Relative compactness and recent common generalizations of metric and locally compact spaces

by

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Abstract. In this paper we introduce the following concept: a topology τ in a non-void set X is compact relative to a topology τ' in X if every ultrafilter convergent in τ' also converges in τ. We develop a theory of relative compactness in the first three sections and obtain a number of earlier and recent results of quite different types as corollaries in the last section.

Introduction. In recent years there have been a number of works which investigate topological spaces relating them to their compact subsets in more or less complicated ways. Some of these are cited in our paper. Remarkable efforts were made to extend A. V. Arhangel'skiĭ's weight addition theorem, A. S. Misčenko's theorem on compact spaces with point countable bases and other results to more general types of spaces (see [2], [7], [9], [11], [14] and [16]).

To do this a number of new concepts were introduced in the last decade, often using certain collections of covers of a topological space related in some sense to compact subsets of the space.

The aim of the present paper is to introduce the concept of relative compactness (see the abstract or Definition 1.1), a new, simple and natural concept, and to study its properties in connection with the most familiar classical concepts of topology. The importance of this investigation lies in the fact that a number of diverse known results on the line mentioned above are easy corollaries to our theorems. (We remark here that one can define relative concepts concerning other compactness-like notions, e.g. realcompactness, in a similar way.)

It seems that most of the concepts mentioned above yield examples of relative compactness. Evidently, the topology τ of any space (X, τ) is compact relative to any topology in X generated by a net (i.e. a netbase) for the topology τ as a subbase. We get a less trivial example considering the topology τ' generated by the union of members of a pluming of (X, τ) as a subbase. Another (even more general) example is the topology generated by a K-net for τ as a subbase (for this concept see I. Juhász [11]). Other examples (strong Σ-nets in R. E. Hodel [8], bases (mod K) in H. R. Bennett and H. W. Martin [4] etc.) can also be obtained.

In the first three sections we give results on relative compactness. The first section is devoted to some basic properties; it is a simple but remarkable fact that relative compactness is preserved under products. In the second section we deal with any T_2 topology τ compact relative to a weaker topology τ' that satisfies a separation axiom T_i. For $i = 2$ we find that τ and τ' coincide. It turns out that for $i = 1$ the character and weight of τ does not exceed those of τ'. Some other results are also given, including an extension of Miščenko's above-mentioned result. In the third section we consider regular T_1 spaces all subspaces of which have topologies compact relative to topologies having character or weight not greater than a given cardinal $\mathfrak{m}$. In the first case we prove that the character of the space is $\leqslant \mathfrak{m}$ if its Souslin number is $\leqslant \mathfrak{m}$. In the second case we prove that the weight of the space is $\leqslant \mathfrak{m}$ if either $\mathfrak{m} = \aleph_0$ or $2^{\mathfrak{m}} = \mathfrak{m}^+$.

In the last section we show the strength of our results by obtaining most of the results of Hodel [7], [8] and Juhász [11] which generalized the results of Arhangel'skiĭ [1], [2] and [3], Miščenko [12], V. I. Ponomarev [16] and J. Nagata [14]. We also obtain here some theorems on perfect maps. We conclude the paper with a result that was first provéd, in the countable case, and with the use of CH, in Arhangel'skiĭ [3].

A remark on terminology and notation. Throughout the paper "base for a topology" or "base for a topology at a point" means an open base. $|T|$ denotes the cardinality of the set T. $\mathrm{cl}_\tau T$ or simply $\mathrm{cl}\,T$ denotes the closure of the set T in the topology τ. $\mathfrak{m}$ always denotes an infinite cardinal.

§ 1. Relative compactness and its basic properties.

DEFINITION 1.1. We say that a topology τ defined in a non-void set X is *compact relative to a topology* τ' in the same set X if every ultrafilter convergent in τ' also converges in τ.

We also say that a topological space (X, τ) is *compact relative to a space* (X, τ') if the topology τ is compact relative to τ'.

Remark. In general no separation axioms will be assumed but τ is often supposed to be Hausdorff.

By the above definition the following propositions are evident.

1. If a topological space (X, τ) is compact relative to a space (X, τ') and τ'' is a topology stronger than τ', then the space (X, τ) is compact relative to (X, τ'').

2. If a topological space (X, τ) is compact relative to a space (X, τ') and τ_0 is a topology weaker than τ, then the space (X, τ_0) is compact relative to (X, τ').

3. The following conditions are equivalent for a topological space (X, τ):

(i) (X, τ) is compact.

(ii) τ is compact relative to any topology defined in X.

(iii) τ is compact relative to a compact topology defined in X.

(iv) τ is compact relative to the indiscrete topology in X.

4. Let $\mathfrak{H}$ be a net for a topological space (X, τ). Then τ is compact relative to the topology in X generated by $\mathfrak{H}$ as a subbase.

In order to apply the results of the next sections less trivial examples of relative compactness will be given in § 4.

PROPOSITION 1.1. *Suppose that a topological space (X, τ) is compact relative to a space (X, τ'), and Y is a closed subset of (X, τ). Then the topology induced in Y by the topology τ is compact relative to the topology induced in Y by τ'.*

Proof. Let $\mathfrak{U}_Y$ be an ultrafilter in Y which converges to a point $y \in Y$ in the topology τ'_Y induced by τ' in Y. Since $\mathfrak{U}_Y$ has the finite intersection property, there is an ultrafilter $\mathfrak{U}$ in X containing it. Evidently y is a limit point of $\mathfrak{U}$ in τ'. Thus $\mathfrak{U}$ converges to a point $x \in X$ in τ. Since the τ-closed set Y is a member of $\mathfrak{U}$, we infer that $x \in Y$. Then $\mathfrak{U}_Y$ converges to x in the topology τ_Y induced by τ in Y.

THEOREM 1.2. *Let $\{(X_i, \tau_i): i \in I\}$ and $\{(X_i, \tau'_i): i \in I\}$ be families of topological spaces such that (X_i, τ_i) is compact relative to (X_i, τ'_i) for all i in I. Then the product space $(\times_{i \in I} X_i, \times_{i \in I} \tau_i)$ is compact relative to $(\times_{i \in I} X_i, \times_{i \in I} \tau'_i)$.*

Remark. If all the topologies τ'_i are the indiscrete topologies in the X_i's, then this theorem yields the Tychonoff product theorem.

Proof. We introduce the notation $X = \times_{i \in I} X_i$, $\tau = \times_{i \in I} \tau_i$, $\tau' = \times_{i \in I} \tau'_i$. Let $\mathfrak{U}$ be a convergent ultrafilter in the space (X, τ'). Then the projection $\mathfrak{U}_i$ of $\mathfrak{U}$ into X_i is an ultrafilter which converges in τ'_i. Since τ_i is compact relative to τ'_i, we infer that $\mathfrak{U}_i$ also converges in τ_i. But since the limit of a filter in a product space is the product of the limits of all projections of it, we conclude that $\mathfrak{U}$ is convergent in τ.

THEOREM 1.3. *If a topological space (X, τ) is compact relative to an $\mathfrak{m}$-Lindelöf space (X, τ'), then the space (X, τ) is itself $\mathfrak{m}$-Lindelöf.*

Proof. Let $\mathfrak{C}$ be a family of closed sets of (X, τ) such that no subfamily of cardinality not greater than $\mathfrak{m}$ has a void intersection. We may assume that $\mathfrak{C}$ is closed under finite intersections. Since τ' is $\mathfrak{m}$-Lindelöf, there is a point $x \in X$ such that the closure of any member of $\mathfrak{C}$ in τ' contains x. Let $\mathfrak{N}'_x$ be the neighbourhood filter of x in τ'. Since $\mathfrak{C} \cup \mathfrak{N}'_x$ has the finite intersection property, there is an ultrafilter $\mathfrak{U}$ in X containing it. Now x is a limit point of $\mathfrak{U}$ in the topology τ'. Thus $\mathfrak{U}$ is also convergent in τ, i.e. the family of the closures in τ of all members of $\mathfrak{U}$ (*a fortiori* $\mathfrak{C}$) has a non-void intersection.

§ 2. Compactness relative to topologies whith separation axioms.

THEOREM 2.1. *If the topology τ of a Hausdorff space (X, τ) is compact relative to a weaker Hausdorff topology τ' in X, then τ and τ' coincide.*

Proof. Let x be an arbitrary point of X. Let us denote the neighbourhood filter of x in τ (resp. in τ') by $\mathfrak{N}_x$ (resp. by $\mathfrak{N}'_x$). Suppose indirectly that there is a member V of $\mathfrak{N}_x$ which is not contained in $\mathfrak{N}'_x$. Then $\mathfrak{N}'_x \cup \{X \setminus V\}$ is a family with the finite intersection property and hence contained in an ultrafilter $\mathfrak{U}$. Since x is a limit point of $\mathfrak{U}$ in the topology τ', it follows that $\mathfrak{U}$ has a limit point y in the topology τ. Considering that τ' is weaker than τ, we infer that $\mathfrak{U}$ converges to y in τ' as well. Since τ' is a Hausdorff topology, it follows that $y = x$. Therefore $\mathfrak{N}_x \subset \mathfrak{U}$. We conclude that V belongs to $\mathfrak{U}$, which contradicts the earlier assumption that $X \setminus V$ belongs to $\mathfrak{U}$.

Remarks 1. It follows from the above theorem that any Hausdorff topology weaker than a compact topology coincides with that compact topology. Further, given a net of cardinality not greater than $\mathfrak{m}$ for a Hausdorff topology, it is easily seen that there is a weaker Hausdorff topology of weight not greater than $\mathfrak{m}$. Thus we infer Arhangel'skiĭ's weight addition theorem:

If a compact Hausdorff space (X, τ) is the union of not more than $\mathfrak{m}$ of its subspaces each of them having a base of cardinality not greater than $\mathfrak{m}$, then (X, τ) also has a base of cardinality not greater than $\mathfrak{m}$.

2. Considering the natural topology τ of the interval $[0, 1]$ and the topology τ' in $[0, 1]$ whose non-trivial closed sets are exactly the finite subsets, it immediately follows that one cannot replace condition T_2 by condition T_1 for τ' in Theorem 2.1. But we show that the weight and character of τ does not exceed those of τ'.

THEOREM 2.2. *If the topology τ of a Hausdorff space (X, τ) is compact relative to a weaker T_1 topology τ' in X, then the weight of τ does not exceed the weight of τ'.*

Proof. Let $\mathfrak{B}'$ be a minimal base of cardinality $\mathfrak{m}$ for the topology τ'. Let $\mathfrak{H}$ be the family of all finite intersections $\bigcap_{i=1}^{n} H_i$ where H_i is a member of $\mathfrak{B}'$ or H_i is the complement of a member of $\mathfrak{B}'$. Evidently, the cardinality of $\mathfrak{H}$ is $\mathfrak{m}$. We show that $\mathfrak{H}$ is a net for the topology τ. To prove this let x be an arbitrary point of X and V an arbitrary neighbourhood of x in τ. Let us denote the collection of all members of $\mathfrak{H}$ containing x by $\mathfrak{H}_x$. Suppose indirectly that no member of $\mathfrak{H}_x$ is contained in V. Then, since $\mathfrak{H}_x$ is closed under finite intersections, we infer that $\mathfrak{H}_x \cup \{X \setminus V\}$ has the finite intersection property, and therefore it is contained in an ultrafilter $\mathfrak{U}$. Since $\mathfrak{H}_x$ contains a neighbourhood base for x in the topology τ', we infer that x is a limit point of $\mathfrak{U}$ in τ'. Thus $\mathfrak{U}$ also converges in the topology τ. Considering that all the τ-closed sets of the form $X \setminus B'$, where B' is a member of the base $\mathfrak{B}'$ for τ' not containing x, belong to $\mathfrak{U}$ and τ' is a T_1 topology, we infer that the only limit point of $\mathfrak{U}$ in the topology τ is x. But then V belongs to $\mathfrak{U}$, which contradicts the assumption that $X \setminus V \in \mathfrak{U}$.

Thus $\mathfrak{H}$ is a net of cardinality not greater than $\mathfrak{m}$. Then by a simple argument (see W. Holsztyński [9]) we can construct a Hausdorff topology τ'' in X having weight not greater than $\mathfrak{m}$ and weaker than τ. Let the topology τ^* be the supremum of τ' and τ''. τ^* is a Hausdorff topology of weight not greater than $\mathfrak{m}$ and is weaker than τ. Thus, by Theorem 2.1, τ^* and τ coincide, so that τ is of weight not greater than $\mathfrak{m}$.

Remarks 1. It follows from our theorem above that a compact space with a point separating open cover $\mathfrak{G}$ (i.e. an open cover such that given any two points x, y in X there is a G in $\mathfrak{G}$ with $x \in G$, $y \notin G$) of cardinality not greater than $\mathfrak{m}$ has weight not greater than $\mathfrak{m}$.

2. If τ' is not necessarily weaker than τ, then in general the statement of Theorem 2.2 does not remain true; take for example a compact T_2 space (X, τ) with $|X| = 2^{\aleph_0}$ and of weight $2^{\aleph_0}$ and any second countable T_1 topology in X as τ'. However, by a slight change of argument in the first part of the proof of Theorem 2.2 we get the following result:

THEOREM 2.3. *If a T_1 space (X, τ) is compact relative to a topological space (X, τ') of weight not greater than $\mathfrak{m}$ and there is a T_1 topology τ'' in X of weight not greater than $\mathfrak{m}$ and weaker than τ, then (X, τ) has a net of cardinality not greater than $\mathfrak{m}$.*

LEMMA 2.4. *Suppose that the topology τ of a hereditarily $\mathfrak{m}$-Lindelöf regular T_1 space (X, τ) is compact relative to a weaker topology τ' in X such that the weight of τ' does not exceed $\mathfrak{m}$. Then there is a regular topology τ'' of weight not greater than $\mathfrak{m}$ in X such that τ'' is weaker than τ and τ is compact relative to τ''.*

Proof. Let $\mathfrak{B}'$ be a base of cardinality not greater than $\mathfrak{m}$ for the topology τ'. Let us define the series $\{\mathfrak{B}_n'\}_{n=0}^{\infty}$ of families of τ-open subsets of X in the following way:

a) $\mathfrak{B}_0' = \mathfrak{B}'$.

b) Suppose that $\mathfrak{B}_n' = \{B_n^{(i)} \colon i \in I_n\}$ is already defined. Since τ is hereditarily $\mathfrak{m}$-Lindelöf, it follows that for all $B_n^{(i)}$ in $\mathfrak{B}_n'$ there is a family $\mathfrak{G}_n^{(i)}$ of cardinality not greater than $\mathfrak{m}$ consisting of τ-open subsets of X such that $B_n^{(i)}$ is the union of all members in $\mathfrak{G}_n^i$ and the closure of any member of $\mathfrak{G}_n^{(i)}$ in the space (X, τ) is contained in $B_n^{(i)}$. Let $\mathfrak{B}_{n+1}$ be the family of all finite intersections of members in $\bigcup_{i \in I_n} \mathfrak{G}_n^{(i)} \cup \mathfrak{B}_n'$. Let $\overline{\mathfrak{B}}_{n+1}$ be the family of all sets of the form $X - \mathrm{cl}_\tau B$ where B is a member of $\mathfrak{B}_{n+1}$. Finally, let $\mathfrak{B}_{n+1}'$ be the union of $\mathfrak{B}_{n+1}$ and $\overline{\mathfrak{B}}_{n+1}$.

Now let us put $\mathfrak{B}'' = \bigcup_{n=0}^{\infty} \mathfrak{B}_n'$.

By using induction it is easily seen that the cardinality of $\mathfrak{B}_n'$ does not exceed $\mathfrak{m}$ ($n = 0, 1, 2, \ldots$). Thus $|\mathfrak{B}''| \leq \mathfrak{m}$. By definition $\mathfrak{B}''$ is a cover of X which is closed under finite intersections. Therefore $\mathfrak{B}''$ is a base for a topology τ'' in X such that τ'' is weaker than τ and stronger than τ'.

It remains to prove that τ'' is a regular topology. Let x be an arbitrary point in X and let B'' be any member of $\mathfrak{B}''$ containing x. Then there is an index n with $B'' \in \mathfrak{B}_n'$. By definition there is a member G in $\mathfrak{B}_{n+1}$ with $x \in \mathrm{cl}_\tau G \subset B''$. Since $X \setminus \mathrm{cl}_\tau G \in \overline{\mathfrak{B}}_{i+1} \subset \mathfrak{B}''$, it follows that $\mathrm{cl}_\tau G$ is closed also in τ''. Thus

$$x \in \mathrm{cl}_{\tau''} G \subset \mathrm{cl}_\tau G \subset B''.$$

We conclude that x has a neighbourhood base consisting of τ''-closed subsets of X, q.e.d.

COROLLARY 1. *If the topology τ of a hereditarily $\mathfrak{m}$-Lindelöf regular T_1 space (X, τ) is compact relative to a weaker T_0 topology τ' in X such that the weight of τ' does not exceed $\mathfrak{m}$, then the weight of (X, τ) is not greater than $\mathfrak{m}$, either.*

Proof. Since regular T_0 spaces are Hausdorff, we conclude by Theorem 2.1 that the topology τ'' constructed in Lemma 2.4 and τ coincide.

Remark. Making use of Theorem 1.3, we obtain the following statement: If the topology of any subspace (Y, τ_Y) of a regular T_1 space (X, τ) is compact relative to a weaker T_0 topology τ'_Y in Y such that the weight of τ'_Y does not exceed $\mathfrak{m}$, then (X, τ) has weight not greater than $\mathfrak{m}$.

Assuming GCH we can omit the assumption that τ_Y is a T_0 topology (see Theorem 3.2).

THEOREM 2.5. *Suppose that the topology τ of a regular T_1 space (X, τ) is compact relative to a weaker topology τ' in X. Let τ have a pseudobase of cardinality not greater than $\mathfrak{m}$ at the point x, and let τ' have a base of cardinality not greater than $\mathfrak{m}$ at the point x. Then the topology τ has a base of cardinality not greater than $\mathfrak{m}$ at the point x.*

Proof. By the regularity of (X, τ) we may assume that there is a family of τ-open subsets $\mathfrak{A} = \{A_i: i \in I\}$ with $|I| \leqslant \mathfrak{m}$ such that the intersection $\bigcap_{i \in I} \mathrm{cl}_\tau A_i$ consists of the single point x. Further, let $\mathfrak{B} = \{V_j: j \in J\}$ be a neighbourhood base of x in τ' with $|J| \leqslant \mathfrak{m}$. Denote by $\mathfrak{M}$ the family of all finite intersections of members of the family $\mathfrak{A} \cup \mathfrak{B}$. Evidently, the cardinality of $\mathfrak{M}$ does not exceed $\mathfrak{m}$. We show that $\mathfrak{M}$ is a base for the topology τ at the point x. Suppose the converse is true, i.e. there is a τ-open neighbourhood V of x in the topology τ such that no member of $\mathfrak{M}$ is contained in it. Then the family $\mathfrak{M} \cup \{X \setminus V\}$ has the finite intersection property and thus it is contained in an ultrafilter $\mathfrak{U}$. Evidently, x is a limit point of $\mathfrak{U}$ in τ', therefore $\mathfrak{U}$ is convergent in τ as well. Since $\mathfrak{A} \subset \mathfrak{U}$, it follows that the limit of $\mathfrak{U}$ in τ consists of the single point x, and this contradicts $X \setminus V \in \mathfrak{U}$.

Remark. It follows from this theorem that the character and pseudocharacter coincide for a compact Hausdorff space.

COROLLARY 1. *If the topology τ of a regular T_1 space (X, τ) is compact relative to a weaker T_1 topology τ' defined in X, then the character of (X, τ) does not exceed the character of (X, τ').*

In order to formulate and prove our next result we need the following definition.

DEFINITION 2.1. The pointwise cardinality of a family $\mathfrak{G}$ of subsets in a non-void set X is the *smallest infinite cardinal* $\mathfrak{m}$ such that every element of X is contained in at most $\mathfrak{m}$ members of $\mathfrak{G}$.

The pointwise weight of a topological space (X, τ) is the smallest infinite cardinal $\mathfrak{m}$ such that there is a base for (X, τ) of pointwise cardinality not greater than $\mathfrak{m}$.

THEOREM 2.6. *If a T_1 space (X, τ) of pointwise weight not greater than $\mathfrak{m}$ is compact relative to a topological space (X, τ') of weight not greater than $\mathfrak{m}$, then the weight of (X, τ) does not exceed $\mathfrak{m}$.*

Proof. The proof is based on the following lemma of Misčenko [12].

If a family $\mathfrak{G}$ of subsets in a non-void set X has pointwise cardinality not greater than $\mathfrak{m}$, then the collection of all finite minimal covers of X by members of $\mathfrak{G}$ has cardinality not greater than $\mathfrak{m}$.

Let $\mathfrak{B}$ be a base of pointwise cardinality not greater than $\mathfrak{m}$ for the topology τ, and let $\mathfrak{B}'$ be a base of cardinality not greater than $\mathfrak{m}$ for the topology τ'. Further, let $\mathfrak{G}$ be the family of all members in $\mathfrak{B}$ and of the complements of all members in $\mathfrak{B}'$. Evidently, $\mathfrak{G}$ has pointwise cardinality not greater than $\mathfrak{m}$. We show that any member B of $\mathfrak{B}$ is contained in a finite (and thus in a finite minimal) cover of X by members of $\mathfrak{G}$. To prove this, let x be an arbitrary point of B. Since $\mathfrak{B}$ is a base for the T_1 topology τ, there is a subfamily $\mathfrak{B}_1$ of $\mathfrak{B}$ such that the union of all members of $\mathfrak{B}_1$ is $X \setminus \{x\}$. Let $\mathfrak{B}_1^*$ be the family of complements of all members of $\mathfrak{B}_1$ and $\mathfrak{B}'_x$ the family of all members in $\mathfrak{B}'$ containing x. By virtue of the definition of $\mathfrak{G}$ it is enough to show that B belongs to the filter $\mathfrak{F}$ generated by $\mathfrak{B}_1^* \cup \mathfrak{B}'_x$. Suppose the converse is true. Then the family $\mathfrak{F} \cup \{X \setminus B\}$ has the finite intersection property, and thus it is contained in an ultrafilter $\mathfrak{U}$. Since $\mathfrak{B}'_x \subset \mathfrak{U}$, it follows that x is a limit point of $\mathfrak{U}$ in τ'. Therefore $\mathfrak{U}$ is also convergent in τ. Making use of the fact that $\mathfrak{B}_1^* \subset \mathfrak{U}$, we infer that the limit of $\mathfrak{U}$ in τ consists of the single point x, which contradicts $X \setminus B \in \mathfrak{U}$.

Now, for each B in $\mathfrak{B}$, let us assign a finite minimal cover of X by members of $\mathfrak{G}$ containing B. This is a map of $\mathfrak{B}$ into the collection $\mathfrak{K}$ of all finite minimal covers of X by members of $\mathfrak{G}$ such that the pre-image of any cover in $\mathfrak{K}$ is finite. Then, by virtue of Misčenko's lemma, we conclude that the cardinality of $\mathfrak{B}$ does not exceed $\mathfrak{m}$.

We can summarize our results obtained in Theorems 2.2, 2.3 and 2.6 in the following theorem:

THEOREM 2.7. *If a T_1 space (X, τ) is compact relative to a topological space (X, τ') of weight not greater than $\mathfrak{m}$ and there is a T_1 topology τ'' in X of pointwise weight not greater than $\mathfrak{m}$ and weaker than τ, then (X, τ') has a net of cardinality not greater than $\mathfrak{m}$.*

If in addition, (X, τ) is a Hausdorff space and τ' is weaker than τ, then (X, τ) has weight not greater than $\mathfrak{m}$.

Proof. By our assumptions τ'' is compact relative to τ'. By applying Theorem 2.6 it follows that the weight of τ'' does not exceed $\mathfrak{m}$. Thus, by Theorem 2.3 there is a net of cardinality not greater than $\mathfrak{m}$ for the topology τ.

If the additional assumptions hold we can apply Theorem 2.2 instead of Theorem 2.3 to find that (X, τ) has weight not greater than $\mathfrak{m}$. (The weaker T_1 topology required in Theorem 2.2 is the supremum of τ' and τ''.)

§ 3. Hereditary properties concerning relative compactness. It is not in general true that if the topology of any subspace of a space (X, τ) is compact relative to a first countable and weaker topology in the same subset of X, then (X, τ) itself satisfies the first axiom of countability. (A convenient counter-example is the one-point compactification of any uncountable discrete space.) However, the above statement is true for spaces with the Souslin condition.

DEFINITION 3.1. The Souslin number of a topological space (X, τ) is the *smallest infinite cardinal* $\mathfrak{m}$ such that there is no family of disjoint open subsets in (X, τ) which has cardinality greater than $\mathfrak{m}$.

THEOREM 3.1. *Let (X, τ) be a regular T_1 space with the Souslin number not greater than $\mathfrak{m}$ and suppose that the topology τ_Y of any subspace (Y, τ_Y) of (X, τ) is compact relative to a weaker topology in Y the character of which does not exceed $\mathfrak{m}$. Then the character of (X, τ) does not exceed $\mathfrak{m}$.*

Proof. Let x be an arbitrary point of X. By Zorn's Lemma there is a maximal family $\mathfrak{G} = \{G_i\colon i \in I\}$ of disjoint τ-open subsets of X such that there is no member G_i in $\mathfrak{G}$ with $x \in \mathrm{cl}_\tau G_i$. Since the Souslin number of (X, τ) is not greater than $\mathfrak{m}$, we infer that $|I| \leqslant \mathfrak{m}$. Let us put $G = \bigcup_{i \in I} G_i$. By the regularity of the topology τ, G is dense in (X, τ). Let $Y = G \cup \{x\}$, and let τ_Y be the topology induced by τ in Y. Then the trace of the family $\{X \setminus \mathrm{cl}_\tau G_i\colon i \in I\}$ of τ-open subsets in Y is evidently a pseudobase of the topology τ_Y at the point x. Thus, by applying Theorem 2.5, τ_Y has a base of cardinality not greater than $\mathfrak{m}$ at the point x. Since Y is dense in (X, τ) and (X, τ) is a regular T_1 space, we infer that τ has a base of cardinality not greater than $\mathfrak{m}$ at the point x.

THEOREM 3.2. *Suppose that $\mathfrak{m} = \aleph_0$ or $2^{\mathfrak{m}} = \mathfrak{m}^+$. Then if the topology of every subspace (Y, τ_Y) of a regular T_1 space (X, τ) is compact relative to a weaker topology τ'_Y in Y such that the weight of τ'_Y does not exceed $\mathfrak{m}$, then (X, τ) has weight not greater than $\mathfrak{m}$.*

Proof. By Theorem 1.3 (X, τ) is a hereditarily $\mathfrak{m}$-Lindelöf space, so that its Souslin number does not exceed $\mathfrak{m}$. Thus we can apply Theorem 3.1 to find that the character of (X, τ) does not exceed $\mathfrak{m}$. It follows either from J. de Groot's result (see [6]) or from a well-known theorem of Arhangel'skiĭ (see 2.21 in [10]) that the cardinality of X does not exceed $2^{\mathfrak{m}}$. Thus the weight of (X, τ) is not greater than $2^{\mathfrak{m}}$. Since (X, τ) is hereditarily $\mathfrak{m}$-Lindelöf, we infer that the cardinality of the family of all open or closed subsets in (X, τ) (*a fortiori* the cardinality of the family of all compact subsets in (X, τ)) does not exceed $2^{\mathfrak{m}}$. Thus X is the union of two disjoint subsets, X_1 and X_2, such that neither X_1 nor X_2 contains a compact subset of (X, τ) of cardinality $2^{\mathfrak{m}}$ (see Hodel [8], for example). It follows from 2.23 of [10] for $\mathfrak{m} = \aleph_0$ and from the assumption $2^{\mathfrak{m}} = \mathfrak{m}^+$ for $\mathfrak{m} > \aleph_0$ that neither X_1 nor X_2 contains a compact subset of (X, τ) of cardinality greater than $\mathfrak{m}$.

We show that if τ_i denotes the topology in X_i $(i = 1, 2)$ induced by τ, then (X_1, τ_1) and (X_2, τ_2) have a net of cardinality not greater than $\mathfrak{m}$.

Let us consider e.g. $i = 1$. By Theorem 1.3 and Theorem 2.4, there is a regular topology τ'_1 in X_1 of weight $\leqslant \mathfrak{m}$ such that τ'_1 is weaker than τ_1, and τ_1 is compact relative to τ'_1. Let us say that two points in X_1 are equivalent iff their neighbourhood filters in τ'_1 coincide. This is clearly an equivalence relation on X_1, and since τ'_1 is a regular topology, it follows that the equivalence class $[x]$ of any point x in X_1 is τ'_1-closed and thus it is τ_1-closed. Now, by Proposition 1.1, the topology in $[x]$ induced by τ_1 is compact relative to the topology in $[x]$ induced by τ'_1. Since the latter is the indiscrete topology in $[x]$, we infer that $[x]$ is a compact subset in (X_1, τ_1). Therefore the cardinality of $[x]$ does not exceed $\mathfrak{m}$.

Let $\mathfrak{C}$ be the family of all equivalence classes defined above. Since the cardinality of any C in $\mathfrak{C}$ does not exceed $\mathfrak{m}$, there is a family $\{(X_1^{(i)}, \tau_1^{(i)})\colon i \in I\}$ of at most $\mathfrak{m}$ disjoint subspaces of (X_1, τ_1) such that $X_1 = \bigcup_{i \in I} X_1^{(i)}$ and the intersection $X_1^{(i)} \cap C$ consists of at most one point for all C in $\mathfrak{C}$ and for all i in I. Since τ'_1 is a regular topology, we infer that the topology in $X_1^{(i)}$ induced by τ'_1 is a T_2 topology. Thus, by applying Theorem 2.3, it follows that $(X_1^{(i)}, \tau_1^{(i)})$ has a net of cardinality not greater than $\mathfrak{m}$. By $|I| \leqslant \mathfrak{m}$ the same holds for the space (X_1, τ_1).

We conclude that (X, τ) has a net of cardinality not greater than $\mathfrak{m}$ and by assumption τ is compact relative to a weaker topology τ' in X such that τ' has weight not greater than $\mathfrak{m}$. Now we can complete the proof in exactly the same way as we did in the proof of Theorem 2.2.

Remark. If we omit the assumption that τ'_Y is weaker than τ_Y in Theorem 3.2, then (X, τ) is only hereditarily $\mathfrak{m}$-separable, i.e. the following proposition holds:

PROPOSITION 3.3. *If the topology of every subspace (Y, τ_Y) of a Hausdorff space (X, τ) is compact relative to a topology in Y of weight not greater than $\mathfrak{m}$, then (X, τ) is hereditarily $\mathfrak{m}$-separable.*

Proof. Let α denote the smallest ordinal of cardinality greater than $\mathfrak{m}$. Suppose that our proposition is not true, i.e. there is a sequence $\{x_\lambda\colon \lambda < \alpha\}$ of points in X such that no x_λ is contained in the closure of $F_\lambda = \{x_\nu\colon \nu < \lambda\}$ in τ. Let $X_1 = \{x_\lambda\colon \lambda < \alpha\}$, and let τ_1 denote the topology in X_1 induced by τ. Since by Theorem 1.3 (X_1, τ_1) is a hereditarily $\mathfrak{m}$-Lindelöf Hausdorff space, we infer that there is a pseudobase $\mathfrak{B}_\lambda$ of cardinality not greater than $\mathfrak{m}$ for the topology τ_1 at each point x_λ. Since $x_\lambda \notin \mathrm{cl}_\tau F_\lambda$, we can assume that no member of $\mathfrak{B}_\lambda$ intersects F_λ. Thus the pointwise cardinality of the family $\mathfrak{B} = \bigcup_{\lambda < \alpha} \mathfrak{B}_\lambda$ in X_1 does not exceed $\mathfrak{m}$. Let τ'_1 be the topology in X_1 generated by $\mathfrak{B}$ as a subbase. τ'_1 is evidently a T_1 topology of pointwise weight not greater than $\mathfrak{m}$. Then, by Theorem 2.7, there is a net of cardinality not greater than $\mathfrak{m}$ for the topology τ_1. Thus the density of X_1 does not exceed $\mathfrak{m}$, which contradicts the definition of X_1.

§ 4. Corollaries and applications. Most of the general results that follow from the above theory of relative compactness are formulated in terms of cardinal invariants. We briefly recall their definitions. We write shortly X for (X, τ) in these definitions.

Denote the weight, pointwise weight, netweight, character, pseudocharacter, Lindelöf degree and Souslin number of a topological space X by $\mathrm{w}(X)$, $\mathrm{pw}(X)$, $\mathrm{n}(X)$, $\chi(X)$, $\psi(X)$, $\mathrm{L}(X)$ and $\mathrm{c}(X)$, respectively. An open cover $\mathfrak{G}$ of the topological space X is called a *separating open cover* if for any pair of distinct points x and y in X, there is a G in $\mathfrak{G}$ such that $x \in G$ and $y \notin G$. The point separating weight of a space X, denoted by $\mathrm{psw}(X)$, is the smallest infinite cardinal $\mathfrak{m}$ such that X has a separating open cover of pointwise cardinality not greater than $\mathfrak{m}$. A collection $\{\mathfrak{G}_i\colon i \in I\}$ of open covers of a topological space X is a pluming for X if the following holds: if $x \in G_i \in \mathfrak{G}_i$ for all i in I, then

(a) $C_x = \bigcap_{i\in I} \operatorname{cl} G_i$ is compact,

(b) $\{\bigcap_{i\in J} \operatorname{cl} G_i\colon J$ is a finite subset of $I\}$ is a "base" for C_x in the sense that given any open set U containing C_x, there is a finite subset J of I with $\bigcap_{i\in J} \operatorname{cl} G_i \subset U$. In Hodel [7] it is shown that every regular T_1 space X has a pluming $\{\mathfrak{G}_i\colon i\in I\}$ with $|I|\leqslant \mathrm{w}(X)$, and a regular T_1 space is a p-space in the sense of Arhangel'skiĭ (see e.g. [2]) if and only if it has a countable pluming as defined above. The pluming degree of a regular T_1 space X, denoted by $\mathrm{pl}(X)$, is the least infinite cardinal $\mathfrak{m}$ such that X has a pluming $\{\mathfrak{G}_i\colon i\in I\}$ with $|I|\leqslant\mathfrak{m}$. A cover $\mathfrak{K}$ with subsets of a space X is called a *K-net* for X if for each point x in X there exists a compact subset C_x of X such that for every neighbourhood U of C_x there is a set N in $\mathfrak{K}$ with $x\in N\subset U$. A *K*-base for the space X is a *K*-net consisting of open subsets in X. The cardinal invariants $\mathrm{Kn}(X)$ and $\mathrm{Kw}(X)$ (*K*-netweight and *K*-weight) are defined as the smallest infinite cardinal $\mathfrak{m}$ such that there is a *K*-net respectively a *K*-base for the space X of cardinality not greater than $\mathfrak{m}$. For a cardinal invariant $f(X)$ let us define the cardinal invariant $f^*(X)$ as the smallest infinite cardinal $\mathfrak{m}$ such that $f(Y)\leqslant\mathfrak{m}$ holds for each subspace Y of X.

PROPOSITION 4.1. *The topology of every topological space (X,τ) is compact relative to the topology τ' in X generated by a K-net for (X,τ) as a subbase.*

Proof. Suppose that an ultrafilter $\mathfrak{U}$ in X has a limit point x in the topology τ'. Then by the definition of a *K*-net there is a compact subset C_x in (X,τ) containing x such that all neighbourhoods of C_x in the topology τ belong to $\mathfrak{U}$. We show that $\mathfrak{U}$ converges to a point of C_x in the topology τ. Suppose the converse is true, i.e. for every point y in C_x there is an open neighbourhood V_y of y in τ such that $V_y\notin\mathfrak{U}$. Since C_x is compact, we infer that there is a finite subfamily $\{V_i\}_{i=1}^s$ of the family $\{V_y\colon y\in C_x\}$ such that $\{V_i\}_{i=1}^s$ covers C_x. Further, since $V_i\notin\mathfrak{U}$ $(i=1,2,\ldots,s)$, we have $\bigcup_{i=1}^s V_i\notin\mathfrak{U}$, which contradicts the earlier assumption that any neighbourhood of C_x in τ belongs to $\mathfrak{U}$.

COROLLARY 1. *Let $\{\mathfrak{G}_i\colon i\in I\}$ be a pluming for a topological space (X,τ). Then τ is compact relative to the topology τ' in X generated by the family $\mathfrak{G} = \bigcup_{i\in I}\mathfrak{G}_i$ as a subbase.*

Proof. Clearly, the family of all finite intersections of members of $\mathfrak{G}$ is a *K*-net (moreover, a *K*-base) for the topology τ.

THEOREM 4.2 (Juhász [11]). *The following relations hold for a topological space*:

(1) $\mathrm{L}(X)\leqslant\mathrm{Kn}(X)$.

(2) $\mathrm{w}(X) = \mathrm{Kn}(X)\cdot\mathrm{pw}(X)$ *if X is a T_1 space.*

(3) $\mathrm{n}(X)\leqslant\mathrm{Kn}(X)\cdot\mathrm{psw}(X)$ *if X is a T_1 space.*

(4) $\mathrm{w}(X) = \mathrm{Kw}(X)\cdot\mathrm{psw}(X)$ *if X is a T_2 space.*

(5) $\mathrm{w}(X) = \mathrm{Kw}(X)\cdot\mathrm{n}(X)$ *if X is a T_2 space.*

(6) $\mathrm{w}(X) = \mathrm{Kw}^*(X)$ *if X is a regular T_1 space and if for $\mathfrak{m} = \mathrm{Kw}^*(X)$ either $\mathfrak{m} = \aleph_0$ or $2^{\mathfrak{m}} = \mathfrak{m}^+$ holds.*

Proof. We shall use Proposition 4.1. First we remark that for any topological space X the inequalities $\mathrm{w}(X)\geqslant\mathrm{Kn}(X)$, $\mathrm{Kw}(X)$, $\mathrm{Kw}^*(X)$, $\mathrm{pw}(X)$, $\mathrm{n}(X)$ are obvious and for T_1 spaces $\mathrm{w}(X)\geqslant\mathrm{psw}(X)$ clearly holds. Then (1) follows from Theorem 1.3, (2) follows from Theorem 2.6, (3) and (4) follow from Theorem 2.7. (5) follows from Theorem 2.1 by the argument used to complete the proof of Theorem 2.2 and, finally, (6) follows from Theorem 3.2.

THEOREM 4.3 (Hodel [7] and [8]). *The following relations hold for a regular T_1 space X*:

(1) $\mathrm{w}(X) = \mathrm{L}(X)\cdot\mathrm{pl}(X)\cdot\mathrm{psw}(X)$.

(2) $\mathrm{w}(X) = \mathrm{L}^*(X)\cdot\mathrm{pl}^*(X)$ *if for $\mathfrak{m} = \mathrm{L}^*(X)\mathrm{pl}^*(X)$ either $\mathfrak{m} = \aleph_0$ or $2^{\mathfrak{m}} = \mathfrak{m}^+$ holds.*

(3) $\mathrm{w}(X) = \mathrm{c}(X)\cdot\mathrm{pl}^*(X)$ *is X is hereditarily paracompact and for $\mathfrak{m} = \mathrm{c}(X)\cdot\mathrm{pl}^*(X)$ either $\mathfrak{m} = \aleph_0$ or $2^{\mathfrak{m}} = \mathfrak{m}^+$ holds.*

Remarks. The next theorem, of Ponomarev [16] and Nagata [14], follows from (1): every Lindelöf p-space with a point countable separating open cover is metrizable. As a special case, (2) implies the following theorem of Arhangel'skiĭ [3]: a space which is hereditarily a Lindelöf p-space is metrizable. If $\mathrm{pl}^*(X) = \aleph_0$, (3) is an answer in the positive to Problem 2 in Arhangel'skiĭ [3] (see Hodel [8]).

Proof. It is clear that the inequalities $\mathrm{w}(X)\geqslant\mathrm{L}(X)$, $\mathrm{pl}(X)$, $\mathrm{psw}(X)$, $\mathrm{L}^*(X)$, $\mathrm{pl}^*(X)$, $\mathrm{c}(X)$ hold for any space X.

To prove (1) let $\{\mathfrak{G}_i\colon i\in I\}$ be a pluming with $|I|\leqslant\mathrm{pl}(X)$ for the space X. We may suppose that $|\mathfrak{G}_i|\leqslant\mathrm{L}(X)$ for each i in I. By Corollary 1 to Proposition 4.1 the topology τ of X is compact relative to the topology τ_1 in X generated by $\mathfrak{G}$ as a subbase.

By the definition of $\mathrm{psw}(X)$ there is a T_1 topology τ_2 weaker than τ in X and such that the pointwise weight of τ_2 does not exceed $\mathrm{psw}(X)$. Let us denote the supremum of τ_1 and τ_2 by τ'. Clearly, τ' is compact relative to τ_1. By Theorem 2.6 the weight of τ' does not exceed $\mathrm{L}(X)\cdot\mathrm{pl}(X)\cdot\mathrm{psw}(X)$. It is obvious that τ is compact relative to the T_1 topology τ'. Applying Theorem 2.2, we obtain (1).

In order to prove (2) it is enough to consider that by the same argument as in the first part of the above proof the conditions of Theorem 3.2 are satisfied.

Finally, it is easily seen that $\mathrm{c}(X) = \mathrm{L}^*(X)$ for any hereditarily paracompact T_2 space (see e.g. Hodel [8]), and so (3) follows from (2).

DEFINITION 4.1. A closed and continuous map from a topological space X onto a topological space Z is called *perfect* if the pre-image of any point in Z is compact in the topology of X.

THEOREM 4.4. a) *Suppose that the topological space X has a perfect map onto the topological space Y. Then*

(1) $\mathrm{L}(X)\leqslant\mathrm{L}(Y)$.

(2) $w(X) \leqslant w(Y) \cdot psw(X)$ *if* X *is a* T_2 *space.*

(3) $\chi(X) \leqslant \chi(Y) \cdot \psi(X)$ *if* X *is a regular* T_1 *space.*

b) *Suppose that any subspace of a regular* T_1 *space* X *has a perfect map onto a topological space of weight not greater than* $\mathfrak{m}$ *and of character not greater than* $\mathfrak{n}$ *(*$\mathfrak{n}$ *is an infinite cardinal). Then*

(4) $w(X) \leqslant \mathfrak{m}$ *if either* $\mathfrak{m} = \aleph_0$ *or* $2^{\mathfrak{m}} = \mathfrak{m}^+$.

(5) $\chi(X) \leqslant c(X) \cdot \mathfrak{n}$.

Proof. First of all we remark that if there is a perfect map of X onto Y, then the topology of X is compact relative to the inverse image of the topology of Y. Indeed, as can easily be seen, the inverse image of any base for the topology of Y is a K-base for the topology of X, so that we can apply Proposition 4.1. Now (1) follows from Theorem 1.3, (2) follows in the simplest way from Theorem 4.2, (3) follows from Theorem 2.5. Further, (4) and (5) follow from Theorem 3.2 and Theorem 3.1, respectively.

Definition 4.2. We say that a topological space X is of *point-*$\mathfrak{m}$ *type* if for every point x in X there is a compact subset of character not greater than $\mathfrak{m}$ in X containing the point x. Spaces of point-$\aleph_0$ type are called *spaces of point-countable type*.

Theorem 4.5. *Let* X *be a regular* T_1 *space with the Souslin number not greater than* $\mathfrak{m}$. *Suppose that each subspace of* X *is of point-*$\mathfrak{m}$ *type. Then the character of* X *does not exceed* $\mathfrak{m}$.

Proof. Let x be an arbitrary point in X and let the subspace $Y = G \cup \{x\}$ be defined in the same way as in the proof of Theorem 3.1. Let C_x be a compact subset of Y containing x such that C_x has a neighbourhood base $\mathfrak{B}_x$ of cardinality not greater than $\mathfrak{m}$ in Y. By Zorn's lemma there is a maximal family $\mathfrak{C}$ of disjoint compact subsets of Y containing C_x such that every C in $\mathfrak{C} \setminus \{C_x\}$ has a neighbourhood base $\mathfrak{B}_C$ of cardinality not greater than $\mathfrak{m}$ and that no member of $\mathfrak{B}_C$ intersects C_x. Let C^* be the union of all members in $\mathfrak{C}$. Since Y is hereditarily point-$\mathfrak{m}$ type, we infer that C^* is dense in Y and thus is also dense in X. By the definition of G the topology of Y (and thus the topology of C^*) has a pseudobase of cardinality not greater than $\mathfrak{m}$ at the point x. From Proposition 4.1 it follows that the topology of C^* induced by the topology of X is compact relative to the topology (of character not greater than $\mathfrak{m}$) generated by the trace of the family $\bigcup_{C \in \mathfrak{C}} \mathfrak{B}_C$ in C^*. Thus, by Theorem 2.5, there is a base of cardinality not greater than $\mathfrak{m}$ for the topology of C^* at the point x. Since C^* is dense in X and X is a regular T_1 space, we conclude that the same holds for the topology of the space X.

It is a pleasant obligation for the author to express his gratitude to Professor Jenö Erdös (Debrecen). We emphasize that the present work could hardly be accomplished without his kind and continuous attention and useful advices.

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Accepté par la Rédaction le 29. 3. 1976