An example of maximal connected Hausdorff space

by

Petr Simon (Praha)

Abstract. An example of maximal connected Hausdorff topology for reals is given.

0. Introduction. A lot of work has been done in the studying of a lattice of topologies over the given set during the last fifteen years (see e.g. [10] for further references). Almost every topological property was discussed from this standpoint of view. In 1968, J. P. Thomas was — as far as the present author knows — the first one, who focused his attention on connected topologies in the paper [14] and who formulated the question: "Does there exist a maximal connected Hausdorff topology on some space other than singleton?"

Various theorems on maximal connected topologies have been proved ([1], [4], [5], [6], [7], [8], [14]), but Thomas' problem still remains unsolved. Some informations were obtained in the opposite direction ("does there exist a non-maximal connected Hausdorff space, having no maximal connected topology finer than the given one?"). It was I. Baggs [1], who gave an example of such a space, a modification of the well-known Roy's space with a dispersion point ([12], [13]); and J. A. Guthrie together with H. E. Stone have described a large class of spaces with this property ([5]).

In 1967–68, P. C. Hammer and W. E. Singletary ([7]) and S. K. Hildebrand ([8]) gave a detailed study of the case of connected topologies for the reals. The above authors have developed extremely useful tools, which enables the present author to claim: A maximal connected Hausdorff topology on an infinite set does exist.

1. Generalities. The notation used here is the standard one, used e.g. in Kelley's book [9]; the topological space is denoted as a pair $\langle X, \mathscr{T}\rangle$, where X is a set, $\mathscr{T}$ is the collection of all open subsets of X. In order to avoid confusions, the symbols $^-$ (closure operator) and ^{0} (interior operator) are replaced by $\mathrm{cl}_{\mathscr{T}}$ and $\mathrm{Int}_{\mathscr{T}}$, where the subscript denotes the topology in question. If $\mathscr{T}$ is some system of subsets of the given set, then the symbol $\langle\mathscr{T}\rangle$ will denote the smallest topology containing the whole $\mathscr{T}$. The phrase "$\langle X, \mathscr{T}\rangle$ is the maximal space with the property V" has a commonly accepted meaning (see e.g. [2]) that the space $\langle X, \mathscr{T}\rangle$ has V and, if $\mathscr{T}'$ is another topology properly containing $\mathscr{T}$, then $\langle X, \mathscr{T}'\rangle$ does not possess V.

Open (resp. closed) intervals of real numbers are denoted by $]\!]a, b[\![$ (resp. $[\![a, b]\!]$).

If X is a set, $\mathcal{K}$ a non-void collection of subsets of X, then a filter $\mathcal{U}$ on X in $\mathcal{K}$ is a system which satisfies

(i) $\emptyset \neq \mathcal{U} \subset \mathcal{K}$,

(ii) if $k<\omega$ and if $U_1, U_2, \ldots, U_k \in \mathcal{U}$, then $U_1 \cap U_2 \cap \ldots \cap U_k \in \mathcal{U}$,

(iii) if $U \in \mathcal{U}$ and if there is a $U' \in \mathcal{K}$, $U' \supset U$, the $U' \in \mathcal{U}$,

(iv) $\emptyset \notin \mathcal{U}$.

A filter $\mathcal{U}$ in $\mathcal{K}$ is called to be an *ultrafilter* in $\mathcal{K}$, if no filter in $\mathcal{K}$ properly contains $\mathcal{U}$. A filter base in $\mathcal{K}$ is a collection satisfying (i), (ii) and (iv). Clearly, a filter base $\mathcal{B}$ in $\mathcal{K}$ is an ultrafilter base in $\mathcal{K}$, if for each $K \in \mathcal{K}$ which meets every member of $\mathcal{B}$ in a non-void member of $\mathcal{K}$ there is some $B \in \mathcal{B}$ with $B \subset K$.

2. Definition. Let $\langle X, \mathcal{G}\rangle$ be a topological space, let $\mathcal{H}$ be a topology on X. We shall call the topology $\mathcal{H}$ to be $\mathcal{G}$*-extremal*, if $\mathcal{H}$ is the largest topology satisfying:

$\mathcal{H} \supset \mathcal{G}$ and

for every point $x \in X$ there is a $\mathcal{H}$-neighborhood base $\mathcal{V}_x$ such that for every $\mathcal{G}$-component C of the set $X-\{x\}$ the system $\{V \cap \mathrm{Int}_{\mathcal{G}} C \mid V \in \mathcal{V}_x\}$ is a base of an ultrafilter in $\mathcal{G}$.

Indeed, such a topology need not exist from various reasons, and if it exists, it need not be connected, even in the case of such "nice" space as a unit square. Despite this, the definition will soon appear to be useful.

Let us mention some straightforward consequences of the definition. If $\mathcal{G}$-extremal topology exists for some space $\langle X, \mathcal{G}\rangle$, then

a) if $x \in X$ and if C is a $\mathcal{G}$-component of $X-\{x\}$, then $x \in \mathrm{cl}_{\mathcal{G}} \mathrm{Int}_{\mathcal{G}} C$,

b) consequently, $\mathrm{Int}_{\mathcal{G}} C$ is non-void and $C \cup \{x\}$ is connected, hence

c) the space $\langle X, \mathcal{G}\rangle$ is connected as a union of connected subsets with a point x in common.

The following lemma is, in fact, known (see [4]), and we list it here only for the sake of completeness.

3. Lemma. *Let $\langle X, \mathcal{G}\rangle$ be a connected space. Denote by $D(\mathcal{G})$ the family of all dense sets in $\langle X, \mathcal{G}\rangle$ and let $\mathcal{D}$ be a filter in $D(\mathcal{G})$. Then the space $\langle X, \langle \mathcal{G} \cup \mathcal{D}\rangle\rangle$ is connected.*

The next lemma is simple, too.

4. Lemma. *Let $\langle X, \mathcal{G}\rangle$ be a topological space and let there exist a $\mathcal{G}$-extremal topology $\mathcal{H}$ on X. Then the set D is dense in $\langle X, \mathcal{H}\rangle$ if and only if D is dense in $\langle X, \mathcal{G}\rangle$.*

Proof. By the definition of $\mathcal{G}$-extremal topology and by the remarks every $\mathcal{H}$-neighborhood of every point contains a $\mathcal{G}$-open subset. Thus D is dense in $\langle X, \mathcal{H}\rangle$ if D is dense in $\langle X, \mathcal{G}\rangle$. The reverse implication is an immediate consequence of the inclusion $\mathcal{H} \supset \mathcal{G}$.

5. Theorem. *Let $\langle X, \mathcal{G}\rangle$ be a connected Hausdorff space such that*

(i) *every component of $X-\{x\}$ belongs to $\mathcal{G}$ for every point $x \in X$,*

(ii) *there exists some connected $\mathcal{G}$-extremal topology on X.*

Then there exists a maximal connected Hausdorff topology $\mathcal{R}$ on X, $\mathcal{R} \supset \mathcal{G}$.

Proof. Denote, as above, by $D(\mathcal{G})$ the set of all dense subsets of $\langle X, \mathcal{G}\rangle$ and let $\mathcal{D}$ be an ultrafilter in $D(\mathcal{G})$. Let $\mathcal{H}$ be the connected $\mathcal{G}$-extremal topology. The topology $\mathcal{R} = \langle \mathcal{H} \cup \mathcal{D}\rangle$ has the desired properties.

I. Since $\mathcal{R} \supset \mathcal{H} \supset \mathcal{G}$, the space $\langle X, \mathcal{R}\rangle$ is obviously Hausdorff. According to Lemma 4, $D(\mathcal{G}) = D(\mathcal{H})$ and since $\langle X, \mathcal{H}\rangle$ is connected, $\langle X, \mathcal{R}\rangle$ is also connected by Lemma 3.

II. For every $x \in X$ denote by $\mathcal{V}_x$ the neighborhood base of x in $\langle X, \mathcal{H}\rangle$, such that for every $\mathcal{G}$-component C of $X-\{x\}$, the system $\{V \cap C \mid V \in \mathcal{V}_x\}$ is an ultrafilter base in $\mathcal{G}$. Denote this base as $\mathcal{U}_x^C$. Then the following holds:

If $M \subset X$, $M \notin \mathcal{R}$, then there exists a point $x \in M$ and a $\mathcal{G}$-component C of $X-\{x\}$, such that for every $U \in \mathcal{U}_x^C$ and for every $D \in \mathcal{D}$ the set $(U \cap D)-(M \cap C)$ is non-empty.

Indeed, if for every point $x \in M$ and for every component C of $X-\{x\}$ there is some $D_C \in \mathcal{D}$ and $U_C \in \mathcal{U}_x^C$ with $(U_C \cap D_C) \subset (M \cap C)$, then the set

$$D = \bigcup \{D_C \cap C \mid C \text{ is a component of } X-\{x\}\}$$

is obviously dense in $\langle X, \mathcal{G}\rangle$ and meets all members of $\mathcal{D}$ in a dense set; since $\mathcal{D}$ is an ultrafilter in $D(\mathcal{G})$, $D \in \mathcal{D}$. The set $\bigcup \{U_C \cap D \mid C \text{ is a component of } X-\{x\}\} \cup \{x\}$ is an $\mathcal{R}$-neighborhood of x contained in M, which contradicts the assumption that $M \notin \mathcal{R}$.

III. It remains to prove that $\mathcal{R}$ is maximal connected. In order to show this, let $\mathcal{R}'$ be a topology on X strictly finer than $\mathcal{R}$. We must find two non-empty disjoint members of $\mathcal{R}'$ which cover X. To this end, pick a set $M \in \mathcal{R}'-\mathcal{R}$ and let x and C be the point of X and the component of $X-\{x\}$ in the topology $\mathcal{G}$, such that for every $U \in \mathcal{U}_x^C$ and for every $D \in \mathcal{D}$ the set $(U \cap D)-(M \cap C)$ is non-void.

Denote $M^C = M \cap C$ and $B = \bigcup \{C' \mid C' \text{ is a } \mathcal{G}\text{-component of } X-\{x\}, C' \neq C\}$. Notice that both M^C and B are members of $\mathcal{R}'$. Let A be the union of all $\mathcal{G}$-open sets G such that $W \cap D \cap M^C \neq \emptyset$ whenever $D \in \mathcal{D}$, $W \in \mathcal{G}$, $\emptyset \neq W \subset G$.

We claim that A does not contain any member of $\mathcal{U}_x^C$: Suppose the contrary. Then $(X-A) \cup (M^C \cap A)$ is dense in $\langle X, \mathcal{G}\rangle$ and meets all members of $\mathcal{D}$ in a dense set, thus ($\mathcal{D}$ is an ultrafilter in $D(\mathcal{G})$) $(X-A) \cup (M^C \cap A) \in \mathcal{D}$. Then the obvious inclusion $((X-A) \cup (M^C \cap A)) \cap A \subset M^C$ contradicts the statement that $(D \cap U)-M^C$ is non-void for every $U \in \mathcal{U}_x^C$, $D \in \mathcal{D}$.

Thus A contains no member of $\mathcal{U}_x^C$, but $\mathcal{U}_x^C$ is an ultrafilter base in $\mathcal{G}$ and $\mathcal{G}$ is closed under finite intersections, so one can find a set $U_0 \in \mathcal{U}_x^C$ with $U_0 \cap A = \emptyset$.

Let D be an arbitrary member of $\mathcal{D}$, let $G \in \mathcal{G}$ be an arbitrary subset of U_0. Since $U_0 \subset X-A$, there is some non-void $W_1 \in \mathcal{G}$, $W_1 \subset G$ and some $D_1 \in \mathcal{D}$ such that $W_1 \cap D_1 \cap M^C = \emptyset$, thus $W_1 \cap D_1 \cap D \cap M^C = \emptyset$. The set $D_1 \cap D$ is dense in $\langle X, \mathscr{G}\rangle$, so it must meet an open set W_1. It follows that

$$W_1 \cap D_1 \cap D \cap (X - M^C) \neq \emptyset$$

and consequently $G \cap D \cap (X - M^C) \neq \emptyset$. Hence the set

$$D' = (U_0 \cap (X - M^C)) \cup (X - U_0)$$

is dense in $\langle X, \mathscr{G}\rangle$ and meets all members of $\mathscr{D}$ in a dense set, so it belongs to $\mathscr{D}$. According to the definition of $\mathscr{R}$, the set $K = (D' \cap U_0) \cup \{x\} \cup B$ belongs to $\mathscr{R}$ and clearly $K \cap M^C = \emptyset$.

Finally, the set $L = M \cup B$ is open in $\langle X, \mathscr{R}'\rangle$ and the set $K \cap L$ is open in $\langle X, \mathscr{R}'\rangle$, too. But the last intersection equals to $B \cup \{x\}$ and the pair C, $B \cup \{x\}$ is a disjoint open cover of the space $\langle X, \mathscr{R}'\rangle$.

We have proved that the space $\langle X, \mathscr{R}'\rangle$ is not connected. Since this result holds for any topology $\mathscr{R}'$ strictly larger than $\mathscr{R}$, the space $\langle X, \mathscr{R}\rangle$ is maximal connected.

It remains to show that the assumptions of Theorem 5 can be non-vacuously satisfied, i.e. we must find some Hausdorff space $\langle X, \mathscr{G}\rangle$ such that for every $x \in X$ every component of $X - \{x\}$ is open and which admits some connected $\mathscr{G}$-extremal topology. It appears that the set of real numbers R is such a space. This will be proved in the following lemma.

6. Lemma. *Let $\mathscr{T}$ be the usual euclidean topology on the set of real numbers R. Then there exists a connected $\mathscr{T}$-extremal topology $\mathscr{S}$ on R.*

Proof. Let $\mathscr{B}$ be the set of all pairs $\langle U, V\rangle$ such that $U, V \in \mathscr{T}$, $U \neq \emptyset \neq V$, $\mathrm{cl}_{\mathscr{T}} U \cup \mathrm{cl}_{\mathscr{T}} V = R$ and $\mathrm{card}(\mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V) = 2^\omega$.

The cardinality of $\mathscr{B}$ does not exceed 2^ω, and we may well-order it: $\mathscr{B} = \{b_\alpha \mid \alpha < 2^\omega\}$.

By an easy transfinite recursion, we shall define points $x_\alpha \in R$ for $\alpha < 2^\omega$ such that:

(i) $x_\alpha \neq x_\beta$ for all $\alpha < \beta < 2^\omega$;

(ii) if $b_\alpha = \langle U, V\rangle$, then $x_\alpha \in \mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V$ for all $\alpha < 2^\omega$.

Suppose x_β have been defined for all $\beta < \alpha$, $\alpha < 2^\omega$, and let $b_\alpha = \langle U, V\rangle$. The cardinality of the set $\{x_\beta \mid \beta < \alpha\}$ is smaller than 2^ω, thus one can find a point $x_\alpha \in (\mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V) - \{x_\beta \mid \beta < \alpha\}$.

For each $x \in R$, applying the maximality principle, one can find two ultrafilters $\mathscr{U}_x^L$, $\mathscr{U}_x^R$ in $\mathscr{T}$ such that

(iii) $\mathscr{U}_x^L \ni]\!]\leftarrow, x[\![$, $\mathscr{U}_x^R \ni]\!]x, \rightarrow[\![$ for all $x \in R$;

(iv) if $F \in \mathscr{U}_x^L \cup \mathscr{U}_x^R$, then $x \in \mathrm{cl}_{\mathscr{T}} F$ for all $x \in R$;

(v) if $x = x_\alpha$ for some $\alpha < 2^\omega$ and if $b_\alpha = \langle U, V\rangle$, then either $U \in \mathscr{U}_x^L$ and $V \in \mathscr{U}_x^R$ or $V \in \mathscr{U}_x^L$ and $U \in \mathscr{U}_x^R$.

If $x \neq x_\alpha$ for all $\alpha < 2^\omega$, an arbitrary ultrafilter in $\mathscr{T}$ which contains all open intervals $]\!]x-r, x[\![$ ($]\!]x, x+r[\![$, respectively) with $r > 0$ has the desired properties and can be denoted as $\mathscr{U}_x^L$ ($\mathscr{U}_x^R$, respectively).

Let $x = x_\alpha$ for some $\alpha < 2^\omega$, and let $b_\alpha = \langle U, V\rangle$. Suppose that

$$]\!]x-r, x[\![\cap U \neq \emptyset \quad \text{and} \quad]\!]x, x+r[\![\cap V \neq \emptyset$$

for each real positive r. In this case, denote by $\mathscr{U}_x^L$ ($\mathscr{U}_x^R$, respectively) some ultrafilter in $\mathscr{T}$ which contains the filter base $\{]\!]x-r, x[\![\cap U \mid r > 0\}$ ($\{]\!]x, x+r[\![\cap V \mid r > 0\}$, resp.).

The second possible case with U, V interchanged is analogous. Clearly, at least one of the possibilities mentioned here must occur.

Define the topology $\mathscr{S}$ as follows: $S \in \mathscr{S}$ iff for every point $x \in S$ there are $U^L \in \mathscr{U}_x^L$ and $U^R \in \mathscr{U}_x^R$ such that $U^L \cup U^R \subset S$. It is self-evident that $\mathscr{S}$ is a topology, that $\mathscr{S} \supset \mathscr{T}$ and that $\mathscr{S}$ is $\mathscr{T}$-extremal.

The topology $\mathscr{S}$ is connected. To show this, let $A, B \in \mathscr{S}$, $A \cup B = R$, $A \neq \emptyset \neq B$. It suffices to verify that $\mathrm{cl}_{\mathscr{S}} A \cap \mathrm{cl}_{\mathscr{S}} B \neq \emptyset$. Denote by $U = \mathrm{Int}_{\mathscr{T}} A$, $V = \mathrm{Int}_{\mathscr{T}} B$; one may quickly check that $\mathrm{cl}_{\mathscr{T}} U \cup \mathrm{cl}_{\mathscr{T}} V = R$, $U \neq \emptyset \neq V$ and, since $\langle R, \mathscr{T}\rangle$ is connected, $\mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V \neq \emptyset$.

If $\mathrm{card}(\mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V) = 2^\omega$, then there is an $\alpha < 2^\omega$ such that $\langle U, V\rangle = b_\alpha$ and $x_\alpha \in \mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V$. For a point x_α, by (v), one of the sets U, V belongs to $\mathscr{U}_{x_\alpha}^L$ and the other to $\mathscr{U}_{x_\alpha}^R$.

If $\mathrm{card}(\mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V) < 2^\omega$, then there is a point $x \in \mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V$ which is isolated in the subspace $\mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V$ (a simple consequence of the well-known fact that every compact, Haussdorff and perfect space is of cardinality at least 2^ω). Thus there is some real $r > 0$ such that $]\!]x-r, x+r[\![\cap \mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V = \{x\}$. It follows that either U is dense in $]\!]x-r, x[\![$ and V is dense in $]\!]x, x+r[\![$, or V is dense in $]\!]x-r, x[\![$ and U is dense in $]\!]x, x+r[\![$. Suppose the contrary. You must find a point $y \neq x$, $y \in]\!]x-r, x+r[\![\cap \mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V$, which contradicts our choice of r. Since $\mathscr{U}_x^L$ and $\mathscr{U}_x^R$ are ultrafilters in $\mathscr{T}$, either $U \in \mathscr{U}_x^L$ and $V \in \mathscr{U}_x^R$, or $V \in \mathscr{U}_x^L$ and $U \in \mathscr{U}_x^R$.

Thus we have verified that there is a point $x \in \mathrm{cl}_{\mathscr{T}} U \cap \mathrm{cl}_{\mathscr{T}} V$ every neighborhood (in the topology $\mathscr{S}$) of which intersects both U and V. Thus

$$x \in \mathrm{cl}_{\mathscr{S}} U \cap \mathrm{cl}_{\mathscr{S}} V \subset \mathrm{cl}_{\mathscr{S}} A \cap \mathrm{cl}_{\mathscr{S}} B\,.$$

This completes the proof.

Applying Theorem 5 and Lemma 6, we obtain the following

7. Theorem. *There exists a maximal connected Hausdorff topology on the set of real numbers.*

Let $\langle X, \mathscr{G}\rangle$ be a connected space, let $x \in X$ and let $\mathscr{C}$ be the collection of all components of $X - \{x\}$. Let $\mathscr{H}$ be any other topology on X such that every $C \in \mathscr{C}$ is connected under $\mathscr{H}$ (this will take place e.g. in the case when $\mathscr{G}$ and $\mathscr{H}$ coincide on C). If $x \in \mathrm{cl}_{\mathscr{H}} C$ for every $C \in \mathscr{C}$, then the space $\langle X, \mathscr{H}\rangle$ is connected, since $X = \bigcup \{C \cup \{x\} \mid C \in \mathscr{C}\}$. This simple observation implies the following two propositions:

8. Proposition. *Let $\langle X, \mathscr{G}\rangle$ be a maximal connected space, let $x \in X$ and let C be a component of $X-\{x\}$. Then $x \in \mathrm{cl}_{\mathscr{G}} C$ if and only if $C \in \mathscr{G}$.*

9. Proposition. *Let $\langle X, \mathscr{G}\rangle$ be a maximal connected space and let $x \in \mathrm{cl}_{\mathscr{G}} C$ whenever $x \in X$ and C is a component of $X-\{x\}$. Then $\mathscr{G}$ is the connected $\mathscr{G}$-extremal topology.*

10. Problems. Let $\langle X, \mathscr{G}\rangle$ be a connected (Hausdorff) space, let $x \in \mathrm{cl}_{\mathscr{G}} C$ whenever C is a component of $X-\{x\}$. Is it true that then there exists a topology $\mathscr{H} \supset \mathscr{G}$ such that $\langle X, \mathscr{H}\rangle$ satisfies the assumptions of Theorem 5?

What can be said about those spaces which contain a couple x, C (C a component of $X-\{x\}$) with $x \notin \mathrm{cl}_{\mathscr{G}} C$?

11. Remarks. A) S. K. Hildebrand ([8], Theorem 4.2) has proved that if $\mathscr{S}$ is a finer connected topology for a unit interval I than $\mathscr{T}$, then any connected subset of I will remain connected under $\mathscr{S}$. This result together with Theorem 7 shows that the conjecture "there is no maximal topology for I having the same connected subsets as $\mathscr{T}$" (Hammer–Singletary, [7]) is false.

B) Consider the case of R^2 with its usual euclidean topology $\mathscr{T}$. Every $\mathscr{T}$-extremal topology on R^2 is disconnected. In order to prove it, let $\mathscr{S}$ be any $\mathscr{T}$-extremal topology on R^2. If $\mathscr{U}_z$ is an $\mathscr{S}$-neighborhood base of a point $z \in R^2$, if we denote by

$$H^+ = \{z \in R^2 |\ z = \langle x, y\rangle, y>0\}, \qquad H^- = \{z \in R^2 |\ z = \langle x, y\rangle, y<0\}$$

and if we define

$$V = \{z \in R^2 |\ H^+ \cup \{z\} \in \mathscr{U}_z\}, \qquad W = \{z \in R^2 |\ H^- \cup \{z\} \in \mathscr{U}_z\},$$

then $V \cup W = R^2$, $V \cap W = \emptyset$, $V \neq \emptyset \neq W$ and both V and W belong to $\mathscr{S}$.

Nevertheless, let $\mathscr{G}$ be the following topology on R^2 (where $\langle R^2, \mathscr{T}\rangle$ is considered as a complex plane): If $z = 0$, its $\mathscr{G}$-neighborhood system will be the same as in $\mathscr{T}$. If $z \neq 0$, then for some real φ and positive real r, $z = re^{i\varphi}$. The $\mathscr{G}$-neighborhood base of z will consist of all sets $\{y |\ y = ue^{i\varphi}, s<u<t\}$ for s, t satisfying the inequality $0<s<r<t$. Obviously $\mathscr{G} \supset \mathscr{T}$ and the space $\langle R^2, \mathscr{G}\rangle$ satisfies all the assumptions of Theorem 5. (The proof of the existence of the connected $\mathscr{G}$-extremal topology is analogous to the proof of Lemma 6.)

C) The very technical proof of Lemma 6 is, perhaps, necessary. It is not true that every $\mathscr{T}$-extremal topology $\mathscr{S}$ on R is connected: Let us sketch an example.

Construct the Cantor discontinuum D by the routine inductive procedure:

$$D_0 = I,$$
$$D_1 = I -]\tfrac{1}{3}, \tfrac{2}{3}[,$$
$$D_2 = D_1 - (]\tfrac{1}{9}, \tfrac{2}{9}[\cup]\tfrac{7}{9}, \tfrac{8}{9}[),$$
$$D_3 = D_2 - (]\tfrac{1}{27}, \tfrac{2}{27}[\cup]\tfrac{7}{27}, \tfrac{8}{27}[\cup]\tfrac{19}{27}, \tfrac{20}{27}[\cup]\tfrac{25}{27}, \tfrac{26}{27}[),$$
$$\dots\dots\dots\dots$$
$$D = \bigcap \{D_i |\ i<\omega\}.$$

Let $A_i = D_i - D_{i+1}$ for $1 \leqslant i < \omega$, $A_0 =]\tfrac{1}{3}, \tfrac{2}{3}[\cup (R-I)$, let

$$P = \bigcup \{A_{2i} |\ i<\omega\}, \qquad Q = \bigcup \{A_{2i+1} |\ i<\omega\}$$

and denote $L = \{x \in R |$ there is some $i<\omega$ such that $x \in \mathrm{cl}_{\mathscr{T}} A_{2i}\}$, $M = R-L$.

The new topology $\mathscr{V}$ will be defined by the point-neighborhood systems as follows: For $x \in L$ (resp. $x \in M$), U is a $\mathscr{V}$-neighborhood of x iff there is some real $r>0$ such that

$$\{x\} \cup (]x-r, x+r[\cap P) \subset U \quad (\text{resp. } \{x\} \cup (]x-r, x+r[\cap Q) \subset U).$$

The topology $\mathscr{V}$ is obviously contained in some $\mathscr{T}$-extremal topology $\mathscr{S}$, and the topology $\mathscr{V}$ is disconnected: $P \cup L$ and $Q \cup M$ are disjoint $\mathscr{V}$-open sets which cover R.

References

[1] I. Baggs, *A connected Hausdorff space which is not contained in a maximal connected space*, Pacific J. Math. 51 (1974), pp. 11–18.

[2] D. E. Cameron, *Maximal and minimal topologies*, Trans. Amer. Math. Soc. 160 (1971), pp. 229–248.

[3] L. Friedler, *Open, connected functions*, Canad. Math. Bull. 16 (1973), pp. 57–60.

[4] J. A. Guthrie, D. F. Reynolds and H. E. Stone, *Connected expansions of topologies*, Bull. Austral. Math. Soc. 9 (1973), pp. 259–265.

[5] — and H. E. Stone, *Spaces whose connected expansions preserve connected subsets*, Fund. Math. 80 (1973), pp. 91–100.

[6] — — *Subspaces of maximally connected spaces*, Notices AMS 18 (1971), p. 672.

[7] P. C. Hammer and W. E. Singletary, *Connectedness-equivalent spaces on the line*, Rend. Circ. Mat. Palermo 17 (2) (1968), pp. 343–355.

[8] S. K. Hildebrand, *A connected topology for the unit interval*, Fund. Math. 61 (1967), pp. 133–140.

[9] J. L. Kelley, *General Topology*, New York 1957.

[10] R. E. Larson and S. J. Andima, *The lattice of topologies: A survey*, Rocky Mountains J. Math. 5 (2) (1975), pp. 177–198.

[11] D. F. Reynolds, *Preservation of connectedness under extension of topologies*, Kyungpook Math. J. 13 (1973), pp. 217–219.

[12] P. Roy, *A countable connected Urysohn space with a dispersion point*, Duke Math. J. 33 (1966), pp. 331–333.

[13] L. A. Steen and J. A. Seebach, Jr., *Counterexamples in Topology*, New York 1970.

[14] J. P. Thomas, *Maximal connected topologies*, J. Austral. Math. Soc. 8 (1968), pp. 700–705.

MATEMATICKÝ ÚSTAV KARLOVY UNIVERSITY
Praha

Accepté par la Rédaction le 22. 3. 1976