Some combinatorics involving ultrafilters

by

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Abstract. This paper briefly discusses the following property of ultrafilters: if μ and λ are cardinals, an ultrafilter U is (μ, λ)-*cohesive* iff given μ sets in U there are λ of them whose intersection is in U. Among other things, it is shown that a p-point over ω is (ω_1, ω)-cohesive, but that this property does not characterize p-points. We can in fact prove the following: if U is a p-point over ω and $\{X_\alpha | \alpha<\omega_1\}\subseteq U$, then for any $\delta<\omega_1$ there is an $S\subseteq\omega_1$ of order type δ so that $\bigcap\{X_\alpha | \alpha \in S\} \in U$. A polarized partition relation is strengthened using this fact. These results have direct generalizations to measurable cardinals, and indeed, the paper is written in this general context.

§ 0. Introduction. In this paper, the following rather general combinatorial property of ultrafilters is considered, mainly in connection with ω and measurable cardinals.

DEFINITION. If μ and λ are cardinals, an ultrafilter $\mathscr{U}$ is (μ, λ)-*cohesive* iff given μ sets in $\mathscr{U}$, there are λ of them whose intersection is still in $\mathscr{U}$.

For those familiar with regularity of ultrafilters, notice that (μ, λ)-cohesion is a strong negation of (μ, λ)-regularity. It is shown that if $\mathscr{U}$ is a p-point in the space βN, then $\mathscr{U}$ is (ω_1, ω)-cohesive. The analogous result for measurable cardinals holds by the same proof. Product ultrafilters are considered in this context, and the situations under various set theoretical hypotheses are also discussed. Finally, a new proof and strengthening of a polarized partition relation is derived.

My set theory is ZFC, and the notation is standard, but I do mention the following: $\alpha, \beta, \gamma, \ldots$ denote ordinals, but $\varkappa$, λ, and μ are reserved for cardinals. If x is a set, $|x|$ denotes its cardinality and $\mathscr{P}x$ its power set; if y is also a set, ${}^{x}y$ denotes the set of functions from x into y; finally, if n is an integer, $[x]^n$ denotes the collection of n-element subsets of x. If z is a set of ordinals, $\bar{z}$ denotes its order type. An ultrafilter over a set I is actually one in the Boolean algebra $\mathscr{P}I$, and is uniform if each of its members has cardinality $|I|$. Finally, ■ indicates the end of a proof.

While working on this paper the author has profited from discussions with Mathias and Prikry.

§ 1. Preliminaries and P-points. Under the GCH, we can make some intial deductions from the following classical result of Sierpiński (which I state in a slightly weakened version relevant to our purposes) — see P_4 of [Si].

1.1. PROPOSITION (Sierpiński). *If* $2^\lambda = \lambda^+$, *there are functions* $f_\alpha\colon \lambda^+\to 2$ *for* $\alpha<\lambda^+$ *so that: whenever* $X\subseteq\lambda^+$ *and* $|X| = \lambda^+$,

$$|\{\alpha<\lambda^+|\ f_\alpha''X\neq 2\}|<\lambda .$$

Prikry showed that the first part of the next theorem follows from Sierpiński's result, and the second is easy enough to see.

1.2. THEOREM. *Suppose* $2^\lambda = \lambda^+$.

(i) (Prikry) *If* $\mathscr{U}$ *is a uniform ultrafilter over* λ^+, *then* $\mathscr{U}$ *is not* (λ^+, λ)*-cohesive.*

(ii) *If* $\mathscr{V}$ *is a uniform ultrafilter over* λ, *then* $\mathscr{V}$ *is not* (λ^+, λ^+)*-cohesive.*

Proof. Let $\{f_\alpha|\ \alpha<\lambda^+\}$ be as in 1.1. For (i), set $A_\alpha^k = \{\xi|\ f_\alpha(\xi) = k\}$ for $k<2$ and $\alpha<\lambda^+$. Suppose that $k<2$ and $S\subseteq\lambda^+$ with $|S| = \lambda$. If $X = \bigcap\{A_\alpha^k|\ \alpha\in S\}$, then $\{\alpha|\ f_\alpha''X = \{k\}\}\supseteq S$. Hence $|X|\leqslant\lambda$ and $X\notin\mathscr{U}$ as $\mathscr{U}$ is uniform. The result now follows, since there must be a $k<2$ for which there are $\lambda^+\alpha$'s so that $A_\alpha^k\in\mathscr{U}$.

To show (ii), set $B_\xi^k = \{\alpha<\lambda|\ f_\alpha(\xi) = k\}$ for $k<2$ and $\xi<\lambda^+$. Suppose that $k<2$ and $Y\subseteq\lambda^+$ with $|Y| = \lambda^+$. Then

$$T = \bigcap\{B_\xi^k|\ \xi\in Y\}\subseteq\{\alpha<\lambda|\ f_\alpha''Y = \{k\}\},$$

and hence $|T|<\lambda$, i.e. $T\notin\mathscr{V}$ as $\mathscr{V}$ is uniform. The result now follows as for (i). ∎

Having made these initial remarks, I now turn to my main concern, the consideration of measurable cardinals $\varkappa$ and the non-trivial cases involving $\varkappa^+$ and $\varkappa$.

1.3. DEFINITION.

(i) $\mathscr{U}$ is a *$\varkappa$-ultrafilter* iff $\mathscr{U}$ is a non-principal, $\varkappa$-complete ultrafilter over $\varkappa$.

(ii) $\varkappa$ is a *measurable cardinal* iff there is a $\varkappa$-ultrafilter.

(iii) $\varkappa$ is *$\varkappa$-compact* iff every $\varkappa$-complete filter over $\varkappa$ can be extended to a $\varkappa$-complete ultrafilter over $\varkappa$.

The non-principal ultrafilters over ω are precisely the ω-ultrafilters, and thus, in this paper I regard ω to be both measurable and ω-compact. Note that $\varkappa$-compactness is just a restricted version of the usual concept of strong compactness, and that it obviously implies the measurability of $\varkappa$. The following is another observation of a negative kind.

1.4. PROPOSITION. *If* $\varkappa$ *is* $\varkappa$*-compact, there is a* $\varkappa$*-ultrafilter which is not* $(2^\varkappa, \varkappa)$*-cohesive.*

Proof. Let $\mathscr{S}\subseteq\mathscr{P}_\varkappa$ be a family of $2^\varkappa$ $\varkappa$-independent sets (see Kunen [Ku 3] for details; the existence of such a family only depends on the fact that $\varkappa^{<\varkappa} = \varkappa$); that is, given any $\mathscr{A}, \mathscr{B}\subseteq\mathscr{S}$ so that $\mathscr{A}\cap\mathscr{B} = \varnothing$ and $|\mathscr{A}|, |\mathscr{B}|<\varkappa$,

$$|\bigcap\mathscr{A}\cap\bigcap\{\varkappa-X|\ X\in\mathscr{B}\}| = \varkappa .$$

Then

$$\mathscr{S}\cup\{\varkappa-\bigcap\mathscr{T}|\ \mathscr{T}\subseteq\mathscr{S} \text{ and } |\mathscr{T}| = \varkappa\}$$

generates a uniform $\varkappa$-complete filter:

Let $\mathscr{A}\subseteq\mathscr{S}$ with $|\mathscr{A}|<\varkappa$, and suppose also that $\lambda<\varkappa$ and for $\alpha<\lambda$, $\mathscr{T}_\alpha\subseteq\mathscr{S}$ are such that $|\mathscr{T}_\alpha| = \varkappa$. It must be shown that

$$(*)\qquad |\bigcap\mathscr{A}\cap\bigcap\{\varkappa-\bigcap\mathscr{T}_\alpha|\ \alpha<\lambda\}| = \varkappa .$$

By the cardinality assumptions, we can inductively choose $X_\alpha\in\mathscr{T}_\alpha-\mathscr{A}$ so that $\alpha<\beta<\lambda$ implies that $X_\alpha\neq X_\beta$. Note that for each $\alpha<\lambda$, $\varkappa-\bigcap\mathscr{T}_\alpha\supseteq\varkappa-X_\alpha$. Thus,

$$\bigcap\mathscr{A}\cap\bigcap\{\varkappa-\bigcap\mathscr{T}_\alpha|\ \alpha<\lambda\}\supseteq\bigcap\mathscr{A}\cap\bigcap\{\varkappa-X_\alpha|\ \alpha<\lambda\},$$

and the set on the right has cardinality $\varkappa$, as $\mathscr{S}$ is independent. Hence, we get $(*)$.

Now by $\varkappa$-compactness of $\varkappa$, let $\mathscr{U}$ be any $\varkappa$-ultrafilter extending the above filter. Clearly the family $\mathscr{S}\subseteq\mathscr{U}$ is a counterexample to the $(2^\varkappa, \varkappa)$-cohesion of $\mathscr{U}$, and we are done. (This example for $\varkappa = \omega$ is just Kunen's example (see 2.8 of [Ku 3]) of an ω-ultrafilter of character 2^ω, i.e. one not generated by less that 2^ω sets of integers.) ∎

Thus, we see that the negative results 1.2 and 1.4 can be culled from previous set theoretical experience; I now turn to the positive results. The following concepts first arose in the study of βN, the Stone–Čech compactification of the integers, which is identifiable with the set of ultrafilters over ω.

1.5. DEFINITIONS.

(i) The *Rudin–Keisler ordering* (RK) on ultrafilters is defined as follows: If $\mathscr{U}$ is an ultrafilter over a set I and $\mathscr{V}$ over J, $\mathscr{V}\leqslant_{RK}\mathscr{U}$ iff there is a function $f\colon I\to J$ so that $\mathscr{V} = f_*(\mathscr{U})$, where

$$f_*(\mathscr{U}) = \{X\subseteq J|\ f^{-1}(X)\in\mathscr{U}\} .$$

If $\mathscr{V}\leqslant_{RK}\mathscr{U}$, then $\mathscr{V}\approx_{RK}\mathscr{U}$ ($\mathscr{V}$ and $\mathscr{U}$ are *isomorphic*) iff $\mathscr{U}\leqslant_{RK}\mathscr{V}$, and $\mathscr{V}<_{RK}\mathscr{U}$ iff $\mathscr{U}\nleqslant_{RK}\mathscr{V}$.

(ii) A $\varkappa$-ultrafilter $\mathscr{U}$ is *minimal* iff it is minimal in the RK ordering, i.e. there is no (non-principal) ultrafilter $\mathscr{V}<_{RK}\mathscr{U}$.

(iii) A $\varkappa$-ultrafilter $\mathscr{U}$ is a *p-point* iff whenever $\{X_\alpha|\ \alpha<\varkappa\}\subseteq\mathscr{U}$, there is a $Y\in\mathscr{U}$ so that $|Y-X_\alpha|<\varkappa$ for each $\alpha<\varkappa$.

See the reference work Comfort–Negrepontis ([CN], especially § 16) for details on these concepts and the general development of the theory of βN. For an analogous development of the theory of $\varkappa$-ultrafiters for $\varkappa>\omega$ with attention to distinctive features and new factors, see Kanamori [Ka]. In the present context, it is not hard to show that if $\mathscr{U}$ is (μ, λ)-cohesive and $\mathscr{V}\leqslant_{RK}\mathscr{U}$, then $\mathscr{V}$ is (μ, λ)-cohesive. For future reference, I collect some known characterizations in the next proposition.

1.6. PROPOSITION.

(i) *The following are equivalent for a* $\varkappa$*-ultrafilter* $\mathscr{U}$:

(a) $\mathscr{U}$ *is minimal.*

(b) $\mathscr{U}$ *is Ramsey: for any* $n<\omega$ *and* $\lambda<\varkappa$, *if* $f\colon [\varkappa]^n\to\lambda$, *there is an* $X\in\mathscr{U}$ *so that* $|f''[X]^n| = 1$.

(c) *$\mathcal{U}$ is selective: if $f \in {}^{\varkappa}\varkappa$ so that $f^{-1}(\{\alpha\}) \notin \mathcal{U}$ for each α, then there is an $X \in \mathcal{U}$ so that $|X \cap f^{-1}(\{\alpha\})| \leqslant 1$ for each α.*

When $\varkappa > \omega$, we can also add:

(d) *There is a normal $\varkappa$-ultrafilter $\mathcal{N}$ so that $\mathcal{N} \approx_{RK} \mathcal{U}$.*

(ii) *The following are equivalent for a $\varkappa$-ultrafilter $\mathcal{U}$:*

(a) *$\mathcal{U}$ is a p-point.*

(b) *$\mathcal{U}$ is almost selective: if $f \in {}^{\varkappa}\varkappa$ so that $f^{-1}(\{\alpha\}) \notin \mathcal{U}$ for each α, then f is almost* 1-1 (mod$\mathcal{U}$), *i.e. there is an $X \in \mathcal{U}$ so that $|X \cap f^{-1}(\{\alpha\})| < \varkappa$ for each α.*

Hence, minimal $\varkappa$-ultrafilters are always p-points. When $\varkappa = \omega$, the converse is not true under CH or Martin's Axiom (MA), but it is not even known whether p-points exist if we do not assume either of these hypotheses. However, Kunen [Ku 1] has shown that there is a model of ZFC without any minimal ω-ultrafilters. When $\varkappa > \omega$, minimal $\varkappa$-ultrafilters always exist (Scott), but it is consistent that all p-points are minimal, and, in fact, all RK-isomorphic to each other (Kunen — see after 2.2 below). Non-minimal p-point $\varkappa$-ultrafilters exist if $\varkappa$ is measurable and a limit of measurable cardinals, but it is still open whether such $\varkappa$-ultrafilters exist when $\varkappa$ is $\varkappa$-compact.

I now proceed to show that if $\mathcal{U}$ is a p-point $\varkappa$-ultrafilter, then $\mathcal{U}$ is $(\varkappa^+, \varkappa)$-cohesive, and, toward this goal, provide a new characterization of p-points which may be of independent interest.

1.7. Definition. If $\mathcal{D}$ is an ultrafilter over some cardinal λ, $\mathcal{D}$ is *coherent* iff whenever $X \in \mathcal{D}$ and $\mathcal{A} \subseteq \mathcal{D}$ so that for each $\alpha < \lambda$,

$$|\{A \in \mathcal{A} | \; X \cap \alpha = A \cap \alpha\}| \geqslant \lambda,$$

then there is a $\mathcal{B} \subseteq \mathcal{A}$ so that $|\mathcal{B}| = \lambda$ and $\bigcap \mathcal{B} \in \mathcal{D}$.

If $\mathcal{D}$ and $\mathcal{E}$ are ultrafilters over λ so that $\mathcal{E} \leqslant_{RK} \mathcal{D}$, then if $\mathcal{D}$ is coherent, so is $\mathcal{E}$. Note that coherence makes sense for an ultrafilter $\mathcal{U}$ over an arbitrary set I, by considering some $\varphi\colon I \leftrightarrow |I|$ and formulating the property for $\varphi_*(\mathcal{U})$ instead.

1.8. Proposition. *If $2^{<\lambda} = \lambda$ and $\mathcal{A} \subseteq \mathcal{P}\lambda$ with $|\mathcal{A}| > \lambda$, then there is an $X \in \mathcal{A}$ so that $|\{A \in \mathcal{A} | \; A \cap \alpha = X \cap \alpha\}| = |\mathcal{A}|$ for every $\alpha < \lambda$.*

Proof. Argue by contradiction, and assume that for each $X \in \mathcal{A}$, there is an $\alpha_X < \lambda$ so that $|\{A \in \mathcal{A} | \; A \cap \alpha_X = X \cap \alpha_X\}| < |\mathcal{A}|$. Surely, there is a $\beta < \lambda$ and a $\mathcal{A}_1 \subseteq \mathcal{A}$ with $|\mathcal{A}_1| = |\mathcal{A}|$ so that $X \in \mathcal{A}_1$ implies $\alpha_X = \beta$. But as $2^\beta < |\mathcal{A}|$, there is an $\mathcal{A}_2 \subseteq \mathcal{A}_1$ with $|\mathcal{A}_2| = |\mathcal{A}|$ so that $X, Y \in \mathcal{A}_2$ imply $X \cap \beta = Y \cap \beta$. This is a contradiction. ∎

The following is now immediate from the definitions and 1.8:

1.9. Corollary. *If $2^{<\lambda} = \lambda$ and $\mathcal{D}$ over λ is coherent, then it is (λ^+, λ)-cohesive.*

With these preliminaries, I now prove the main result. The $(\varkappa^+, \varkappa)$-cohesion of normal $\varkappa$-ultrafilters for $\varkappa > \omega$ was first proved by Solovay.

1.10. Theorem. *If $\mathcal{U}$ is a $\varkappa$-ultrafilter, then $\mathcal{U}$ is a p-point iff $\mathcal{U}$ is coherent.*

Thus, p-point $\varkappa$-ultrafilters are $(\varkappa^+, \varkappa)$-cohesive, and in particular, p-points in βN are (ω_1, ω)-cohesive.

Proof. Suppose first that $\mathcal{U}$ is coherent and $\{X_\xi | \; \xi < \varkappa\} \subseteq \mathcal{U}$. We must find a $Y \in \mathcal{U}$ so that $|Y - X_\xi| < \varkappa$ for each $\xi < \varkappa$. By taking successive intersections, we can assume henceforth that $\xi < \zeta < \varkappa$ implies $X_\zeta \subseteq X_\xi$.

Set $Y_\xi = X_\xi \cup \xi$ for $\xi < \varkappa$. Then for each $\alpha < \varkappa$,

$$|\{\xi < \varkappa | \; Y_\xi \cap \alpha = \alpha\}| = \varkappa$$

and so by coherence, there is a $T \subseteq \varkappa$ with $|T| = \varkappa$ so that $Y = \bigcap \{Y_\xi | \; \xi \in T\} \in \mathcal{U}$. Now given any $\gamma < \varkappa$, let $\delta \geqslant \gamma$ so that $\delta \in T$. By the definition of the Y_ξ's and the fact that the X_ξ's were descending, we have $|Y_\delta - Y_\gamma| < \varkappa$. Hence, $|Y - Y_\gamma| < \varkappa$ and the result follows.

Conversely, suppose that $\mathcal{U}$ is a p-point, and $X \in \mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{U}$ with $|\mathcal{A}| = \varkappa$ so that for each $\alpha < \varkappa$,

$$(*) \qquad |\{A \in \mathcal{A} | \; X \cap \alpha = A \cap \alpha\}| = \varkappa.$$

We must establish the existence of a $\mathcal{B} \subseteq \mathcal{A}$ so that $|\mathcal{B}| = \varkappa$ and $\bigcap \mathcal{B} \in \mathcal{U}$.

Since $\mathcal{U}$ is a p-point, there is a $Y \in \mathcal{U}$ so that $|Y - A| < \varkappa$ for every $A \in \mathcal{A}$. For each $A \in \mathcal{A}$, with $A \neq X$, let I_A be the half-open interval of ordinals $[\gamma_A, \delta_A)$, where

$$\gamma_A = \bigcup \{\alpha | \; X \cap \alpha = A \cap \alpha\},$$

and

$$\delta_A = \text{least } \delta \geqslant \gamma_A \text{ so that } Y - \delta \subseteq A.$$

Notice that I_A may be empty; in any case, $|I_A| < \varkappa$.

By $(*)$ for every $\varrho < \varkappa$, there is an $A \in \mathcal{A}$ so that $\varrho < \gamma_A$. Hence, by induction we can choose an $\mathcal{A}' \subseteq \mathcal{A}$ so that $|\mathcal{A}'| = \varkappa$ and if $A, B \in \mathcal{A}'$ with $A \neq B$, then $I_A \cap I_B = \emptyset$. Now we can find some $\mathcal{B} \subseteq \mathcal{A}'$ so that $|\mathcal{B}| = \varkappa$ and

$$Z = \bigcup \{I_A | \; A \in \mathcal{B}\} \notin \mathcal{U}.$$

Thus, $X \cap Y \cap (\varkappa - Z) \in \mathcal{U}$.

Suppose now that $\beta \in X \cap Y \cap (\varkappa - Z)$, and $A \in \mathcal{B}$. As $\beta \notin I_A$, either $\beta < \gamma_A$ or $\delta_A \leqslant \beta$. If $\beta < \gamma_A$, then $\beta \in X$ implies $\beta \in A$ by the definition of γ_A. If $\delta_A \leqslant \beta$, then $\beta \in Y$ implies $\beta \in A$ by the definition of δ_A. Hence, in either case, $\beta \in A$. We have thus shown that $X \cap Y \cap (\varkappa - Z) \subseteq \bigcap \mathcal{B}$. This establishes that $\bigcap \mathcal{B} \in \mathcal{U}$, and the proof is complete. ∎

In § 2, it is shown that $(\varkappa^+, \varkappa)$-cohesion does not characterize p-points, and in § 3, a refinement of the argument for 1.10 is given.

§ 2. Product ultrafilters.

Let us first recall some further definitions.

2.1. Definitions. Let $\mathcal{D}$ be an ultrafilter over I, and $\mathcal{E}_i$ ultrafilters over J for $i \in I$.

(i) The $\mathscr{D}$-sum of $\langle \mathscr{E}_i | \ i \in I \rangle$ is the ultrafilter $\mathscr{D} \sum \mathscr{E}_i$ over $I \times J$ defined by

$$X \in \mathscr{D} \sum \mathscr{E}_i \quad \text{iff} \quad \{i | \ \{j | \ \langle i, j\rangle \in X\} \in \mathscr{E}_i\} \in \mathscr{D}.$$

(ii) When each $\mathscr{E}_i = a$ fixed $\mathscr{E}$ in (i), we get the *product* of $\mathscr{D}$ and $\mathscr{E}$, denoted $\mathscr{D} \times \mathscr{E}$. For $0 < n < \omega$, $\mathscr{U}^n$ is defined by induction: $\mathscr{U}^1 = \mathscr{U}$ and $\mathscr{U}^{n+1} = \mathscr{U} \times \mathscr{U}^n$.

Notice that if $\mathscr{D}$ and $\mathscr{E}_\alpha$ for $\alpha < \varkappa$ are all $\varkappa$-ultrafilters, then $\mathscr{U} = \mathscr{D} \sum \mathscr{E}_\alpha$ is RK-isomorphic to a $\varkappa$-ultrafilter, but not a p-point, since $\pi \colon \varkappa \times \varkappa \to \varkappa$, the projection onto the first coordinate, cannot be almost 1-1 (mod$\mathscr{U}$). The next propositions show that cohesion is preserved under the taking of sums and products of $\varkappa$-ultrafilters under suitable conditions, and thus, that this concept does not characterize p-points.

2.2. Proposition. *Suppose $\mathscr{U}$ is a minimal $\varkappa$-ultrafilter. If $\mathscr{U}$ is (μ, λ)-cohesive, then $\mathscr{U}^n$ is (μ, λ)-cohesive for each $n < \omega$. Hence, each $\mathscr{U}^n$ is always $(\varkappa^+, \varkappa)$-cohesive.*

Proof. Let $\Delta_n = \{\langle \alpha_1, \alpha_2, \ldots, \alpha_n\rangle | \ \alpha_1 < \alpha_2 < \ldots < \alpha_n < \varkappa\}$. It is not hard to establish the following characterization of minimal $\varkappa$-ultrafilters, using the Ramsey condition:

A $\varkappa$-ultrafilter $\mathscr{D}$ is minimal iff for any n, $\{X^n | \ X \in \mathscr{D}\} \cup \{\Delta_n\}$ generates $\mathscr{D}^n$, i.e. for any $A \in \mathscr{D}^n$, there is an $X \in \mathscr{D}$ such that $X^n \cap \Delta_n \subseteq A$.

Hence, that $\mathscr{U}$ is (μ, λ)-cohesive certainly implies that $\mathscr{U}^n$ is (μ, λ)-cohesive for each $n < \omega$. An appeal to 1.9 now yields the full conclusion of the proposition. ■

Kunen [Ku2] showed that in $L[\mathscr{U}]$, the inner model constructed from a normal $\varkappa$-ultrafilter over $\varkappa > \omega$, each $\varkappa$-ultrafilter is RK-isomorphic to $(\mathscr{U} \cap L[\mathscr{U}])^n$ for some $n < \omega$. Hence, 2.2 immediately shows that if it is consistent that there is a measurable cardinal $\varkappa > \omega$, then it is consistent that such a cardinal $\varkappa$ exists and every $\varkappa$-ultrafilter is $(\varkappa^+, \varkappa)$-cohesive. Thus, 1.4 is yet another way of showing that $\varkappa$ cannot be $\varkappa$-compact in $L[\mathscr{U}]$.

The proof of the following result does not generalize for $\varkappa > \omega$.

2.3. Proposition. *Suppose that $\mathscr{U}$ and $\mathscr{V}_n$ for $n < \omega$ are all (ω_1, ω_1)-cohesive ω-ultrafilters. Then $\mathscr{U} \sum \mathscr{V}_n$ is (ω_1, ω)-cohesive.*

Proof. For any $S \subseteq \omega \times \omega$ and $n < \omega$, set $(S)_n = \{i | \ \langle n, i\rangle \in S\}$ for the purposes of this proof. Also, if $S \in \mathscr{U} \sum \mathscr{V}_n$, let $S^* = \{n | \ (S)_n \in \mathscr{V}_n\}$. Thus, $S^* \in \mathscr{U}$.

Now let $\mathscr{A} \subseteq \mathscr{U} \sum \mathscr{V}_n$ with $|\mathscr{A}| = \omega_1$. By the (ω_1, ω_1)-cohesion of $\mathscr{U}$, there is an $\mathscr{A}' \subseteq \mathscr{A}$ so that $|\mathscr{A}'| = \omega_1$ and $K = \bigcap \{A^* | \ A \in \mathscr{A}'\} \in \mathscr{U}$.

By induction on the ascending enumeration of K, we can define $\mathscr{B}_n \subseteq \mathscr{A}'$ for $n \in K$ with the following properties:

(a) $m < n$ implies $\mathscr{B}_n \subseteq \mathscr{B}_m$,

(b) $R_n = \bigcap \{(A)_n | \ A \in \mathscr{B}_n\} \in \mathscr{V}_n$, and

(c) $|\mathscr{B}_n| = \omega_1$.

Choose $S_n \in \mathscr{B}_n$ for $n \in K$ so that $m < n$ and $m, n \in K$ imply $S_m \neq S_n$. For each $n \in K$, we have

$$T_n = R_n \cap \bigcap \{(S_m)_n | \ m < n \text{ and } m \in K\} \in \mathscr{U}_n.$$

Hence, by the construction,

$$\bigcup_{n \in K} \{n\} \times T_n \subseteq \bigcap \{S_n | \ n \in K\} \in \mathscr{U} \sum \mathscr{V}_n.$$

The proof is complete. ■

We know from 1.2(ii) that CH implies that no ω-ultrafilter is (ω_1, ω_1)-cohesive. However, the previous proposition is not vacuous under Martin's Axiom. Booth [Bo] showed that MA implies the existence of minimal ω-ultrafilters $\mathscr{U}$ with the following property: for any $\mu < 2^\omega$ and $\mathscr{A} \subseteq \mathscr{U}$ so that $|\mathscr{A}| = \mu$, there is a $Y \in \mathscr{U}$ so that $|Y - X| < \omega$ for every $X \in \mathscr{A}$. Thus, when μ is uncountable, there is a finite set s so that $Y - s$ is contained in μ members of $\mathscr{A}$, and hence, $\mathscr{U}$ is (μ, μ)-cohesive. It is also clear from Booth's work how to get non-minimal p-points under MA which are still (μ, μ)-cohesive for $\omega_1 \leqslant \mu < 2^\omega$. On the other hand, Solomon [So] showed that MA and $2^\omega > \omega_1$ also imply the existence of minimal ω-ultrafilters which are not (ω_1, ω_1)-cohesive.

§ 3. Polarized partition relations. This section is devoted to showing that a refinement of the proof of 1.10 yields a strengthened, ultrafilter related, version of a known polarized partition relation for measurable cardinals. Let us first recall the definitions of the relevant versions of the polarized partition symbol of Erdös and Hajnal, and also specify a modification. Recall that if x is a set of ordinals, $\bar{x}$ denotes its order type.

3.1. Definitions.

(i) The polarized partition symbol

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \to \begin{pmatrix} \gamma \\ \delta \end{pmatrix}^{m,n}_{\lambda}$$

where $m, n < \omega$, denotes the following statement: whenever $F \colon [\alpha]^m \times [\beta]^n \to \lambda$, there are $A \subseteq \alpha$ and $B \subseteq \beta$ so that $\bar{A} = \gamma$ and $\bar{B} = \delta$, and $|F''([A]^m \times [B]^n)| = 1$.

(ii) When "n" in the symbol is replaced by "$<\omega$", we mean the following statement: whenever $F_n \colon [\alpha]^m \times [\beta]^n \to \lambda$ for each $n < \omega$, there are $A \subseteq \alpha$ and $B \subseteq \beta$ so that $\bar{A} = \gamma$ and $\bar{B} = \delta$, and for all $n < \omega$, $|F_n''([A]^m \times [B]^n)| = 1$.

(iii) When "δ" in the symbol (either in context (i) or (ii)) is replaced by "$\in \mathscr{A}$" where $\mathscr{A}$ is a set, we mean that the Y specified is a member of $\mathscr{A}$ (instead of $\bar{Y} = \delta$).

The following result strengthens a known polarized partition relation. The reader is referred to Hajnal [H] and Choodnovsky [Ch] for the previous efforts in this direction. In particular, a question asked in passing in [H] (top of p. 44) is now answered positively.

3.2. Theorem. *Let $\varkappa \geqslant \omega$ be a measurable cardinal.*

(i) *If $\mathscr{U}$ is a p-point $\varkappa$-ultrafilter, then*

$$\begin{pmatrix} \varkappa^+ \\ \varkappa \end{pmatrix} \to \begin{pmatrix} \eta \\ \in \mathscr{U} \end{pmatrix}^{1,1}_{\lambda}$$

for any $\eta<\varkappa^+$ *and* $\lambda<\varkappa$.

(ii) *If* $\mathscr{U}$ *is a minimal* $\varkappa$*-ultrafilter, then*

$$\binom{\varkappa^+}{\varkappa} \to \binom{\eta}{\in \mathscr{U}}^{1,n}_{\lambda}$$

for any $\eta<\varkappa^+$, $\lambda<\varkappa$, *and* $n<\omega$. *When* $\varkappa>\omega$, *the "n" can be replaced by "$<\omega$".*

Proof. If $\mathscr{U}$ is a $\varkappa$-ultrafilter, $\lambda<\varkappa$, and $F\colon \varkappa^+\times\varkappa\to\lambda$, for each $\xi<\varkappa^+$ there is an $X_\xi\in\mathscr{U}$ and a $\beta_\xi<\lambda$ so that $F''(\{\xi\}\times X_\xi)=\beta_\xi$. Also, $\beta_\xi=$ fixed β for $\varkappa^+$ ξ's. Hence, to show (i), it suffices to show the following: If $\mathscr{U}$ is a p-point and $\{X_\xi|\ \xi<\varkappa^+\}\subseteq\mathscr{U}$, then for any $\eta<\varkappa^+$ there is a $B\subseteq\varkappa^+$ with $\bar{B}=\eta$ and $\bigcap\{X_\xi|\ \xi\in B\}\in\mathscr{U}$.

The refinement to get (ii) is just an initial application of the Ramsey property of minimal $\varkappa$-ultrafilters in the above argument, and the final remark in (ii) follows from an application next, for each $\xi<\varkappa^+$, of the countable completeness of $\varkappa$-ultrafilters for $\varkappa>\omega$.

Thus, suppose that $\mathscr{U}$ is a p-point and $\{X_\xi|\ \xi<\varkappa^+\}\subseteq\mathscr{U}$. By Proposition 1.8, there is a $Y\in\mathscr{U}$ so that for each $\alpha<\varkappa$,

$$|\{\xi<\varkappa^+|\ Y\cap\alpha=X_\xi\cap\alpha\}|=\varkappa^+ .$$

We can surely define ordinals $f(\zeta)<\varkappa^+$ for $\zeta<\varkappa^+$ by induction so that the following are satisfied:

(i) f is a normal function, i.e. f is strictly increasing and continuous at limits.

(ii) For any $\zeta<\varkappa^+$ and $\alpha<\varkappa$, $|\{\xi|\ f(\zeta)\leqslant\xi<f(\zeta+1)$ and $Y\cap\alpha=X_\xi\cap\alpha\}|=\varkappa$.

Now fix an $\eta<\varkappa^+$, where, to avoid trivialities, we assume $\varkappa\leqslant\eta$. Since $\mathscr{U}$ is a p-point, there is a $Z\in\mathscr{U}$ so that $|Z-X_\xi|<\varkappa$ for any $\xi<f(\eta+\eta+1)$. Define (possibly empty) intervals I_ξ for $\xi<f(\eta+\eta+1)$ as in the proof of 1.9: $I_\xi=[\gamma_\xi,\delta_\xi)$, where

$$\gamma_\xi=\bigcup\{\alpha|\ Y\cap\alpha=X_\xi\cap\alpha\},$$

and

$$\delta_\xi=\text{least } \delta\geqslant\gamma_\xi \text{ so that } Z-\delta\subseteq X_\xi .$$

Let $\varphi\colon \varkappa\leftrightarrow\eta+\eta$ be a bijection. By induction, we can choose $\xi_\alpha<\varkappa^+$ for $\alpha<\varkappa$ as follows: If ξ_β for $\beta<\alpha$ have been chosen, let ξ_α be such that:

(a) $f(\varphi(\alpha))\leqslant\xi_\alpha<f(\varphi(\alpha)+1)$, and

(b) $I_{\xi_\alpha}\cap I_{\xi_\beta}=\varnothing$ for $\beta<\alpha$.

By the definition of the intervals I_ξ, the condition (b) can always be met because of the property (ii) of the function f.

Clearly, $\{\xi_\alpha|\ \alpha<\varkappa\}$ has order type $\eta+\eta$. By splitting it into two parts each of type η, it is seen that there must be a $B\subseteq\{\xi_\alpha|\ \alpha<\varkappa\}$ so that $\bar{B}=\eta$ and

$$T=\bigcup\{I_\xi|\ \xi\in B\}\notin\mathscr{U} .$$

Hence, like in the proof of 1.10,

$$Y\cap Z\cap(\varkappa-T)\subseteq\bigcap\{X_\xi|\ \xi\in B\},$$

and so since this last set is in $\mathscr{U}$, the proof is complete. ■

3.3. COROLLARY (Galvin for $\varkappa=\omega$, unpublished Choodnovsky [Ch]). *If* $\varkappa\geqslant\omega$ *is measurable, then*

$$\binom{\varkappa^+}{\varkappa} \to \binom{\eta}{\varkappa}^{1,n}_{\lambda}$$

for any $\eta<\varkappa^+$, $\lambda<\varkappa$, *and* $n<\omega$. *When* $\varkappa>\omega$, *the "n" can be replaced by "$<\omega$".*

Proof. For $\varkappa>\omega$, the result is immediate from 3.2, since normal $\varkappa$-ultrafilters always exist. But, as remarked after 1.6 there are models of ZFC without any minimal ω-ultrafilters. However, the following strategem is available:

A minimal ω-ultrafilter $\mathscr{U}$ can always be added to any model of ZFC by an ω-closed notion of forcing. (For example, say that p is a *condition* iff p is a countable collection of infinite subsets of ω with the finite intersection property and that a condition q is stronger than p iff $q\supseteq p$. Notice that if $m,n<\omega$ and $f\colon[\omega]^n\to m$, any condition p can be extended to one which contains a homogeneous set for f: first let $Y\subseteq\omega$ be infinite so that $|Y-X|$ is finite for every $X\in p$, and use Ramsey's theorem to get an infinite homogeneous subset Z of Y for f. Then $p\cup\{Z\}$ is stronger than p.)

Thus, the forcing adds no new countable sequences of ordinals, and ω_1 is preserved as a cardinal. Hence, for any $F\colon\omega_1\times[\omega]^n\to m$ with $m,n<\omega$, 3.2 can be applied in the extension using $\mathscr{U}$, and any resultant "homogeneous" set for F, being countable, must already exist in the ground model. ■

Galvin's proof of 3.3 for $\varkappa=\omega$ apparently did not generalize, and both the proof of Choodnovsky [Ch], and of Hajnal [H] for the weaker statement with η replaced by $\varkappa$, relied on developing a tree and showing that a long branch exists. The present proof yields more information, being a thinning process which works by keeping the needed large sets in an ultrafilter. In the paper Baumgartner and Hajnal [BH] another proof of 3.3 for $\varkappa=\omega$ is outlined which, like the one I give, depends on a forcing and absoluteness argument. But their forcing is one to make MA true in the extension, and hence not ω-closed, and thus a more involved argument was needed to show absoluteness.

3.4. Interestingly enough, when $\varkappa>\omega$ the well-foundedness of ultrapowers can be used to yield a simpler proof of the main assertion of 3.2 (and hence, 1.10):

Let $\varkappa>\omega$, and again, $\mathscr{U}$ a p-point $\varkappa$-ultrafilter with $\{X_\xi|\ \xi<\varkappa^+\}\subseteq\mathscr{U}$. For a fixed η, $\varkappa\leqslant\eta<\varkappa^+$, we want to find a $B\subseteq\varkappa^+$ with $\bar{B}=\eta$ so that $\bigcap\{X_\xi|\ \xi\in B\}\in\mathscr{U}$. Just as before, we can suppose that there is a $Y\in\mathscr{U}$ so that

$$|\{\xi<\varkappa^+|\ Y\cap\alpha=X_\xi\cap\alpha\}|=\varkappa^+$$

for every $\alpha<\varkappa$.

By well-foundedness, let $h\in{}^{\varkappa}\varkappa$ be a "least" non-constant function, i.e. one so that for any $\alpha<\varkappa$, $h^{-1}(\{\alpha\})\notin\mathscr{U}$, but so that if $g\in{}^{\varkappa}\varkappa$ and $\{\xi<\varkappa|\ g(\xi)<h(\xi)\}\in\mathscr{U}$,

then $g^{-1}(\{\beta\}) \in \mathcal{U}$ for some $\beta < \varkappa$. Since $\mathcal{U}$ is a p-point, we can assume that h is almost 1-1, i.e. for each $\alpha < \varkappa$, $|h^{-1}(\{\alpha\})| < \varkappa$.

Let $\tau: \varkappa \leftrightarrow \eta$ be a bijection. We can define ordinals $f(\zeta) < \varkappa^+$ for $\zeta < \eta$ by induction so that the following are satisfied:

(i) f is strictly increasing.

(ii) $Y \cap (\alpha+1) = X_{f(\zeta)} \cap (\alpha+1)$ for any α so that $h(\alpha) \leqslant \tau^{-1}(\zeta)$. (Recall h is almost 1-1.)

It now suffices to show that $T = \bigcap \{X_{f(\zeta)} | \zeta < \eta\} \in \mathcal{U}$. If not, $Z = Y \cap (\varkappa - T) \in \mathcal{U}$. On Z we can then define a function g by

$$g(\alpha) = \tau^{-1} \text{ of the least } \zeta \text{ so that } \alpha \notin X_{f(\zeta)}.$$

If $\alpha \in Z$ and $h(\alpha) \leqslant g(\alpha)$, then $Y \cap (\alpha+1) = X_{f(\zeta)} \cap (\alpha+1)$ where $\tau^{-1}(\zeta) = g(\alpha)$. But $\alpha \in Y$ so that $\alpha \in X_{f(\zeta)}$, contradicting the definition of g. Hence, $\alpha \in Z$ implies $g(\alpha) < h(\alpha)$, and thus $g^{-1}(\{\gamma\}) \in \mathcal{U}$ for some $\gamma < \varkappa$. But this set is disjoint from $X_{f(\tau(\gamma))} \in \mathcal{U}$, an evident contradiction. Thus, this proof is complete. ■

This argument enables us to make the following observation about closed unbounded sets.

3.5. Proposition. *Suppose $\lambda^{<\lambda} = \lambda$ and C_α for $\alpha < \lambda^+$ are closed unbounded subsets of λ. Then for any $\eta < \lambda^+$, there is a $B \subset \lambda^+$ with $\bar{B} = \eta$ so that $\bigcap \{C_\alpha | \alpha \in B\}$ is still closed unbounded in λ.*

Proof. Mimic the argument of 3.4 with the identity function: $\lambda \to \lambda$ in the role of h, and use the normality of the ideal of non-stationary subsets of λ at the appropriate places. ■

§ 4. Open questions. I conclude the paper with two typical open questions.

4.1. Question. Is it provable in ZFC alone that there is a (ω_1, ω)-cohesive ω-ultrafilter?

4.2. Question. Is it consistent that there is a $\varkappa > \omega$ and a $\varkappa$-ultrafilter $\mathcal{U}$ which is $(\varkappa^+, \varkappa^+)$-cohesive? $2^\varkappa > \varkappa^+$. If there were such a $\varkappa$-ultrafilter, then by 1.2(ii), Silver first showed that the consistency of the existence of a measurable cardinal $\varkappa > \omega$ so that $2^\varkappa > \varkappa^+$ follows from a large cardinal assumption (2-extendibility).

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