An example of maximal connected Hausdorff space

by

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Abstract. An example of maximal connected Hausdorff topology for reals is given.

0. Introduction. A lot of work has been done in the studying of a lattice of topologies over the given set during the last fifteen years (see e.g. [10] for further references). Almost every topological property was discussed from this standpoint of view. In 1968, J. P. Thomas was — as far as the present author knows — the first one, who focused his attention on connected topologies in the paper [14] and who formulated the question: "Does there exist a maximal connected Hausdorff topology on some space other than singleton?"

Various theorems on maximal connected topologies have been proved ([11], [4], [5], [6], [7], [8], [14]), but Thomas' problem still remains unsolved. Some informations were obtained in the opposite direction ("does there exist a non-maximal connected Hausdorff space, having no maximal connected topology finer than the given one?").

It was L. Baggs [1], who gave an example of such a space, a modification of the well-known Roy's space with a dispersion point ([12], [13]); and J. A. Guthrie together with H. E. Stone have described a large class of spaces with this property ([5]).

In 1967–68, P. C. Hammer and W. E. Singletary ([7]) and S. K. Hildebrand ([8]) gave a detailed study of the case of connected topologies for the reals. The above authors have developed extremely useful tools, which enables the present author to claim: A maximal connected Hausdorff topology on an infinite set does exist.

1. Generalities. The notation used here is the standard one, used e.g. in Kelley's book [9]; the topological space is denoted as a pair $\langle X, \mathcal{F} \rangle$, where $X$ is a set, $\mathcal{F}$ is the collection of all open subsets of $X$. In order to avoid confusions, the symbols $\overline{\cdot}$ (closure operator) and $\circ$ (interior operator) are replaced by $\text{cl}_\mathcal{F}$ and $\text{Int}_\mathcal{F}$, where the subscript denotes the topology in question. If $\mathcal{F}$ is some system of subsets of the given set, then the symbol $\langle \mathcal{F} \rangle$ will denote the smallest topology containing the whole $\mathcal{F}$. The phrase $\langle X, \mathcal{F} \rangle$ is the maximal space with the property $\mathcal{V}$ has a commonly accepted meaning (see e.g. [2]) that the space $\langle X, \mathcal{F} \rangle$ has $\mathcal{V}$ and, if $\mathcal{F}'$ is another topology properly containing $\mathcal{F}$, then $\langle X, \mathcal{F}' \rangle$ does not possess $\mathcal{V}$.

Open (resp. closed) intervals of real numbers are denoted by $]a, b[$ (resp. $[a, b]$).
If $X$ is a set, $\mathcal{X}$ a non-void collection of subsets of $X$, then a filter $\mathcal{F}$ on $X$ in $\mathcal{X}$ is a system which satisfies

(i) $\emptyset \notin \mathcal{F}$ in $\mathcal{X}$,
(ii) if $k \in \omega$ and $U_1, U_2, \ldots, U_k \notin \mathcal{F}$, then $U_1 \cap U_2 \cap \cdots \cap U_k \notin \mathcal{F}$,
(iii) if $U \in \mathcal{F}$ and if there is a $U' \in \mathcal{X}$, then $U' \supseteq U$, the $U' \in \mathcal{F}$,
(iv) $\emptyset \notin \mathcal{F}$.

A filter $\mathcal{F}$ in $\mathcal{X}$ is called to be an ultraliter in $\mathcal{X}$, if no filter in $\mathcal{X}$ properly contains $\mathcal{F}$. A filter base $\mathcal{B}$ in $\mathcal{X}$ is a collection satisfying (i), (ii) and (iv). Clearly, a filter base $\mathcal{B}$ in $\mathcal{X}$ is an ultrafilter base in $\mathcal{X}$, if for each $K \in \mathcal{X}$ which meets every member of $\mathcal{B}$ in a non-void member of $\mathcal{X}$ there is some $B \in \mathcal{B}$ with $B \subseteq K$.

2. DEFINITION. Let $(X, \mathcal{A})$ be a topological space, let $\mathcal{F}$ be a topology on $X$. We shall call the topology $\mathcal{F}$ to be $\mathcal{F}$-extremal, if $\mathcal{F}$ is the largest topology satisfying:

$\mathcal{F} \supseteq \mathcal{A}$

and

for every point $x \in X$ there is a $\mathcal{F}$-neighborhood base $\mathcal{F}_x$ such that for every $\mathcal{F}$-component $C$ of the set $X \setminus \{x\}$ the system $\{V \cap \text{Int}_C C \mid V \in \mathcal{F}_x\}$ is a base of an ultrafilter in $\mathcal{F}$.

Indeed, such a topology need not exist from various reasons, and if it exists, it need not be connected, even in the case of such "nice" space as a unit square. Despite this, the definition will soon appear to be useful.

Let us mention some straightforward consequences of the definition. If $\mathcal{F}$-extremal topology exists for some space $(X, \mathcal{X})$, then:

a) if $x \in X$ and if $C$ is a $\mathcal{F}$-component of $X \setminus \{x\}$, then $x \in \text{cl}_{\mathcal{X}} \text{Int}_C C$.

b) consequently, $\text{Int}_C C$ is non-void and $C \cup \{x\}$ is connected.

c) the space $(X, \mathcal{F})$ is connected as a union of connected subsets with a point $x$ in common.

The following lemma is, in fact, known (see [4]), and we list it here only for the sake of completeness.

3. LEMMA. Let $(X, \mathcal{A})$ be a connected space. Denote by $D(\mathcal{A})$ the family of all dense sets in $(X, \mathcal{A})$ and let $D$ be a filter in $D(\mathcal{A})$. Then the space $(X, \langle D \cup \mathcal{A} \rangle)$ is connected.

The next lemma is simple, too.

4. LEMMA. Let $(X, \mathcal{A})$ be a topological space and let there exist a $\mathcal{F}$-extremal topology $\mathcal{F}$ on $X$. Then the set $D$ is dense in $(X, \mathcal{F})$ if and only if $D$ is dense in $(X, \mathcal{A})$.

Proof. By the definition of $\mathcal{F}$-extremal topology and by the remarks every $\mathcal{F}$-neighborhood of every point contains a $\mathcal{F}$-open subset. Thus $D$ is dense in $(X, \mathcal{F})$ if $D$ is dense in $(X, \mathcal{A})$.

5. THEOREM. Let $(X, \mathcal{A})$ be a connected Hausdorff space such that

(i) every component of $X \setminus \{x\}$ belongs to $\mathcal{A}$ for every point $x \in X$,
(ii) there exists some connected $\mathcal{A}$-extremal topology on $X$.

Then there exists a maximal connected Hausdorff topology $\mathcal{A}$ on $X, \mathcal{A} \supseteq \mathcal{A}$.

Proof. Denote, as above, by $D(\mathcal{A})$ the set of all dense subsets of $(X, \mathcal{A})$ and let $\mathcal{B}$ be an ultrafilter in $D(\mathcal{A})$. Let $\mathcal{F}$ be the connected $\mathcal{A}$-extremal topology. The topology $\mathcal{A} = \langle \mathcal{A} \cup \mathcal{B} \rangle$ has the desired properties.

I. Since $\mathcal{A} \supseteq \mathcal{A} \supseteq \mathcal{A}$, the space $(X, \mathcal{A})$ is obviously Hausdorff. According to Lemma 4, $D(\mathcal{A}) = D(\mathcal{A})$ and since $(X, \mathcal{A})$ is connected, $(X, \mathcal{A})$ is also connected by Lemma 3.

II. For every $x \in X$ denote by $\mathcal{V}_x$ the neighborhood base of $x$ in $(X, \mathcal{A})$, such that for every $\mathcal{A}$-component $C$ of $X \setminus \{x\}$, the system $\{V \cap C \mid V \in \mathcal{V}_x\}$ is an ultrafilter base in $\mathcal{A}$. Denote this base as $\mathcal{V}_x^\mathcal{A}$. Then the following holds:

If $M \subseteq X, M \notin \mathcal{A}$, then there exists a point $x \in M$ and a $\mathcal{A}$-component $C$ of $X \setminus \{x\}$ such that for every $U \in \mathcal{V}_x^\mathcal{A}$ and for every $D \in \mathcal{A}$ the set $(U \cap D \cap \mathcal{M} \cap \mathcal{A})$ is non-empty.

Indeed, if for every point $x \in M$ and for every component $C$ of $X \setminus \{x\}$ there is some $D_C \in \mathcal{A}$ and $U \in \mathcal{V}_x^\mathcal{A}$ with $(U \cap D_C) \subseteq (M \cap \mathcal{C})$, then the set

$$D = \bigcup \{D_C \cap C \mid C \text{ is a component of } X \setminus \{x\}\}$$

is obviously dense in $(X, \mathcal{A})$ and meets all members of $\mathcal{B}$ in a dense set; since $\mathcal{B}$ is an ultrafilter in $D(\mathcal{A})$, $D \ni D$. The set $\bigcup \{U \cap D \cap C \mid C \text{ is a component of } X \setminus \{x\}\} \cup \{x\}$ is an $\mathcal{A}$-neighborhood of $x$ contained in $M$, which contradicts the assumption that $M \notin \mathcal{A}$.

III. It remains to prove that $\mathcal{A}$ is maximal connected. In order to show this, let $\mathcal{B}$ be a topology on $X$ strictly finer than $\mathcal{A}$. We must find two non-empty disjoint members of $\mathcal{B}$ which cover $X$. To this end, pick a set $M \in \mathcal{B} \setminus \mathcal{A}$ and let $x \in C$ be the point of $X$ and the component of $X \setminus \{x\}$ in the topology $\mathcal{B}$, such that for every $U \in \mathcal{V}_x^\mathcal{A}$ and for every $D \in \mathcal{A}$ the set $(U \cap D \cap \mathcal{M} \cap \mathcal{A})$ is non-empty.

Denote $M^\mathcal{B} = M \cap C$ and $B = \bigcup \{C \cap \mathcal{B} \mid C \text{ is a component of } X \setminus \{x\}\} \cup \{x\}$. Notice that both $M^\mathcal{B}$ and $B$ are members of $\mathcal{B}$. Let $A$ be the union of all $\mathcal{B}$-open sets $G$ such that $W \cap D \cap M^\mathcal{B} \subseteq \emptyset$ whenever $D \in \emptyset, W \subseteq W, A \cap W = \emptyset$.

We claim that $A$ does not contain any member of $\mathcal{B}$; Suppose the contrary. Then $(X \setminus A) \cup (M^\mathcal{B} \cap A)$ is dense in $(X, \mathcal{B})$ and meets all members of $\mathcal{B}$ in a dense set, thus $\mathcal{B}$ is an ultrafilter in $D(\mathcal{B})$, $(X \setminus A) \cup (M^\mathcal{B} \cap A) \subseteq \emptyset$. Then the obvious inclusion $(X \setminus A) \cup (M^\mathcal{B} \cap A) \subseteq \emptyset$ contradicts the statement that $(D \cap U) \cap M^\mathcal{B}$ is non-empty for every $U \in \mathcal{V}_x^\mathcal{A}, D \in \mathcal{A}$.

Thus $A$ contains no member of $\mathcal{B}$, but $\mathcal{V}_x^\mathcal{A}$ is an ultrafilter base in $\mathcal{B}$ and $\mathcal{B}$ is closed under finite intersections, so one can find a set $U_0 \in \mathcal{V}_x^\mathcal{A}$ with $U_0 \cap A = \emptyset$.

Let $D$ be an arbitrary member of $\mathcal{B}$, let $G \ni \emptyset$ be an arbitrary subset of $U_0$. Since $U_0 \subseteq X \setminus A$, there is some non-void $W_1 \in \mathcal{B}, W_1 \subseteq G$ and some $D_1 \in \mathcal{A}$ such that $W_1 \cap D_1 \cap M^\mathcal{B} = \emptyset$, thus $W_1 \cap D_1 \cap D \cap M^\mathcal{B} = \emptyset$. The set $D_1 \cap D$ is...
dense in \( (X, \mathcal{G}) \), so it must meet an open set \( W \). It follows that
\[ W_1 \cap D_1 \cap D \cap (X - M^c) \neq \emptyset \]
and consequently \( G \cap D \cap (X - M^c) \neq \emptyset \). Hence the set
\[ D' = (U_0 \cap (X - M^c)) \cup (X - U_0) \]
is dense in \( (X, \mathcal{G}) \) and meets all members of \( \mathcal{G} \) in a dense set, so it belongs to \( \mathcal{G} \).
According to the definition of \( \mathcal{G} \), the set \( K = (D' \cup U_0) \cup \{ x \} \cup B \) belongs to \( \mathcal{G} \) and clearly \( K \cap M^c = \emptyset \).
Finally, the set \( L = M \cup B \) is open in \( (X, \mathcal{G'}) \) and the set \( K \cap L \) is open in \( (X, \mathcal{G'}) \), too. But the last intersection equals to \( B \cup \{ x \} \) and the pair \( C, B \cup \{ x \} \) is a disjoint open cover of the space \( (X, \mathcal{G'}) \).

We have proved that the space \( (X, \mathcal{G'}) \) is not connected. Since this result holds for any topology \( \mathcal{G'} \) strictly larger than \( \mathcal{G} \), the space \( (X, \mathcal{G}) \) is maximal connected.

It remains to show that the assumptions of Theorem 5 can be non-vacuously satisfied, i.e. we must find some Hausdorff space \( (X, \mathcal{G}) \) such that for every \( x \in X \) every component of \( X - \{ x \} \) is open and which admits some connected \( \mathcal{G} \)-extremal topology. It appears that the set of real numbers \( \mathbb{R} \) is such a space. This will be proved in the following lemma.

6. Lemma. Let \( \mathcal{F} \) be the usual euclidean topology on the set of real numbers \( \mathbb{R} \). Then there exists a connected \( \mathcal{F} \)-extremal topology \( \mathcal{G} \) on \( \mathbb{R} \).

Proof. Let \( \mathcal{G} \) be the set of all pairs \( (U, V) \) such that \( U, V \in \mathcal{F}, U \neq \emptyset \neq V, cl_r U \cap cl_r V = \emptyset \), and \( \text{card}(cl_r U \cup cl_r V) = 2^n \).

The cardinality of \( \mathcal{G} \) does not exceed \( 2^n \), and we may well-order it: \( \mathcal{G} = \{ b \alpha : \alpha < 2^n \} \).

By an easy transfinite recursion, we shall define points \( x_\alpha \in R \) for \( \alpha < 2^n \) such that:

(i) \( x_\alpha \neq x_\beta \) for all \( \alpha < \beta < 2^n \);
(ii) if \( b_\alpha = \langle U, V \rangle \), then \( x_\alpha \in cl_r U \cup cl_r V \) for all \( \alpha < 2^n \).

Suppose \( x_\alpha \) have been defined for all \( \beta < \alpha < 2^n \), and let \( b_\alpha = \langle U, V \rangle \). The cardinality of the set \( \{ x_\beta : \beta < \alpha \} \) is smaller than \( 2^n \), thus one can find a point \( x_\beta \in (cl_r U \cap cl_r V) - \{ x_\alpha \} \).

For each \( x \in R \), applying the maximality principle, one can find two ultrafilters \( \mathcal{U}_a, \mathcal{U}_b \) in \( \mathcal{F} \) such that

(iii) \( \mathcal{U}_a \ni \{ x_\alpha : \alpha < \beta < 2^n \} \) for all \( \alpha \in R \);
(iv) if \( F \in \mathcal{U}_a \cup \mathcal{U}_b \), then \( x \in cl_r F \) for all \( x \in R \);
(v) if \( x = x_\alpha \) for some \( \alpha < 2^n \), and if \( b_\alpha = \langle U, V \rangle \), then either \( U \in \mathcal{U}_a \) and \( V \in \mathcal{U}_b \) or \( U \in \mathcal{U}_b \) and \( V \in \mathcal{U}_a \).

If \( x \neq x_\alpha \) for all \( \alpha < 2^n \), an arbitrary ultrafilter in \( \mathcal{F} \) which contains all open
8. PROPOSITION. Let \( \langle X, \mathcal{G} \rangle \) be a maximal connected space, let \( x \in X \) and let \( C \) be a component of \( X - \{x\} \). Then \( x \notin cl_{\mathcal{G}} C \) if and only if \( C \notin \mathcal{G} \).

9. PROPOSITION. Let \( \langle X, \mathcal{G} \rangle \) be a maximal connected space and let \( x \in cl_{\mathcal{G}} C \) whenever \( x \in X \) and \( C \) is a component of \( X - \{x\} \). Then \( \mathcal{G} \) is the connected \( \mathcal{G} \)-extremal topology.

10. PROBLEMS. Let \( \langle X, \mathcal{G} \rangle \) be a connected (Hausdorff) space, let \( x \in cl_{\mathcal{G}} C \) whenever \( x \in X \) and \( C \) is a component of \( X - \{x\} \). Is it true that then there exists a topology \( \mathcal{H} = \mathcal{G} \) such that \( \langle X, \mathcal{H} \rangle \) satisfies the assumptions of Theorem 5?

What can be said about those spaces which contain a couple \( x, C \) (\( C \) a component of \( X - \{x\} \)) with \( x \notin cl_{\mathcal{G}} C \)?

11. Remarks. A) S. K. Hildebrand [8], Theorem 4.2] has proved that if \( \mathcal{G} \) is a finer connected topology for a unit interval \( I \) than \( \mathcal{G} \), then any connected subset of \( I \) will remain connected under \( \mathcal{G} \). This result together with Theorem 7 shows that the conjecture "there is no maximal topology for \( I \) having the same connected subsets as \( \mathcal{G} \)" (Hammer-Singletary, [7]) is false.

B) Consider the case of \( R^2 \) with its usual euclidean topology \( \mathcal{F} \). Every \( \mathcal{F} \)-extremal topology on \( R^2 \) is disconnected. In order to prove it, let \( \mathcal{G} \) be any \( \mathcal{F} \)-extremal topology on \( R^2 \). If \( \mathcal{G} \subseteq \mathcal{F} \), and we denote by

\[
H^+ = \{ z \in R^2 | z = (x, y), y > 0 \}, \quad H^- = \{ z \in R^2 | z = (x, y), y < 0 \}
\]

and if we define

\[
V = \{ z \in R^2 | H^+ \cup \{ z \} \in \mathcal{G} \}, \quad W = \{ z \in R^2 | H^- \cup \{ z \} \in \mathcal{G} \},
\]

then \( V \cup W = R^2 \), \( V \cap W = \emptyset \), \( V \neq \emptyset \neq W \) and both \( V \) and \( W \) belong to \( \mathcal{F} \).

Nevertheless, let \( \mathcal{G} \) be the following topology on \( R^2 \) (where \( \langle R^2, \mathcal{G} \rangle \) is considered as a complex plane): If \( z = 0 \), its \( \mathcal{G} \)-neighborhood system will be the same as in \( \mathcal{F} \). If \( z \neq 0 \), then for some real \( \varphi \) and positive real \( r \), \( z = re^{i\varphi} \). The \( \mathcal{G} \)-neighborhood base of \( z \) will consist of all sets \( \{ y | y = re^{i\varphi}, s \leq r, t \leq r \} \) for \( s, t \) satisfying the inequality \( 0 < s, r < t \). Obviously \( \mathcal{G} \subseteq \mathcal{F} \) and the space \( \langle R^2, \mathcal{F} \rangle \) satisfies all the assumptions of Theorem 5. (The proof of the existence of the connected \( \mathcal{F} \)-extremal topology is analogous to the proof of Lemma 6.)

C) The very technical proof of Lemma 6 is, perhaps, necessary. It is not true that every \( \mathcal{F} \)-extremal topology \( \mathcal{G} \) on \( R \) is connected: Let us sketch an example.

Construct the Cantor discontinuum \( D \) by the routine inductive procedure:

\[
D_0 = I, \\
D_1 = I - \{a, b\}, \\
D_2 = D_1 - \{\{a, b\}, \{a, b\} \cup \{a, b\} \}, \\
D_3 = D_2 - \{\{a, b, c\}, \{a, b, c\} \cup \{a, b, c\} \}, \\
D = \bigcap \{D_i | i < \omega\}.
\]

Let \( A_1 = D_{i} - D_{i+1} \) for \( 1 \leq i < \omega \), \( A_0 = \{a, b\} \cup \{a, b\} \), let

\[
\mathcal{F} = \{ \bigcup A_i | i < \omega \}, \quad \mathcal{Q} = \{ \bigcup A_{i+1} | i < \omega \}
\]

and denote \( L = \{ x \in R | x \notin \bigcup \{ A_i | i < \omega \} \} \). Then \( L \cap \{ x \in R | x \notin \bigcup \{ A_{i+1} | i < \omega \} \} \) is a \( \mathcal{F} \)-neighborhood of \( x \) iff there is some real \( r > 0 \) such that

\[
\{ x | x \notin \{a, b, c\} \} \cup \{ x \notin \{a, b, c\} \} \subseteq U.
\]

The topology \( \mathcal{F} \) is obviously contained in some \( \mathcal{F} \)-extremal topology \( \mathcal{G} \), and the topology \( \mathcal{F} \) is disconnected: \( P \cup L \) and \( Q \cup M \) are disjoint \( \mathcal{F} \)-open sets which cover \( R \).

References