(\mathfrak{B})$ *into the corresponding class of* $\mathscr{L}(\mathfrak{A})$. $\mathscr{WS}(\mathfrak{A})$ *or* $\mathscr{S}(\mathfrak{A})$.

It is less obvious, and still unsolved, as to whether the following statement is true in general or, if not generally true, for what classes of rings and homomorphisms, it is true:

If M is $\mathrm{pro}(\mathscr{M}od\text{-}\mathfrak{B})$ is such that $\mathrm{pro}(U^{\varphi})\mathfrak{M}$ is in $\mathscr{WS}(\mathfrak{A})$ (resp. $\mathscr{S}(\mathfrak{A})$) then M is in $\mathscr{WS}(\mathfrak{B})$ (resp. $\mathscr{S}(\mathfrak{B})$).

The corresponding statement for $\mathscr{L}$ is true. The converse problem is perhaps more fruitful. For a specific example, consider $\mathfrak{A} = \mathbf{Z}$, the integers, $\mathfrak{B} = \mathbf{Q}$, the rationals and

$$\varphi\colon \mathbf{Z}\to\mathbf{Q}$$

the obvious homomorphism. Further let M be any system of abelian groups of finite rank, then $\mathrm{pro}(F^{\varphi})M$ is a system of finite dimensional rational vector spaces and hence is weakly stable; however it is easy to construct examples in which $\lim^{(1)}M \neq 0$, so $\mathrm{pro}(F^{\varphi})$ creates stability and weak stability.

Since $\mathrm{pro}(F^{\varphi})$ is a left adjoint to $\mathrm{pro}(U^{\varphi})$, it preserves direct limits, but the other process used in the construction of $\mathscr{WS}$ from $\mathscr{S}$, namely that of extension, has a more complicated behaviour under $\mathrm{pro}(F^{\varphi})$, since, in general, $?\otimes_{\mathfrak{A}}\mathfrak{B}$ is not an exact functor. So to start with we shall restrict ourselves to studying stability, but, even in this case, it will be convenient to impose the condition that $\mathfrak{B}$ is a flat $\mathfrak{A}$-algebra and hence that $?\otimes_{\mathfrak{A}}\mathfrak{B}$ is, in fact, exact. To see why this is so, consider a system M in $\mathrm{pro}(\mathscr{M}od\text{-}\mathfrak{A})$ such that $\mathrm{pro}(F^{\varphi})M$ satisfies (ML). Can one find a condition on M that will ensure that M itself satisfies (ML)? For such a condition to be of use, it must, of course, be sufficiently dissimilar from the condition (ML) itself.

If we suppose $M\colon \mathscr{I}\to\mathscr{M}od\text{-}\mathfrak{A}$, then given i in $\mathscr{I}$ and $j\geqslant i$, there is a short exact sequence

$$0\to p_i^j M(j)\to M(i)\to \mathrm{Coker}\, p_i^j\to 0.$$

Since F^{φ} is right exact, this sequence becomes, on applying F^{φ},

$$\to\mathrm{Tor}^1_{\mathfrak{A}}(\mathrm{Coker}\, p_i^j, \mathfrak{B})\to p_i^j M(j)\otimes_{\mathfrak{A}}\mathfrak{B}\to M(i)\otimes_{\mathfrak{A}}\mathfrak{B}\to\mathrm{Coker}\, p_i^j\otimes_{\mathfrak{A}}\mathfrak{B}\to 0.$$

Comparison with the corresponding sequence for $\mathrm{pro}(F^{\varphi})M = M\otimes_{\mathfrak{A}}\mathfrak{B}$ shows that the last two terms are the same. Making the assumption that

$$\mathrm{Tor}^1_{\mathfrak{A}}(\mathrm{Coker}\, p_i^j, \mathfrak{B}) = 0,$$

we get that $p_i^j M(j)\otimes_{\mathfrak{A}}\mathfrak{B}\cong(p_i^j\otimes_{\mathfrak{A}}\mathfrak{B})(M(j)\otimes_{\mathfrak{A}}\mathfrak{B})$. If this is true for all $j\geqslant i$, then we can find an $f(i)>i$ such that

$$(p_i^j\otimes_{\mathfrak{A}}\mathfrak{B})(M(j)\otimes_{\mathfrak{A}}\mathfrak{B}) = (p_i^{f(i)}\otimes_{\mathfrak{A}}\mathfrak{B})\big(M(f(i))\otimes_{\mathfrak{A}}\mathfrak{B}\big)$$

for all $j>f(i)$, and combining with the previous isomorphism gives us:

$$p_i^j M(j)\otimes_{\mathfrak{A}}\mathfrak{B}\cong p_i^{f(i)}M(f(i))\otimes_{\mathfrak{A}}\mathfrak{B}.$$

This condition looks as if it is getting near to (ML) for the system M itself. Thus it is convenient to impose the condition that $\mathfrak{B}$ is a flat $\mathfrak{A}$-algebra, since requiring that M satisfy $\mathrm{Tor}^1_{\mathfrak{A}}(p_i^j M(j), \mathfrak{B}) = 0$ for all $j\geqslant i$ would seem rather cumbersome; it must, however, be noted that this condition is not necessary, merely convenient for our limited purposes here. In fact if we make the further restriction that φ is an epimorphism, the situation simplifies considerably due to the following well known results (see Section 13 of Stenström's lecture notes [15] for example).

If $\varphi\colon \mathfrak{A}\to\mathfrak{B}$ is a flat epimorphism then the family, $\mathfrak{F}$, of right ideals, $\mathfrak{I}$, of $\mathfrak{A}$ such that $\varphi(\mathfrak{I})\mathfrak{B} = \mathfrak{B}$, is a perfect topology on $\mathfrak{A}$. The corresponding ring of quotients, $\mathfrak{A}_{\mathfrak{F}}$, is isomorphic to $\mathfrak{B}$ and the canonical map $\mathfrak{A}\to\mathfrak{A}_{\mathfrak{F}}$ is isomorphic to φ. To $\mathfrak{F}$, there corresponds a torsion radical, $t(\ )$, and a torsion theory t and $\mathfrak{A}_{\mathfrak{F}}$ are related by

$$t(\mathfrak{M}) = \mathrm{Ker}(\mathfrak{M}\to\mathfrak{M}\otimes_{\mathfrak{A}}\mathfrak{A}_{\mathfrak{F}})$$

for all modules, $\mathfrak{M}$.

We can now state a result which gives a partial answer to the problem of the creation of stability by $\mathrm{pro}(\boldsymbol{F}^{\varphi})$.

PROPOSITION 3.2 *Suppose* $\varphi\colon \mathfrak{A}\to\mathfrak{B}$ *is a flat epimorphism of rings. If* $\boldsymbol{M}\colon \mathscr{I}\to\mathscr{M}\mathrm{od}\text{-}\mathfrak{A}$ *is a pro-*$\mathfrak{A}$ *module, such that* $\mathrm{pro}(\boldsymbol{F}^{\varphi})\boldsymbol{M}$ *satisfies* (ML) *and, for each* i, *and all* $j>i$, $t(\mathrm{Coker}\,p_i^j) = 0$, *then* $\boldsymbol{M}$ *satisfies* (ML).

Proof. From the above discussion we obtain two diagrams:

$$\begin{array}{ccccc}
0 & & 0 & & 0\\
\downarrow & & \downarrow & & \downarrow\\
0\to t(p_i^jM(j)) & \xrightarrow{\alpha} & t(M(i)) & \to & t(\mathrm{Coker}\,p_i^j)\\
\downarrow & & \downarrow & & \downarrow\\
0\to p_i^jM(j) & \to & M(i) & \to & \mathrm{Coker}\,p_i^j\to 0\\
\downarrow & & \downarrow & & \downarrow\\
0\to p_i^jM(j)\otimes_{\mathfrak{A}}\mathfrak{B} & \to & M(i)\otimes_{\mathfrak{A}}\mathfrak{B} & \to & \mathrm{Coker}\,p_i^j\otimes_{\mathfrak{A}}\mathfrak{B}\to 0
\end{array}$$

and a corresponding one for $f(i)$. By assumption $t(\mathrm{Coker}\,p_i^j) = 0$ so α is an isomorphism; similarly $t\big(p_i^{f(i)}M(f(i))\big) = t(M(i))$. Thus both the torsion and torsion free parts of $p_i^jM(j)$ and $p_i^{f(i)}M(f(i))$ are naturally isomorphic. Since $p_i^jM(j)$ is a submodule of $p_i^{f(i)}M(f(i))$ for $j>f(i)$, it follows that $p_i^jM(j)$ and $p_i^{f(i)}M(f(i))$ coincide.

In a more or less dual fashion, one gets the corresponding result for (EM).

PROPOSITION 3.3. *Suppose* $\varphi\colon \mathfrak{A}\to\mathfrak{B}$ *is a flat epimorphism of rings. If* $\boldsymbol{M}\colon \mathscr{I}\to\mathscr{M}\mathrm{od}\text{-}\mathfrak{A}$ *is, a pro-*$\mathfrak{A}$*-module such that* $\mathrm{pro}(\boldsymbol{F}^{\varphi})\boldsymbol{M}$ *satisfies* (EM) *and for each* $j>i_0$, $t(\mathrm{Ker}\,p_{i_0}^j) = 0$ *then* $\boldsymbol{M}$ *satisfies* (EM).

Combining these results gives us a result on stability. Concerning weak stability, if $\boldsymbol{M} = \mathbf{colim}\,\boldsymbol{M}_{\alpha}$, the resulting Coker $p_{i,\alpha}^j$ and $\mathrm{Ker}\,p_{i,\alpha}^j$ can be used as follows:

PROPOSITION 3.4. *If* $\varphi\colon \mathfrak{A}\to\mathfrak{B}$ *is a flat epimorphism, and* $\boldsymbol{M}$ *in* $\mathrm{pro}(\mathscr{M}\mathrm{od}\text{-}\mathfrak{A})$ *is a direct limit of* pro $\mathfrak{A}$*-modules,* $\boldsymbol{M} = \mathrm{colim}\,\boldsymbol{M}_{\alpha}$ *such that* $\mathrm{pro}(\boldsymbol{F}^{\varphi})\boldsymbol{M}$ *is weakly stable and* $\mathrm{pro}(\boldsymbol{F}^{\varphi})\boldsymbol{M}_{\alpha}$ *is stable for each* α, *then as long as, for each* i, $j>i$, $t(\mathrm{Coker}\,p_i^j) = 0$ *and* $t(\mathrm{Ker}\,p_i^j) = 0$, $\boldsymbol{M}$ *is weakly stable.*

Proof. Extending from previous results, one needs only to note that the idempotent radical associated with a perfect topology commutes with direct limits.

Finally, if φ is a flat epimorphism $\mathrm{pro}(\boldsymbol{F}^{\varphi})$ preserves extensions. If φ is a faithfully flat epimorphism then all extensions come from extensions. Thus we get

THEOREM 3.5. *If* $\varphi\colon \mathfrak{A}\to\mathfrak{B}$ *is a faithfully flat epimorphism,* $\mathrm{pro}(\boldsymbol{F}^{\varphi})\boldsymbol{M}$ *is weakly stable and for each* $i, j>i$, $t(\mathrm{Coker}\,p_i^j) = 0$ *and* $t(\mathrm{Ker}\,p_i^j) = 0$, *then,* $\boldsymbol{M}$ *is weakly stable as a pro* $\mathfrak{A}$*-module.*

It is probable that, in many cases, the Coker/Ker condition can be considerably weakened, for instance, by adding "essentially" at strategic points.

4. The weakly stable socle, $\bar{s}(\ )$. Earlier we saw that, due to the fact that, in general, $\mathscr{L}$ and $\mathscr{W}\mathscr{S}$ are not closed under quotients, we cannot obtain a torsion radical by assigning, to each pro-$\mathfrak{A}$-module $\boldsymbol{M}$, its maximal, weakly stable or or $\mathscr{L}$-subobject. We can overcome this difficulty if we restrict ourselves to simple systems.

A pro $\mathfrak{A}$-module $\boldsymbol{M}$ is said to be *simple* if, whenever $\boldsymbol{N}$ is a subobject of $\boldsymbol{M}$ in $\mathrm{pro}(\mathrm{Mod}\text{-}\mathfrak{A})$, then either

(i) the monomorphism $\boldsymbol{N}\to\boldsymbol{M}$ is an isomorphism, or

(ii) the monomorphism $\boldsymbol{N}\to\boldsymbol{M}$ is the zero morphism and hence $\boldsymbol{N}$ is essentially zero.

It should be recalled that the statement "N is essentially zero" does not imply that $N(i) \doteq 0$ for all indices i, but only that, given any i, there is a $j>i$ with $p_i^j = 0$.

Although simplicity of $\boldsymbol{M}$ does give very strong limitations on the properties $\boldsymbol{M}$ can have, it does not, by itself, seem to imply stability, we therefore introduce the class of stable, simple pro $\mathfrak{A}$-modules, which will be denoted by $\mathscr{S}\mathscr{S}(\mathfrak{A})$, or, more briefly, $\mathscr{S}\mathscr{S}$ is no confusion can arise.

Given $\boldsymbol{M}$ in $\mathrm{pro}(\mathscr{M}\mathrm{od}\text{-}\mathfrak{A})$, we can assign to $\boldsymbol{M}$, the sum of all simple stable subobjects of $\boldsymbol{M}$, which we denote by $\boldsymbol{s}(\boldsymbol{M})$. The assignment of $\boldsymbol{s}(\boldsymbol{M})$ to $\boldsymbol{M}$ is functorial and defines an idempotent preradical,

$$s\colon \mathrm{pro}(\mathscr{M}\mathrm{od}\text{-}\mathfrak{A})\to\mathrm{pro}(\mathscr{M}\mathrm{od}\text{-}\mathfrak{A}).$$

Now we proceed to form the associated torsion radical, $\bar{s}(\ )$, following the procedure of [15], Chapter 1.

Define, recursively $s_1 = s$ and, for any ordinal, β, let s_{β} be defined by

(i) if β is not a limit ordinal, $s_{\beta}(M)$ is given by

$$s_{\beta}(M)/S_{\beta-1}(M) = s\big(M/S_{\beta-1}(M)\big)$$

or

(ii) if β is a limit ordinal, $s_{\beta} = \mathbf{colim}_{\alpha<\beta}\, s_{\alpha}$.

Since $s_1(M)$ is in $\mathscr{L}$ and s_{β} is defined in (i) by extension and in (ii) by a special direct limit, it follows that $s_{\beta}(M)$ is always in $\mathscr{L}$. Now define $\bar{s}(M) = \mathbf{colim}_{\beta}\, s_{\beta}(M)$.

Again $\bar{s}(M)$ is a special direct limit of systems in $\mathscr{L}$ and hence it is itself in $\mathscr{L}$. Since all this is performed within $\mathscr{L}$ it follows that

$$\mathbf{lim}^{(i)}\bar{s}(M) = 0 \quad \text{for} \quad i>0.$$

We thus have a short exact sequence,

$$0\to\bar{s}(M)\xrightarrow{t}M\xrightarrow{u}M/\bar{s}(M)\to 0$$

in pro($\mathcal{M}$od-$\mathfrak{A}$), where $\lim M$ is an extension of $\lim s(M)$ by $\lim M/\bar{s}(M)$ and, for $i>0$,

$$\lim{}^{(i)} M \overset{u^{(i)}}{\to} \lim{}^{(i)} M/\bar{s}(M)$$

is an isomorphism, $\bar{s}$ defines a torsion class which we will denote by $\mathcal{WSS}(\mathfrak{A})$, the weakly stable "semi-simple" systems.

The ideal situation would be if

$$\lim \bar{s}(M) \overset{i^*}{\to} \lim M$$

was also an isomorphism, i.e. if $\lim M/\bar{s}(M)$ was zero, since then we would have that any pro $\mathfrak{A}$-module was the extension of a weakly stable system by a zero limit system and could ask when this extension was split, à la Teply [16]. In general, all one can say is the following.

THEOREM 4.1. *Any pro $\mathfrak{A}$-module, M, is an extension of a weakly stable semisimple pro $\mathfrak{A}$-module, $\bar{s}(M)$, by a pro $\mathfrak{A}$-module, N, which satisfies:*

(i) $\lim^{(i)} M \cong \lim^{(i)} N$ *for* $i>0$,

(ii) $\mathrm{soc}(\lim N) = 0$,

where soc( ) *denotes the socle preradical in* $\mathcal{M}$od-$\mathfrak{A}$.

Proof. The only thing left to prove is Property (ii). Suppose $\mathfrak{S}$ is a nonzero simple module of $\lim N$, then denoting the canonical map from $\lim N$ to N in pro($\mathcal{M}$od-$\mathfrak{A}$) by

$$\mu: \lim N \to N,$$

we have that, since $\mathfrak{S}$ is nonzero and simple, $\mu_i(\mathfrak{S}) \cong \mathfrak{S}$ for cofinally many indices i; that is, given any i there is some $j>i$ with $\mu_j(\mathfrak{S}) = \mathfrak{S}$. If j, k are two such indices, $k>j$, then, again because $S \neq 0$,

$$p_j^k: \mu_k(\mathfrak{S}) \to \mu_j(\mathfrak{S})$$

is an isomorphism, at least, for cofinally many k, j. Hence N contains a subsystem isomorphic to the stable system, $c(\mathfrak{S})$. This contradicts the maximality of $\bar{s}(M)$.

If we restrict the class of rings being considered, we can get improved results.

A ring $\mathfrak{A}$ is said to be *right semi-Artinian* if $\mathfrak{M}$ is in $\mathcal{M}$od-$\mathfrak{A}$, $\mathfrak{M} \not\cong 0$, implies $\mathrm{soc}(M) \not\cong 0$.

COROLLARY 4.2. *If $\mathfrak{A}$ is right semi-Artinian then any pro $\mathfrak{A}$-module M is an extension of a $\mathcal{WSS}(\mathfrak{A})$-object, $\bar{s}(M)$, with the same limit as M, by a zero limit pro $\mathfrak{A}$-module with the same derived limits as M.*

Jensen [9] p. 92 mentions the following result:

If M is a pro $\mathfrak{A}$-module in which each $M(i)$ is Artinian, then M is essentially zero if and only if $\lim M = 0$.

The proof is based on Bourbaki [3].

Combining this result with 4.2 we get

COROLLARY 4.3. *For any right semi-Artinian ring $\mathfrak{A}$, any pro $\mathfrak{A}$-module M: $\mathcal{I} \to \mathcal{M}$od-$\mathfrak{A}$ in which each $M(i)$ is Artinian is weakly stable.*

This result is probably not the best possible one since Jensen shows that any pro $\mathfrak{A}$-module M for which $M(i)$ is Artinian is in $\mathcal{L}$, ([9] p. 57), no matter what the ring is.

5. Stability results. Although a combination of the results of Sections 3 and 4 will give us some of the results of this section, with virtually no extra work, they do not provide "stability results" only "weak stability results" for the rings concerned. Since we will need these stronger stability arguments elsewhere, we have included this section which also gives elementary proofs of some special cases of theorems proved above.

Let $\mathfrak{A}$ be a semisimple ring. Then any finitely generated right $\mathfrak{A}$-module $\mathfrak{P}$ has a direct decomposition,

$$\mathfrak{P} \cong \mathfrak{S}_1^{n_1} \oplus \ldots \oplus \mathfrak{S}_r^{n_r},$$

where $\mathfrak{S}_i$ are pairwise non-isomorphic simple right $\mathfrak{A}$-modules. In the corresponding decomposition for $\mathfrak{A}$ itself there is at least one representative of each isomorphism class of simple right $\mathfrak{A}$-modules. We shall assume that the same is true of any given decomposition of any module, even the zero module, in as much as redundant summands of form $\mathfrak{S}_i^{n_i}$ for $n_i = 0$ will be included. With this convention we can assign r separate "dimensions" to any given module $\mathfrak{P}$; for $\mathfrak{P}$, given by the above decomposition, we have

$$d_i(\mathfrak{P}) = n_i \quad \text{for} \quad i = 1, 2, \ldots, r.$$

If $\mathfrak{A}$ is a field, this reverts to the single vector space dimension of $\mathfrak{P}$ over $\mathfrak{A}$.

Suppose $M: \mathcal{I} \to \mathcal{M}$od-$\mathfrak{A}$ is a pro $\mathfrak{A}$-module, we say that M is *essentially of bounded dimension* if there is an integer n such that given any index i, there is an index $j>i$ with

$$d_s(p_i^j M(j)) < n \quad \text{for} \quad s = 1, 2, \ldots, n.$$

If M is essentially of bounded dimension, then M is isomorphic to a system, M', in which each $M'(i)$ has bounded dimension, i.e. there is an integer, n, for which

$$\sup_i d_s(M(i)) < n, \quad s = 1, 2, \ldots, n.$$

If we relax this condition so that each $p_i^j M(j)$ is of finite type, then we shall say M is *essentially of finite type* and in this case there is an isomorphic, M', for which each $M'(i)$ is of finite type.

For M essentially of finite type, we can use 4.3 to conclude that M is weakly stable. If we require that M is essentially of bounded dimension, then we can show that it is stable.

PROPOSITION 5.1. *If $\mathfrak{A}$ is semi-simple, and M a pro $\mathfrak{A}$-module which is essentially of bounded dimension, then M is stable.*

Proof. First we can replace M by an isomorphic pro $\mathfrak{A}$-module for which $d_s(M(i))$ is bounded. We require that M satisfies (EM) and (ML).

Suppose i is fixed, then the family $\{p_i^j(M(j))\}$ contains a minimal element, $p_i^{f(i)}M(f(i))$, and hence, for all $j>f(i)$, $p_i^jM(j) = p_i^{f(i)}M(f(i))$. Note this uses only the fact that $M(i)$ is of finite type, and hence Artinian, since $\mathfrak{A}$ is Artinian.

Now we may assume that every structure map p_i^j in M is epimorphic. Let i be fixed and, for $j>i$ and $s = 1, 2, \ldots, n$, let $k_s(j, i) = d_s(\operatorname{Ker} p_i^j)$. For each s, $k_s(j, i)$ increases with "increasing" j and hence, for some i_s in $\mathscr{I}$, we must achieve a maximum, i.e.

(i) $k_s(i_s, i) \geqslant k_s(j, i)$ for all $j>i$,

(ii) $k_s(i_s, i) = k_s(j, i)$ for all $j \geqslant i_s$.

(Here we are making use of the fact that p_i^j is epimorphic to check that k_s is increasing with j.) Since $\mathscr{I}$ is cofiltering we can find an i_0 greater than all the i_s's and then $k_s(j, i) = k_s(i_0, i)$ for all $j \geqslant i_0$ and each $s = 1, 2, \ldots, n$. It follows that $\operatorname{Ker} p_{i_0}^j$ must be zero for each $j \geqslant i_0$, i.e. that M is cofinally (and hence essentially) monomorphic. Since M satisfies both (EM) and (ML), it is stable.

COROLLARY 5.2. *Any pro-vector space of bounded dimension is stable.*

Since we have a stability result for semi-simple rings, we can apply Propositions 3.2 and 3.3 to gain information about stability over more general rings with semisimple quotient rings.

Firstly we need to recall some facts from ring theory (see Sandomierski [14] or Stenström [15]).

A submodule, $\mathfrak{L}$, of a right $\mathfrak{A}$-module, $\mathfrak{M}$, is called an *essential submodule* of $\mathfrak{M}$ if $\mathfrak{L} \cap \mathfrak{N} \neq 0$ for all non zero submodules, $\mathfrak{N}$, of $\mathfrak{M}$. A right ideal of $\mathfrak{A}$ is *essential* in $\mathfrak{A}$ if it is essential as a submodule of $\mathfrak{A}$. If $\mathfrak{M}$ is any right $\mathfrak{A}$ module,

$$\mathfrak{Z}(\mathfrak{M}) = \{m \in \mathfrak{M} \mid (0:m) \text{ is essential in } \mathfrak{A}\}$$

is the *singular submodule* of $\mathfrak{M}$.

The ring $\mathfrak{A}$ is called *right non-singular* if $\mathfrak{Z}(\mathfrak{A}) = 0$, where $\mathfrak{A}$ is considered as a right $\mathfrak{A}$-module. $\mathfrak{A}$ is said to be right finite dimensional if no right ideal can be written as a direct sum of infinitely many non-zero right ideals of $\mathfrak{A}$.

Sandomierski, [15], showed that, if $\mathfrak{A}$ is a right nonsingular, right finite dimensional ring, then the maximal, right quotient ring, $\mathfrak{Q}_{\mathfrak{A}}$, of $\mathfrak{A}$ is semi-simple; moreover,

$$\mathfrak{Z}(\mathfrak{M}) \cong \operatorname{Ker}(\mathfrak{M} \to \mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{Q}_{\mathfrak{A}})$$

and $\mathfrak{A} \to \mathfrak{Q}_{\mathfrak{A}}$ is a flat epimorphism.

For $\mathfrak{A}$ a right nonsingular and right finite dimensional ring, let $\mathfrak{M}$ be any right $\mathfrak{A}$-module, then $\mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{Q}_{\mathfrak{A}}$ is a module over the semi-simple, Artinian ring, $\mathfrak{Q}_{\mathfrak{A}}$, we shall say that the sth $\mathfrak{Q}_{\mathfrak{A}}$-rank of $\mathfrak{M}$ is the integer

$$rk_s(\mathfrak{M}, \mathfrak{Q}_{\mathfrak{A}}) = d_s(\mathfrak{M} \otimes_{\mathfrak{A}} \mathfrak{Q}_{\mathfrak{A}}).$$

If $M: \mathscr{I} \to \mathscr{M}od\text{-}\mathfrak{A}$ is a pro $\mathfrak{A}$-module, then if there is some integer, n, such that for each $s = 1, 2, \ldots, n$

$$\sup_i rk_s(M(i), \mathfrak{Q}_{\mathfrak{A}}) < n$$

then M will be said to be of *bounded* $\mathfrak{Q}_{\mathfrak{A}}$*-rank*.

THEOREM 5.3. *If* $M: \mathscr{I} \to \mathscr{M}od\text{-}\mathfrak{A}$ *is of bounded* $\mathfrak{Q}_{\mathfrak{A}}$*-rank and for each index i, and* $j>i$, $\mathfrak{Z}(\operatorname{Coker} p_i^j) = 0$ *and* $\mathfrak{Z}(\operatorname{Ker} p_i^j) = 0$, *then* M *is stable.*

Proof. This is a direct corollary of 3.2 and 3.3 since, for $\varphi: \mathfrak{A} \to \mathfrak{Q}_{\mathfrak{A}}$, $t = \mathfrak{Z}$ is the torsion radical.

A particular case of a nonsingular ring is an integral domain; $\mathfrak{Z}$-torsion reverts in this case to being classical torsion, T, of modules over an integral domain and $\mathfrak{Q}_{\mathfrak{A}}$ is the field of fractions. Thus we get as a special case of the above.

COROLLARY 5.4. *If* $\mathfrak{A}$ *is a commutative integral domain,* $M: \mathscr{I} \to \mathscr{M}od\text{-}\mathfrak{A}$ *is of bounded torsion free rank and for each index i and* $j>i$, $T(\operatorname{Coker} p_i^j) = 0$ *and* $T(\operatorname{Ker} p_i^j) = 0$ *then* M *is stable.*

We will need this form of the result in the next section.

Whilst with commutative integral domains, it is worth noting that localization techniques will generally create stability. Thus, restricting ourselves to localizing the integers at a prime ideal (p), if $M: \mathscr{I} \to \mathscr{M}od\text{-}\mathbf{Z}$ is a pro-abelian group, then as long as the kernels and cokernels are "cofinally" without p-torsion and the constituent abelian groups have finitely generated p-torsion and bounded torsion free rank, the localized system $M_{(p)}$ of $\mathbf{Z}_{(p)}$-modules will be stable. Corresponding comments go through for weak stability.

6. A possible extension to inverse systems of nilpotent groups. A start has been made recently (by Michael Barr [2]) on the study of torsion theories in non-abelian categories; much work has also been put into the study of derived functors in non-abelian cases. Here we content ourselves with looking at stability of inverse systems of nilpotent groups; nilpotent groups present themselves for this generalisation since, in a sense, they are themselves a generalisation of abelian groups and many of the techniques go through (such as localization) to the nilpotent situation.

Firstly we fix some notation:

Let $\mathfrak{G}$ be a group then subgroups $\Gamma^i\mathfrak{G}$ are defined inductively by

$$\Gamma^1\mathfrak{G} = \mathfrak{G} \quad \text{and} \quad \Gamma^{i+1}\mathfrak{G} = [\mathfrak{G}, \Gamma^i\mathfrak{G}].$$

$\mathfrak{G}$ is nilpotent of class c if $\Gamma^c\mathfrak{G} \neq 1$, but $\Gamma^{c+1}\mathfrak{G} = 1$. We write $\operatorname{nil}\mathfrak{G} = c$. Of course $\mathfrak{G}$ is nilpotent of class 1 if and only if $\mathfrak{G}$ is Abelian.

If $\mathfrak{G}$ is nilpotent of class c, there is a functorial exact sequence,

$$1 \to \Gamma^c\mathfrak{G} \to \mathfrak{G} \to \mathfrak{G}/\Gamma^c\mathfrak{G} \to 1,$$

where $\operatorname{nil}(\Gamma^c\mathfrak{G})$ and $\operatorname{nil}(\mathfrak{G}/\Gamma^c\mathfrak{G})$ are both less than c. (This provides the basis for an inductive argument.)

We will denote the full subcategory of the category of groups, consisting of the nilpotent groups of class c or less, by $\mathcal{N}_c$.

Let p be any prime number.

Suppose $\mathfrak{G}$ is a group and $\mathfrak{H}$ a subgroup. $x \in \mathfrak{G}$ will be said to be a *p-torsion element modulo* $\mathfrak{H}$ if $x^{p^n} \in \mathfrak{H}$ for some n; it is *torsion modulo* $\mathfrak{H}$ if $x^n \in \mathfrak{H}$ for some n. If $\mathfrak{H} \lhd \mathfrak{G}$ then x is *torsion mod* $\mathfrak{H}$ if and only if $x\mathfrak{H}$ is a torsion element of $\mathfrak{G}/\mathfrak{H}$. Finally we need the following lemma, proof of which is omitted as it is straightforward.

LEMMA. *Let*

$$\begin{array}{ccccccccc} 1 \to & \mathfrak{A}_0 & \to & \mathfrak{B}_0 & \to & \mathfrak{C}_0 & \to 1 \\ & \downarrow \varphi_{\mathfrak{A}} & & \downarrow \varphi_{\mathfrak{B}} & & \downarrow \varphi_{\mathfrak{C}} & \\ 1 \to & \mathfrak{A}_1 & \to & \mathfrak{B}_1 & \to & \mathfrak{C}_1 & \to 1 \end{array}$$

be a commutative diagram of groups with exact rows, then, if $\operatorname{Ker}\varphi_{\mathfrak{B}}$ *has no p-torsion and* $\mathfrak{B}_1$ *has no p-torsion modulo* $\varphi_{\mathfrak{B}}(\mathfrak{B}_0)$, $\operatorname{Ker}\varphi_{\mathfrak{A}}$ *and* $\operatorname{Ker}\varphi_{\mathfrak{C}}$ *have no p-torsion and* $\mathfrak{A}_1$ *(resp.* $\mathfrak{C}_1$*) has no p-torsion modulo* $\varphi_{\mathfrak{A}}(\mathfrak{A}_0)$ *(resp.* $\varphi_{\mathfrak{A}}(\mathfrak{C}_0)$*).*

We denote by $\operatorname{pro}(\mathcal{N}_c)$ the category of cofiltered projective (i.e. inverse) systems of class c nilpotent groups. Let $G\colon \mathcal{I} \to \mathcal{N}_c$ be an object in $\operatorname{pro}(\mathcal{N}_c)$. If $c = 1$, G is of *bounded rank* if $\sup \operatorname{rank} G(i) \otimes \mathbf{Q} < \mathcal{N}$ for some integer $\mathcal{N}$. Assuming the definition is made for objects in $\operatorname{pro}(\mathcal{N}_{c-1})$, we say G in $\operatorname{pro}(\mathcal{N}_c)$ has *bounded rank* if both $\Gamma^c G$ and $G/\Gamma^c G$ have bounded rank.

THEOREM 6.1. *Let* G *in* $\operatorname{pro}(\mathcal{N}_c)$ *and suppose* G *to be of bounded rank, then, if given any index* i*, and* $j > i$, $\operatorname{Ker} p_i^j$ *has no torsion and* $G(i)$ *has no torsion modulo* $p_i^j G(j)$, G *is stable.*

Proof. If $c = 1$, the theorem is a special case of 5.4. So assume the theorem holds in $\operatorname{pro}(\mathcal{N}_{c-1})$. If G is of bounded rank and the Kernel/Cokernel condition is satisfied then, in the exact sequence

$$1 \to \Gamma^c G \to G \to G/\Gamma^c G \to 1$$

$\Gamma^c G$ and $G/\Gamma^c G$ are of bounded rank and also satisfy the Kernel/Cokernel condition by the lemma. Thus we have the commutative diagram

$$\begin{array}{ccccccc} 1 \to & \Gamma^c G & \to & G & \to & G/\Gamma^c G & \to 1 \\ & \cong \uparrow & & \uparrow \mu & & \cong \uparrow & \\ 1 \to & \lim \Gamma^c G & \to & \lim G & \to & \lim G/\Gamma^c G & \to 1 \end{array}$$

and hence μ is an isomorphism. Note the bottom row is exact precisely because $\Gamma^c G$ is stable.

COROLLARY 6.2. *If* G *in* $\operatorname{pro}(\mathcal{N}_c)$ *is of bounded rank and each* $G(i)$ *is uniquely divisible, then* G *is stable.*

Proof. In this case for $c = 1$ we have Corollary 5.2. The proof then proceeds, as in 6.1, by induction.

Results analogous to the "change of rings" results of Section 3 can be obtained for localizing objects of $\operatorname{pro}(\mathcal{N}_c)$ in the sense of Warfield's Notes [18] or Hilton [7]. However these are left to the reader to formulate.

References

[1] M. Artin and B. Mazur, *Etale Homotopy*, Lecture Notes in Math. 100, Berlin–Heidelberg–New York 1969.

[2] M. Barr, *Non-abelian torsion theories*, Canad. J. Math 25 (1973), pp. 1224–1237.

[3] N. Bourbaki, *Topologie Générale*, Paris 1961.

[4] J. Duskin, *Pro-objects*, Séminaire Heidelberg–Strasbourg, Année 1966–67, Exposé 6, I.R.M.A., Strasbourg.

[5] A. Grothendieck, *Technique de descente et théorèmes d'existence en géometrie algébrique*, Séminaire Bourbaki 1959/1960, exposé 190–195, Tome 12, Paris, Secrétairiat mathématique, 1960.

[6] P. J. Hilton and U. Stammbach, *A course in Homological Algebra*, Graduate Texts in Math. 4, Berlin–Heidelberg–New York 1971.

[7] — *Localization and cohomology of nilpotent groups*, Math. Z. 132 (1973), pp. 263–286.

[8] C. U. Jensen, *On the vanishing of* $\lim^{(i)}$, J. Algebra 15 (1970), pp. 151–166.

[9] — *Les foncteurs dérivés de* lim *et leurs applications en théorie des modules*, Lecture Notes in Math. 254, Berlin–Heidelberg–New York 1972.

[10] O. A. Laudal, *Sur la limite projective et la théorie de la dimension*, Séminaire de Topologie et de Géometrie differentielle (C. Ehresmann) Paris, Faculté des Sciences, 1961.

[11] — *Cohomologie-locale* Applications Math. Scandinav. 12 (1963), pp. 147–162.

[12] T. Porter, *Čech homotopy II*, J. London Math. Soc. 2, Ser. 6 (1973), pp. 667–675.

[13] — *Stability results for topological spaces*, Math. Z. 140 (1974), pp. 1–21.

[14] F. L. Sandomierski, *Semi-simple maximal quotient rings*, Trans. Amer. Math. Soc. 128 (1967), pp. 112–120.

[15] B. Stenström, *Ring and modules of quotients*, Lecture Notes in Math. 237, Berlin–Heidelberg–New York 1971.

[16] M. Teply, *The torsion submodule of a cyclic module splits off*, Canad. J. Math. 24 (1972), pp. 450–464.

[17] J. L. Verdier, *Equivalence essentielle des systèmes projectifs*, C. R. Acad. Sci., Paris 261 (1965), pp. 4950–4953.

[18] D. B. Warfield, *Localization of nilpotent groups*, University of Washington, unpublished mimeographed notes.

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