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Note on decompositions of metrizable spaces II

by

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Abstract. This paper is a continuation of the author's paper [16]. We improve some results from [16] and investigate the special decompositions of metrizable spaces introduced in [16] which establish close relations between A. H. Stone's [20] property $\sigma\mathrm{Lw}(<\mathfrak{t})$ and stationary sets of ordinals. On this ground we construct decompositions of Baire spaces $B(\mathfrak{t})$ which yield results on absolutely $\mathfrak{t}$-analytic spaces (considered by A. H. Stone [19]) and give, under an additional set theoretic axiom, the negative answer to a question raised in [16]. Connections between these topics and non-separable theory of Borel sets are also investigated.

This paper is a continuation of our paper [16]. In the first section we prove a theorem on σ-discrete reduction which improves a result from [16] and a proposition on completely additive-Borel families which extends an important R. W. Hansell's theorem [9]; these results together give a reduction theorem in non-separable theory of Borel sets which yields a selection theorem.

In the second section we investigate the special decompositions of metrizable spaces introduced in [16] (we call them "natural") which allow to establish close relations between $\sigma\mathrm{Lw}(<\mathfrak{t})$ property (considered by A. H. Stone [20]) and the notion of stationary sets of ordinals and we consider the class of mappings preserving σ-discretness which is closely related to these topics.

In the third section we apply some of results of Section 2 to obtain special decompositions of $B(\mathfrak{t})$ (i.e. the countable product of discrete spaces of cardinality $\mathfrak{t}$) which generalize the classical F. Bernstein's decompositions of irrationals $B(\aleph_0)$ into totally imperfect sets. These decompositions yield a theorem on absolutely $\mathfrak{t}$-analytic spaces (introduced by A. H. Stone [19]) and, under an additional set theoretic axiom, provide an example which settles a problem raised by the author in [16].

The author wishes to thank W. G. Fleissner for the first draft of his paper [6] from which the author has learned the axiom $E(\omega_2)$ and some related ideas used in Section 3.5.

Notation and terminology. Our topological terminology follows [3] and [12]; set theoretic terminology is taken from [13] — with the only exception — a regular cardinal is always understanding to be uncountable. *By a space we shall mean in this paper always a metrizable space.* Given a space X we denote by ϱ a metric agreeing with the topology on X and we write $K(A, \varepsilon) = \{x \in X\colon \varrho(x, A) < \varepsilon\}$, where $A \subset X$ and $\varepsilon > 0$; the symbol $w(X)$ stands for the weight of X. We say that a family of sets $\mathscr{A}$ is a *refinement* of a family of sets $\mathscr{E}$, provided that $\bigcup \mathscr{A} = \bigcup \mathscr{E}$ and for every $A \in \mathscr{A}$ there exists $E \in \mathscr{E}$ with $A \subset E$. Given a family of sets $\mathscr{E}$ and a set A we denote by $\mathscr{E}|A$ the restriction of $\mathscr{E}$ to A, i.e. the family $\{E \cap A\colon E \in \mathscr{E}\}$; if $\mathscr{E}$ is a disjoint family, a set S is said to be a *selector* for $\mathscr{E}$ if S contains exactly one element of each non-empty member of $\mathscr{E}$. Given an ordinal λ we denote by $W(\lambda)$ the set of all ordinals less than λ with the order topology and the same set with the discrete topology is denoted by $D(\lambda)$; $\mathrm{Lim}(\lambda)$ stands for the set of all limit ordinals less than λ and $C(\lambda)$ is the set of all sequential ordinals from $W(\lambda)$ (i.e. $C(\lambda) = \{\xi < \lambda\colon \mathrm{cf}(\xi) = 0\}$). A subset S of the space $W(\lambda)$ is said to be *stationary* (in $W(\lambda)$) if S intersects each closed cofinal subset of $W(\lambda)$; the reader is referred to [11] for basic properties of stationary sets. Given a cardinal $\mathfrak{t}$ we denote by $B(\mathfrak{t})$ the product of countably many copies of the discrete space of cardinality $\mathfrak{t}$ ($B(\mathfrak{t})$ is called Baire space of weight $\mathfrak{t}$; see [19; Sec. 2]). A space is said to be *of σ-local weight less than* $\mathfrak{t}$ (abbreviated $\sigma\mathrm{Lw}(<\mathfrak{t})$) if X is the union of countably many sets of local weight $<\mathfrak{t}$; this notion was introduced and investigated by A. H. Stone [20] (cf. also [16]). Finally, $\omega(\mathfrak{t})$ denote the initial ordinal of cardinality $\mathfrak{t}$ and N stand for the set of natural numbers.

1. The theorem on σ-discrete reduction and completely additive-Borel families.

1.1. Given a family $\mathscr{E}$ of subsets of a space X, a set A is said to be *$\mathscr{E}$-discrete* if for every $a \in A$ there exists a set $E_a \in \mathscr{E}$ with $A \cap E_a = \{a\}$.

DEFINITION. A family $\mathscr{E}$ of subsets of a space X is said to be *weakly discrete*, provided that every $\mathscr{E}$-discrete set is σ-discrete.

Weak discreteness of a disjoint family $\mathscr{E}$ means exactly that each selector for $\mathscr{E}$ is σ-discrete.

Remark. Let us notice that if a disjoint and weakly discrete family $\mathscr{E}$ consists of σ-discrete sets, then the union $\bigcup \mathscr{E}$ is σ-discrete; this is a particular case of [16; Theorem 2] but it can be easily verified directly.

The following theorem improves [16; Remark 2] (for $\mathfrak{t} = \aleph_0$).

THEOREM (On σ-discrete reduction). *Every weakly discrete point-countable covering $\mathscr{E}$ of a space X consisting of sets of weight $\leqslant \aleph_1$ has a σ-discrete refinement.*

Proof. Let $\mathscr{E} = \{E_\xi\colon \xi < \gamma\}$,

(1.1.1) $$A_\xi = E_\xi \setminus \bigcup_{\alpha<\xi} E_\alpha \quad \text{and} \quad K = \{\xi < \gamma\colon A_\xi \neq \emptyset\}.$$

The family $\mathscr{A} = \{A_\xi\colon \xi \in K\}$ is a refinement of the family $\mathscr{E}$; we shall show that $\mathscr{A}$ has a σ-discrete refinement. To this purpose it is enough to verify, by virtue of [16; Remark 2] (where $\mathfrak{t} = \aleph_0$), that each selector for $\mathscr{A}$ is σ-discrete. Assume on the contrary that there exists a selector for $\mathscr{A}$

(1.1.2) $$S = \{a_\xi \in A_\xi\colon \xi \in K\}$$

which is not σ-discrete. For every $\xi \leqslant \gamma$ let us put $S(\xi) = \{a_\alpha\colon \alpha \in K \cap W(\xi)\}$ and let (notice that $S = S(\gamma)$)

(1.1.3) $$\tau = \min\{\xi\colon S(\xi) \text{ is not } \sigma\text{-discrete}\}.$$

Let $\lambda = \omega_{\mathrm{cf}(\tau)}$ (see [13]); by the definition of τ we have $\mathrm{cf}(\tau) > 0$ and thus λ is a regular ordinal.

We shall prove that if $\varphi\colon L \to K$, where $L \subset W(\lambda)$, is a strictly increasing function with $\lim_{\alpha \in L} \varphi(\alpha) = \tau$, then the set

(1.1.4) $$T = \{a_\xi\colon \xi \in \varphi(L)\} \text{ is } \sigma\text{-discrete}.$$

Given $\xi \in K$ let us put

(1.1.5) $$K(\xi) = \{\eta < \tau\colon a_\xi \in E_\eta\}.$$

The set $K(\xi)$ is countable, as the family $\mathscr{E}$ is point-countable; by regularity of λ and the properties of the function φ one can define a strictly increasing and continuous function $\mu\colon W(\lambda) \to W(\tau)$ satisfying the conditions

(1.1.6) $$\mu(0) = 0, \quad \lim_{\alpha<\lambda} \mu(\alpha) = \tau,$$
$$\mu(\alpha+1) > \sup\left[\bigcup \{K(\xi)\colon \xi \in \varphi(L) \cap W(\mu(\alpha))\}\right].$$

The function μ splits the set T into σ-discrete sets

$$T_\alpha = \{a_\xi\colon \xi \in \varphi(L) \cap [\mu(\alpha), \mu(\alpha+1))\} \subset S(\mu(\alpha+1));$$

to end the proof of (1.1.4) it is enough to verify by Remark that each selector for the decomposition $\{T_\alpha\colon \alpha < \lambda\}$ is σ-discrete. Let us choose, for every non-empty T_α, a point $a_{\varphi(\alpha)} \in T_\alpha$ (thus $\mu(\alpha) \leqslant \varphi(\alpha) < \mu(\alpha+1)$) and let W be the selector obtained in this way. Let $W_0 = \{a_{\varphi(\alpha)} \in W\colon \alpha \text{ is even}\}$ and $W_1 = \{a_{\varphi(\alpha)} \in W\colon \alpha \text{ is odd}\}$. We shall show that each of the sets W_0, W_1 is $\mathscr{E}$-discrete and thus — by our assumption — σ-discrete. To this end let $a_{\varphi(\alpha)}$ and $a_{\varphi(\beta)}$ be points of W_i with $\varphi(\alpha) < \varphi(\beta)$ (equivalently — $\alpha < \beta$). By (1.1.1) we have $a_{\varphi(\alpha)} \in E_{\varphi(\alpha)}$ and $a_{\varphi(\beta)} \in E_{\varphi(\beta)} \setminus E_{\varphi(\alpha)}$. Moreover, we have $\varphi(\alpha) < \mu(\alpha+1) < \mu(\alpha+2) \leqslant \mu(\beta) \leqslant \varphi(\beta)$ and by (1.1.6) it follows that $\varphi(\beta) > \sup K(\varphi(\alpha))$ which gives, accordingly to (1.1.5), $a_{\varphi(\alpha)} \notin E_{\varphi(\beta)}$. Hence, choosing for every $a_{\varphi(\alpha)} \in W_i$ the set $E_{\varphi(\alpha)} \in \mathscr{E}$ we obtain $W_i \cap E_{\varphi(\alpha)} = \{a_{\varphi(\alpha)}\}$. This completes the proof of (1.1.4).

Now, let us choose a strictly increasing continuous function $\nu\colon W(\lambda) \to W(\tau)$ such that $\nu(0) = 0$ and $\lim_{\alpha<\lambda} \nu(\alpha) = \tau$. Let us consider the decomposition of the set $S(\tau)$ into σ-discrete sets

$$S_\alpha = \{a_\xi\colon \xi \in K \cap [\nu(\alpha), \nu(\alpha+1))\} \subset S(\nu(\alpha+1)).$$

Let T be a selector for the decomposition $\{S_\alpha\colon \alpha < \lambda\}$. Let us put $L = \{\alpha\colon S_\alpha \neq \emptyset\}$ and let us choose for every $\alpha \in L$ an ordinal $\varphi(\alpha) \in K \cap [\nu(\alpha), \nu(\alpha+1))$ with

$S_\alpha \cap T = \{a_{\varphi(\alpha)}\}$. The function $\varphi\colon L \to K$ is strictly increasing and $\lim_{\alpha \in L} \varphi(\alpha) = \tau$. By virtue of (1.1.4) we infer that the set $T = \{a_\alpha : \alpha \in \varphi(L)\}$ is σ-discrete. Using Remark ones again we conclude that the set $S(\tau)$ is σ-discrete, contrary to (1.1.3). This proves that the set (1.1.2) is σ-discrete and completes the proof.

1.2. A family $\mathscr{A}$ of subsets of a space X is *completely additive-Borel* (respectively — *analytic*) provided that the union $\bigcup \mathscr{E}$ of each family $\mathscr{E} \subset \mathscr{A}$ is a Borel set (respectively — analytic set) in X. These notions were introduced by R. W. Hansell [9], where the deep theorem is proved that every disjoint completely additive-analytic family in a complete space has a σ-discrete refinement. We shall establish a result which reinforces the Hansell's theorem (see Remark; for another extensions of this theorem see [14]). Our proof will be a version of the reasons used in the proof of Theorem 1 in [14] which are closely related to the original Hansell's idea.

PROPOSITION. *Let $\mathscr{A}$ be a completely additive-Borel family of subsets of a complete space X. For every $A \in \mathscr{A}$ let us put $A^* = A \setminus \bigcup \{B \in \mathscr{A}\colon B \neq A\}$; then the family $\mathscr{A}^* = \{A^*\colon A \in \mathscr{A}\}$ has a σ-discrete refinement.*

Remark. The proof given bellow uses in fact the weaker assumption that for every $\mathscr{S} \subset \mathscr{A}$ the union $\bigcup \mathscr{S}$ and its complement $X \setminus \bigcup \mathscr{S}$ are analytic in X (i.e. the set $\bigcup \mathscr{S}$ is an extended Borel set in the sense of Hansell [10]). Under this assumption Proposition improves Hansell's Theorem 2 from [9].

Proof. We shall use the following notation: D stands for the two-point space $\{0, 1\}$; for every finite or infinite sequence $s = (i_1, i_2, \ldots)$ and an $n \in N$ we write $s|n = (i_1, \ldots, i_n)$; given a finite sequence $s = (i_1, \ldots, i_n)$ and an $m \in N$ we write $s;m = (i_1, \ldots, i_n, m)$ (cf. [13, Ch. XI, § 5]).

Let $M = \boldsymbol{P}_i X_i$ be the countable product where $X_0 = X$ and $X_i = N^N$ for $i \geq 1$; let ϱ_i be a complete metric on X_i and let $\varrho = \sum_i 2^{-i} \min(\varrho_i, 1)$ be the complete metric on M. Fix a point $a \in N^N$, let

(1.2.1) $$M_k = \{(x_i) \in M\colon x_i = a \text{ for } i > k\},$$

(we identify X with M_0) and let $p_k\colon M \to M_k$ stands for the projection. For every $\mathscr{S} \subset \mathscr{A}$ we put

(1.2.2) $$L(\mathscr{S}) = \bigcup \mathscr{S} \setminus \bigcup (\mathscr{A} \setminus \mathscr{S}).$$

We argue indirectly supposing that the family $\mathscr{A}$ has not any σ-discrete refinement. For every $k \in N$ and $s \in D^k$ we define inductively a family $\mathscr{A}_s \subset \mathscr{A}$ and a closed set $F_s \subset M_k$ satisfying the following conditions:

(1.2.3) $$\mathscr{A}_{s;0} \cup \mathscr{A}_{s;1} = \mathscr{A}_s \quad \text{and} \quad \mathscr{A}_{s;0} \cap \mathscr{A}_{s;1} = \emptyset,$$

(1.2.4) $$F_{s;0} \cap F_{s;1} = \emptyset,$$

(1.2.5) $$p_k(F_{s;i}) \subset F_s \quad \text{and} \quad \operatorname{diam} F_s < 2^{-k}, \quad \text{whenever} \quad s \in D^k,$$

(1.2.6) $$p_0(F_s) \subset L(\mathscr{A}_s),$$

(1.2.7) the family $\{p_0^{-1}(A^*)\colon A \in \mathscr{A}_s\} | F_s$ has not any σ-discrete refinement.

Since the construction differs from the construction given in the proof of Theorem 1 in [14] only in minor details — we omit it.

Let us put

(1.2.8) $$Z = \bigcup_{s \in D^N} \bigcap_k p_k^{-1}(F_{s|k}).$$

Using (1.2.4) and (1.2.5) one can easily verify that assigning to each sequence $s \in D^N$ the unique point z_s of the intersection $\bigcap_k p_k^{-1}(F_{s|k})$ we define a homeomorphism of the Cantor space D^N onto Z. Let $x_s = p_0(z_s)$ and

(1.2.9) $$p_0(Z) = \{x_s\colon s \in D^N\} = C.$$

For every $k \in N$ and $s \in D^N$ we have by (1.2.6)

(1.2.10) $$x_s \in L(\mathscr{A}_{s|k}).$$

Let us choose for every $s \in D^N$ a set $A_s \in \mathscr{A}$ with

(1.2.11) $$x_s \in A_s.$$

From (1.2.10) and (1.2.2) we infer that

(1.2.12) $$A_s \in \bigcap_k \mathscr{A}_{s|k}.$$

Given two distinct sequences $s, t \in D^N$ there is an $k \in N$ with $t|k \neq s|k$ and hence, by (1.2.10), (1.2.3) and (1.2.12) we have $x_s \notin \bigcup \mathscr{A}_{t|k}$, i.e.

(1.2.13) $$x_s \notin A_t \quad \text{if} \quad s \neq t.$$

This implies, by (1.2.11), that $x_s \neq x_t$ for distinct s and t, and hence the set C is of cardinality $2^{\aleph_0}$; using again (1.2.11) and (1.2.13) we infer that every set

$$\{x_s\colon s \in E \subset D^N\} = C \cap \bigcup \{A_s\colon s \in E\}$$

is a Borel set in C. Both of these facts together give a contradiction which completes the proof.

COROLLARY. *Every completely additive-Borel family $\mathscr{E}$ of subsets of a complete space X is weakly discrete.*

Proof. Let A be an $\mathscr{E}$-discrete set, choose for every $a \in A$ a set E_a with $A \cap E_a = \{a\}$ and let us put $\mathscr{A} = \{E_a\colon a \in A\}$. The set A is a selector for the family $\mathscr{A}^*$ defined in the proposition and hence A is σ-discrete by this proposition.

1.3. THEOREM. *Every completely additive-Borel, point-countable family consisting of subsets of weight $\leq \aleph_1$ of a complete space X has a σ-discrete refinement.*

Proof. This follows immediately from Theorem 1.1 and Corollary 1.2.

The above theorem yields a result on Borel selectors; to establish this result let us recall a few notions: given a space Y we denote by 2^Y the family of all non-empty closed subsets of Y; a function $F\colon X \to 2^Y$ is of class α if the set

$$\{x\colon F(x) \cap U \neq \emptyset\}$$

is of additive Borel class α whenever U is an open set; a selector for a function $F\colon X\to 2^Y$ is a function $f\colon X\to Y$ with $f(x)\in F(x)$ for $x\in X$.

COROLLARY. *Let X be a complete space of weight $\leqslant\aleph_1$ and let Y be an arbitrary complete space. Then every function $F\colon X\to 2^Y$ of class $\alpha>0$ with separable values $F(x)$ admits a selector, Borel measurable of class α.*

The proof of this result is quite parallel to that given in [14; Sec. 3] and hence we omit it; the reader is also referred to [14] for related results and discussion.

The author does not know whether the weight restriction on X in the theorem and the corollary is necessary.

Remark. W. G. Fleissner [5] proved a deep theorem of set theory which implies that under the Gödel Axiom of Constructibility (V = L) every non-σ-discrete space X contains a subset which is not an F_σ-set in X (see [17; VIII (5)]). On the ground of this result and Theorem 1.1 we obtain the following statement.

Assume (V = L) *and let $\mathscr{E}$ be a point-countable covering of a space X consisting of sets of weight $\leqslant\aleph_1$. If the union of every subfamily of the family $\mathscr{E}$ is an F_σ-set in X, then $\mathscr{E}$ has a σ-discrete refinement.*

Indeed, the Fleissner's result yields that the family $\mathscr{E}$ is weakly discrete. Let us notice that the above statement is independent of the usual axioms for set theory (see [17; IV (3)]); the author does not know whether the weight restriction is essential.

1.4. EXAMPLE. Let $F_1\subset\ldots\subset F_\xi\subset\ldots\subset B(\aleph_1)$, where $\xi<\omega_1$, be closed, separable subspaces of $B(\aleph_1)$. The family $\mathscr{E}=\{F_\xi\colon \xi<\omega_1\}$ is weakly discrete and the union of every subfamily of $\mathscr{E}$ is an F_σ-set in $B(\aleph_1)$. However there is no σ-discrete refinement of the family $\mathscr{E}$, as the space $B(\aleph_1)$ is not $\sigma\mathrm{Lw}(<\aleph_1)$ (see [20]). This shows that the assumption of point-countability in the theorems of this section was necessary. We shall show in Example 3.5 that the weight restriction in Theorem 1.1 was necessary too.

2. Natural decompositions of spaces and σ-discreteness preserving mappings.

2.1. Let X be a space of a regular weight $\mathfrak{t}$, let $\lambda=\omega(\mathfrak{t})$ be the initial ordinal of cardinality $\mathfrak{t}$ and let $\{X_\xi\colon \xi<\lambda\}$ be a sequence of closed subsets of the space X satisfying the following conditions (cf. [16; (3)-(6)]):

(2.1.1) $$X_1\subset\ldots\subset X_\xi\subset\ldots\subset X \quad\text{and}\quad w(X_\xi)<w(X) \quad\text{for}\quad \xi<\lambda,$$

(2.1.2) $$X=\bigcup_{\xi<\lambda}X_\xi \quad\text{and}\quad X_\xi=\overline{\bigcup_{\alpha<\xi}X_\alpha} \quad\text{whenever}\quad \xi\in\mathrm{Lim}(\lambda).$$

Let us put

(2.1.3) $$\mathscr{P}=\{P_\xi\colon \xi<\lambda\} \quad\text{where}\quad P_\xi=X_\xi\setminus\bigcup_{\alpha<\xi}X_\alpha,$$

(2.1.4) $$\Gamma(\mathscr{P})=\{\xi\in\mathrm{Lim}(\lambda)\colon P_\xi\neq\emptyset\}.$$

Any such a decomposition $\mathscr{P}$ we shall call a natural decomposition of the space X (related to the family $\{X_\xi\colon \xi<\lambda\}$). Evidently, each space X of regular weight $\mathfrak{t}$ has many natural decompositions (see [16]); however, every two natural decompositions of X coincide apart from a set which is $\sigma\mathrm{Lw}(<\mathfrak{t})$, as it will be shown in Corollary 2.3.

The notation introduced in this subsection will be used throughout the whole Section 2.

2.2. Let us adopt the notation introduced in the previous subsection. Let, for simplicity, Γ stand for $\Gamma(\mathscr{P})$. For every $K\subset W(\lambda)$ we write

(2.2.1) $$X(K)=\bigcup\{P_\xi\colon \xi\in K\},$$

(2.2.2) $$\mathscr{E}(K)=\{P_\xi\colon \xi\in\Gamma\cap K\}.$$

The following theorem slightly improves Theorem 1 in [16] and it can be proved by arguments quite parallel to that given in [16]; therefore we omit this proof.

THEOREM. *For every set $K\subset W(\lambda)$ the following conditions are equivalent:*

(i) *the set $X(K)$ is $\sigma\mathrm{Lw}(<\mathfrak{t})$,*

(ii) *there exists a selector for $\mathscr{E}(K)$ which is $\sigma\mathrm{Lw}(<\mathfrak{t})$,*

(iii) *the set $\Gamma\cap K$ is not stationary in $W(\lambda)$.*

COROLLARY (cf. [16; Remark 5]). *Let S be a selector for a natural decomposition $\mathscr{P}$. For every set $K\subset W(\lambda)$ the following conditions are equivalent:*

(i) *the set $S\cap X(K)$ is $\sigma\mathrm{Lw}(<\mathfrak{t})$,*

(ii) *the set $\Gamma\cap K$ is not stationary in $W(\lambda)$.*

Proof. The set $S\cap X(K)$ is a selector for the family $\{P_\xi\colon \xi\in K\}$ containing the family $\mathscr{E}(K)$. Thus by virtue of the theorem we have the following implications: ($S\cap X(K)$ is $\sigma\mathrm{Lw}(<\mathfrak{t})$) $\Rightarrow$ (there exists a selector for $\mathscr{E}(K)$ which is $\sigma\mathrm{Lw}(<\mathfrak{t})$) $\Rightarrow$ ($\Gamma\cap K$ is not stationary) $\Rightarrow$ ($X(K)$ is $\sigma\mathrm{Lw}(<\mathfrak{t})$) $\Rightarrow$ ($S\cap X(K)$ is $\sigma\mathrm{Lw}(<\mathfrak{t})$).

2.3. DEFINITION (cf. [16]). A mapping $f\colon X\to Y$ is said to be a *d-isomorphism* if it is a bijection and both f and f^{-1} take σ-discrete sets to σ-discrete sets (i.e. $f(A)$ is σ-discrete iff A is σ-discrete); if $f\colon X\to f(X)\subset Y$ is a d-isomorphism, we say that f is a *d-embedding*.

LEMMA. *Let $f\colon X\to X'$ be a d-isomorphism of a space X of a regular weight $\mathfrak{t}$ onto a space X' and let $\mathscr{P}=\{P_\xi\colon \xi<\lambda\}$ and $\mathscr{P}'=\{P'_\xi\colon \xi<\lambda\}$, where $\lambda=\omega(\mathfrak{t})$, be natural decompositions of X and X' respectively, related respectively to families $\{X_\xi\colon \xi<\lambda\}$ and $\{X'_\xi\colon \xi<\lambda\}$ (see 2.1). Then the set $\{\xi<\lambda\colon f(P_\xi)\neq P'_\xi\}$ is not stationary in $W(\lambda)$.*

Proof. We shall show that each of the sets

$$\{\xi<\lambda\colon f(P_\xi)\not\subset P'_\xi\} \quad\text{and}\quad \{\xi<\lambda\colon f^{-1}(P'_\xi)\not\subset P_\xi\}$$

is not stationary; by the symmetry it is enough to verify that the set

(2.3.1) $$K=\{\xi\in\mathrm{Lim}(\lambda)\colon f(P_\xi)\not\subset P'_\xi\} \text{ is not stationary.}$$

Let us define two strictly increasing continuous functions $\Phi,\Psi\colon W(\lambda)\to W(\lambda)$ such that

(2.3.2) $$fX_{\Phi(\xi)}\subset X'_{\Psi(\xi+1)}\subset fX_{\Phi(\xi+1)} \quad\text{for}\quad \xi<\lambda.$$

For this purpose we put $\Phi(0) = \Psi(0) = 0$,

$$\Psi(\xi+1) = \min\{\alpha > \Psi(\xi): f X_{\Phi(\xi)} \subset X'_\alpha\},$$
$$\Phi(\xi+1) = \min\{\alpha > \Phi(\xi): f^{-1} X'_{\Psi(\xi)} \subset X_\alpha\},$$

and for a limit $\xi \in \mathrm{Lim}(\lambda)$

$$\Phi(\xi) = \sup\{\Phi(\alpha): \alpha < \xi\}, \quad \Psi(\xi) = \sup\{\Psi(\alpha): \alpha < \xi\}.$$

It is easy to verify that

(2.3.3) $\quad f P_{\Phi(\xi)} \subset P'_{\Psi(\xi)} \cup (X'_{\Psi(\xi+1)} \setminus X'_{\Psi(\xi)})$ for every $\xi \in \mathrm{Lim}(\lambda)$.

Let C be the set of the common fixed points of mappings Φ, Ψ, i.e.

(2.3.4) $\quad C = \{\xi < \lambda: \Phi(\xi) = \xi = \Psi(\xi)\}$.

The set C is closed and cofinal in $W(\lambda)$ (see [13; Ch. VII, § 3]); to prove (2.3.1) it suffices to show that the set

(2.3.5) $\quad C \cap K = K'$ is not stationary.

Let us choose for every $\xi \in K'$, using (2.3.1), (2.3.3) and (2.3.4), a point $x_\xi \in P_\xi$ such that $f(x_\xi) = a_\xi \in X'_{\Psi(\xi+1)} \setminus X'_\xi$ and let us consider the sets $A = \{a_\xi: \xi \in K'\}$ and $A_n = \{a_\xi \in A: \varrho(a_\xi, X'_\xi) > 1/n\}$. Since the distance between two distinct points of each set A_n is at least $1/n$ and $A = \bigcup_n A_n$ we infer that the set A is σ-discrete. Therefore the set $\{x_\xi: \xi \in K'\} = f^{-1}(A)$ is σ-discrete and it implies (2.3.5) by Corollary 2.2.

Corollary. *Let $\mathscr{P} = \{P_\xi: \xi < \lambda\}$ and $\mathscr{P}' = \{P'_\xi: \xi < \lambda\}$ be natural decompositions of a space X of regular weight $\mathfrak{t}$. Then the set $K = \{\xi < \lambda: P_\xi \neq P'_\xi\}$ is not stationary in $W(\lambda)$ and there exists a set A which is $\sigma\mathrm{Lw}(<\mathfrak{t})$ such that $\mathscr{P}|(X \setminus A) = \mathscr{P}'|(X \setminus A)$.*

Proof. The first part of the corollary follows immediately from the lemma, where f is the identity on X; for the proof of the second part it is enough to take

$$A = \bigcup\{P_\xi: \xi \in K\} \cup \bigcup\{P'_\xi: \xi \in K\}$$

and use Theorem 2.2.

Proposition. *Let X be a space of regular weight $\mathfrak{t}$ and let $\mathscr{P}$ be a natural decomposition of the space X. For every d-embedding $f: A \to X$ of a subset A of X into X the set $K = \{\xi < \lambda: f(P_\xi \cap A) \not\subset P_\xi\}$ is not stationary in $W(\lambda)$.*

Remark. Under the assumption of the proposition it follows from Theorem 2.2 that the set $\bigcup\{P_\xi: f(P_\xi \cap A) \not\subset P_\xi\}$ is $\sigma\mathrm{Lw}(<\mathfrak{t})$.

Proof. First, let us assume that $A = X$ and let $X' = f(A)$. Let $\{X_\xi: \xi < \lambda\}$ be the family of sets designating the natural decomposition $\mathscr{P}$ (see 2.1) and let $\mathscr{P}' = \{P'_\xi: \xi < \lambda\}$ be the natural decomposition of the space X' related to the family $\{X'_\xi: \xi < \lambda\}$ defined by $X'_{\xi+1} = X_{\xi+1} \cap X'$ and $X'_\xi = \overline{\bigcup_{\alpha<\xi} X'_\alpha} \cap X'$ for $\xi \in \mathrm{Lim}(\lambda)$.

It is easy to verify (cf. 2.1) that

(2.3.6) $\quad P'_\xi \subset P_\xi$ whenever $\xi \in \mathrm{Lim}(\lambda)$.

From the lemma we infer that the set $L = \{\xi < \lambda: f(P_\xi) \neq P'_\xi\}$ is not stationary, and since by (2.3.6) we have $K \cap \mathrm{Lim}(\lambda) \subset L$, we conclude that the set K is not stationary. The case of arbitrary A we shall derive from the case just considered by means of the following remark (which requires only the assumption that f is invertible):

(2.3.7) there exists a disjoint decomposition $A = C_0 \cup C_1 \cup \ldots$ such that $f(C_0) \subset C_0$ and $f(C_{i+1}) \cap C_{i+1} = \emptyset$ for $i \geqslant 0$.

To prove (2.3.7) we define $A_0 = A$, $A_{i+1} = f^{-1}(A_i \cap f(A_i))$ and put $C_0 = \bigcap_i A_i$ and $C_{i+1} = A_i \setminus A_{i+1}$ for $i \geqslant 0$.

Now, let us define

$$f_0(x) = \begin{cases} x & \text{if } x \notin C_0, \\ f(x) & \text{if } x \in C_0, \end{cases}$$

$$f_{i+1}(x) = \begin{cases} x & \text{if } x \notin C_{i+1} \cup f(C_{i+1}), \\ f(x) & \text{if } x \in C_{i+1}, \\ f^{-1}(x) & \text{if } x \in f(C_{i+1}). \end{cases}$$

The mapping f_0 is a d-embedding and f_{i+1} are d-isomorphisms defined on the whole X. By the case considered before each set $K_i = \{\xi < \lambda: f_i(P_\xi) \not\subset P_\xi\}$ is not stationary. Since $A = \bigcup_i C_i$ we have

$$f(P_\xi \cap A) = \bigcup_i f(P_\xi \cap C_i) = \bigcup_i f_i(P_\xi \cap C_i) \subset \bigcup_i f_i(P_\xi),$$

and hence $K \subset \bigcup_i K_i$ which completes the proof.

2.4. A space X is *chaotic* (see [15]) if no two non-empty disjoint open sets are homeomorphic; we say that X is *d-chaotic* if we can replace homeomorphism by d-isomorphism in this definition. For results about chaotic spaces the reader is referred to [15]; let us notice that E. S. Berney [1] proved that the real line contains, under the continuum hypothesis, a chaotic space.

Corollary. *Every space X of regular weight $\mathfrak{t}$ which is not $\sigma\mathrm{Lw}(<\mathfrak{t})$ contains a subspace Z of cardinality $\mathfrak{t}$ such that every d-embedding $f: U \to Z$ of an open non-empty subspace U of Z into Z has a fixed point; in particular the space Z is chaotic and even d-chaotic.*

Proof. Let $\mathscr{P} = \{P_\xi: \xi < \lambda\}$, where $\lambda = \omega(\mathfrak{t})$, be a natural decomposition of the space X, let $\Gamma = \Gamma(\mathscr{P})$ (see 2.1) and let $E = \{x_\xi \in P_\xi: \xi \in \Gamma\}$ be a subselector for $\mathscr{P}$. By Theorem 2.2 the space E is not $\sigma\mathrm{Lw}(<\mathfrak{t})$; let Z be the nowhere $\sigma\mathrm{Lw}(<\mathfrak{t})$ kernel of the space E defined by A. H. Stone [20; 2.2. (1)]. Thus

(2.4.1) $\quad$ the space Z is non-empty and no non-empty subset of Z is $\sigma\mathrm{Lw}(<\mathfrak{t})$.

Let us take a non-empty open subset U of Z. By Corollary 2.2 (where $S = E$) it follows that the set

(2.4.2) $K = \{\xi \in \Gamma:\ x_\xi \in U\}$ is stationary.

Given a d-embedding $f\colon U \to Z$ the set $L = \{\xi < \lambda:\ f(P_\xi \cap U) \not\subset P_\xi\}$ is not stationary by Proposition 2.3 and hence there exists an ordinal $\xi \in K \setminus L$; thus

$$f(x_\xi) = f(P_\xi \cap U) \subset P_\xi \cap Z = \{x_\xi\},$$

i.e. $f(x_\xi) = x_\xi$.

Remark. Using Corollary one can easily construct, for a given regular cardinal $\mathfrak{t}$, a family $\mathscr{E}$ consisting of $2^{\mathfrak{t}}$ spaces such that no member of $\mathscr{E}$ can be d-embedded in any other member of $\mathscr{E}$.

3. Decompositions of Baire spaces $B(\mathfrak{t})$ and absolutely $\mathfrak{t}$-analytic spaces.

3.1. Throughout Section 3 we shall use the following notation. For any ordinal ξ let

(3.1.1) $$B(\xi) = D(\xi)^N,$$

i.e. $B(\xi)$ is the Baire space of sequences of ordinals less than ξ, let

(3.1.2) $$B_\xi = B(\xi) \setminus \bigcup_{\alpha<\xi} B(\alpha)$$

and let for every set of ordinals K

(3.1.3) $$B(K) = \bigcup \{B_\xi:\ \xi \in K\}.$$

We shall identify the space $B(\mathfrak{t})$ with the Baire space of sequences of ordinals less than $\omega(\mathfrak{t})$, i.e.

(3.1.4) $$B(\mathfrak{t}) = B(\lambda) \quad \text{where} \quad \lambda = \omega(\mathfrak{t}).$$

It is easy to see that $\mathscr{B} = \{B_\xi:\ \xi < \lambda\}$ is a natural decomposition of the space $B(\mathfrak{t})$ (see [16; Example]) and (see (2.1.4))

(3.1.5) $$\Gamma(\mathscr{B}) = C(\lambda).$$

3.2. Lemma. *Let $\lambda = \omega(\mathfrak{t})$, where $\mathfrak{t}$ is a regular cardinal, and let $K \subset C(\lambda)$ be a stationary set; then the space $B(K)$ intersects each subspace of the space $B(\lambda)$ homeomorphic (in fact — d-isomorphic) to the space $B(\lambda)$.*

Proof. Let $f\colon B(\lambda) \to E$ be a d-isomorphism onto a subspace E of the space $B(\lambda)$. Accordingly to Proposition 2.3 (where $A = B(\lambda) = X$) the set

$$L = \{\xi < \lambda:\ f(B_\xi) \not\subset B_\xi\} \subset C(\lambda)$$

is not stationary and therefore there exists an ordinal $\xi \in K \setminus L$. We have then $E \cap B(K) \supset f(B_\xi) \cap B_\xi = f(B_\xi)$ and $f(B_\xi) \neq \emptyset$ by (3.1.5).

3.3. Definition. We say that a space X is *totally $\mathfrak{t}$-imperfect* if X does not contain topologically the space $B(\mathfrak{t})$ (cf. [12; § 40, I]).

Theorem (On generalized F. Bernstein's decomposition; cf. [12; § 40, I, Th. 1]). *Baire space $B(\mathfrak{t})$ of regular weight $\mathfrak{t}$ can be split into $\mathfrak{t}$ disjoint subspaces each of which has the totally $\mathfrak{t}$-imperfect complement.*

Proof. By virtue of R. Solovay's theorem [18; Theorem 9] the stationary set $C(\lambda)$, where $\lambda = \omega(\mathfrak{t})$, can be split into $\mathfrak{t}$ disjoint stationary sets K_α, $\alpha < \lambda$. The family $\{B(K_\alpha):\ \alpha < \lambda\}$ gives the desired decomposition in view of Lemma 3.2.

Remark 1. Recently, W. G. Fleissner [7] considered independently the decomposition $\{B(K_\alpha):\ \alpha < \lambda\}$ used in the above proof for the other purpose; Fleissner showed that each $B(K_\alpha)$ is a Baire space (cf. Lemma 3.2) whereas for $\alpha \neq \beta$ the product $B(K_\alpha) \times B(K_\beta)$ is of first category.

Remark 2. Let us notice that for some cardinals $\mathfrak{t}$ the classical proof of the Bernstein's theorem (see [13; Ch. VII, § 8, 3] and [12]) cannot be applied, even if we want to have two-element decomposition; this is the case for example if $\mathfrak{t} = (2^{\aleph_0})^+$.

3.4. A space X is said to be *absolutely $\mathfrak{t}$-analytic*, provided that for some (equivalently — for every) complete space Y containing the space X there exists a closed set $F \subset Y \times B(\mathfrak{t})$ which maps onto X under the projection parallel to the second axis. For the properties of absolutely $\mathfrak{t}$-analytic sets the reader is referred to [19; Sec. 8] (see also [2]); for $\mathfrak{t} = \aleph_0$ we have the classical notion of absolutely analytic spaces. In the sequel we shall need the following result which is contained implicite in the proof of Theorem 4 in A. H. Stone's paper [21] (cf. also [12; § 36]).

Lemma 1. *Let $f\colon B(\mathfrak{t}^+) \to X$ be a continuous mapping into a space X which takes discrete sets to $\mathfrak{t}$-discrete sets and $w(f^{-1}(x)) \leqslant \mathfrak{t}$ for every $x \in X$; then the space X contains topologically $B(\mathfrak{t}^+)$.*

Lemma 2. *If an absolutely $\mathfrak{t}$-analytic space X is not $\sigma\mathrm{Lw}(<\mathfrak{t}^+)$ then X contains topologically $B(\mathfrak{t}^+)$.*

Proof. It is easy to verify (using arguments parallel to that given in the proof of Lemma 2 in [14]) that the projection $p\colon Y \times B(\mathfrak{t}) \to Y$, where Y is an arbitrary space, takes discrete sets to $\mathfrak{t}$-discrete sets and it preserves $\sigma\mathrm{Lw}(<\mathfrak{t}^+)$-property.

Let Y be a completion of the space X and let $F \subset Y \times B(\mathfrak{t})$ be a closed set with $p(F) = x$. Since X is not $\sigma\mathrm{Lw}(<\mathfrak{t}^+)$ we infer from the above remark that F is not $\sigma\mathrm{Lw}(<\mathfrak{t}^+)$ and hence, by A. H. Stone's theorem [20; 2.2. (2)], F contains a subset B homeomorphic to $B(\mathfrak{t}^+)$; using the initial remark ones again we infer that the mapping $f = p|B$ satisfies the assumption of Lemma 1 and by virtue of this lemma the proof is completed.

Remark 1. (a) One can prove (by minor modifications of given arguments) that if X is an absolutely $\mathfrak{m}$-analytic space and X is not $\sigma\mathrm{Lw}(<\mathfrak{n})$ for a regular cardinal $\mathfrak{n} > \mathfrak{m}$, then X contains topologically $B(\mathfrak{n})$; see also [20] and [4].

(b) If $\mathfrak{t}$ is a sequential cardinal and X is an absolutely $\mathfrak{t}$-analytic space which is not $\sigma\mathrm{Lw}(<\mathfrak{t})$, then X contains topologically $B(\mathfrak{t})$ (the proof is similar to that of Lemma 2).

(c) Under the assumption that $\mathfrak{t}^{\aleph_0} = \mathfrak{t}$ Lemma 2 was proved (by other means) by M. M. Čoban [2; Corollary 11.3] and by R. C. Freiwald [8].

Lemma 3. *Baire space $B(\mathfrak{t}^+)$ contains a subspace which is not absolutely $\mathfrak{t}$-analytic.*

Proof. Let us consider a disjoint decomposition $B(\mathfrak{t}^+) = A_0 \cup A_1$ into two totally $\mathfrak{t}^+$-imperfect sets (see Theorem 3.3). Since the space $B(\mathfrak{t}^+)$ is not $\sigma\mathrm{Lw}(<\mathfrak{t}^+)$ (see [20]) either A_0 or A_1 is not $\sigma\mathrm{Lw}(<\mathfrak{t}^+)$ and it remains to apply Lemma 2.

Theorem. *If every subspace of a space X is absolutely $\mathfrak{t}$-analytic then X is $\sigma\mathrm{Lw}(<\mathfrak{t}^+)$.*

Proof. Indeed, in the opposite case the space X would contain, by virtue of Lemma 2, a space homeomorphic to $B(\mathfrak{t}^+)$ which would contradict however to Lemma 3.

Remark 2. For $\mathfrak{t} = \aleph_0$ Theorem gives A. G. Elkin's result [4].

Corollary. *Assuming the generalized continuum hypothesis the following statements about a space X are equivalent*:

(a) *every subspace of X is absolutely $\mathfrak{t}$-analytic,*

(b) *X is $\mathfrak{t}$-discrete.*

Proof. Obviously (b)$\Rightarrow$(a). Conversely, if (a) holds then X is $\sigma\mathrm{Lw}(<\mathfrak{t}^+)$ by Theorem and (b) follows from A. H. Stone's result [19; Theorem 24] that Corollary is true under the additional assumption that $w(X) \leqslant \mathfrak{t}$.

Remark 3. (a) Our Corollary is closely related to A. H. Stone's question raised in [19; 8.5].

(b) Under the assumption that $\mathfrak{t} < \mathfrak{t}^{\aleph_0}$ (if we assume the generalized continuum hypothesis this is equivalent to the condition that $\mathfrak{t}$ is a sequential cardinal) Corollary was proved by M. M. Čoban [2; Theorem 13.3] and R. C. Freiwald [8] (see also [20; 3.4, Remark]); let us notice that our proof differs essentialy from the proofs given in these papers.

3.5. The aim of this subsection is to construct, under an additional set theoretic axiom $E(\omega_2)$, a special decomposition of the Baire space $B(\aleph_2)$ which settles in the negative Question 3 raised by the author in [16]. The author learned the axiom $E(\omega_2)$ and the idea of the proof of lemma bellow from the first version of W. G. Fleissner's paper [6]; the reader is also referred to [6] for informations about consistency of $E(\omega_2)$ and related bibliographic notes.

We shall derive our main result from Lemma 3.2 and some facts about $\sigma\mathrm{Lw}(<\mathfrak{t})$-property; the reader is referred to [6; Remark 3.10] for a direct proof.

$E(\omega_2)$ is the assertion that there exists a set $\Sigma \subset C(\omega_2)$ stationary in $W(\omega_2)$ such that for every $\xi < \omega_2$ the intersection $\Sigma \cap W(\xi)$ is not stationary in $W(\xi)$.

This assertion is consistent with the usual axioms for set theory. We shall use in the sequel the notation introduced in subsection 3.1.

Lemma. *If Σ witnesses $E(\omega_2)$ and $x_\xi \in B_\xi$ for every $\xi \in \Sigma$ then for every $\alpha < \omega_2$ the set $A_\alpha = \{x_\xi \colon \xi \in \Sigma,\ \xi \leqslant \alpha\}$ is σ-discrete.*

Proof (cf. the proof of 3.1.d in [6]). We proceed by induction relatively to $\alpha < \omega_2$. For $\alpha = 0$ we have $A_\alpha = \emptyset$; let us assume that A_α is σ-discrete for $\alpha < \xi < \omega_2$. If $\xi = \alpha+1$ then the set $A_\xi = A_\alpha$ is σ-discrete; assume that $\xi \in \mathrm{Lim}(\omega_2)$. In this case there exists a set $C \subset W(\xi)$, a closed and cofinal in $W(\xi)$, such that $C \cap \Sigma = \emptyset$. Let us put $\Phi(\alpha) = \sup\{\beta \in C\colon \beta \leqslant \alpha\}$ for every $\alpha \in W(\xi)$. We have then

$$\Phi(\alpha) < \alpha \quad \text{for every } \xi \in \Sigma \cap W(\xi)\,. \tag{3.5.1}$$

Let $U(\alpha, n) = K(B_\alpha, 1/n)$ and let us put for every $n \in N$ and $\alpha \in W(\xi)$

$$A_{\alpha n} = A_\alpha \setminus U(\Phi(\alpha), n) \quad \text{and} \quad A_n = \bigcup \{A_{\alpha n}\colon \alpha < \xi\}\,.$$

Using (3.5.1) one can easily verify that $A_\xi \subset \bigcup_n A_n \cup \{x_\xi\}$; it remains to prove that each set A_n is σ-discrete. Since the open sets $U(\alpha, n)$, where $\alpha < \xi$, cover the set A_n it is enough to verify that each set $U(\alpha, n) \cap A_n$, where $\alpha < \xi$, is σ-discrete. Let us fix an ordinal $\alpha < \xi$; there exists $\gamma \in C \cap W(\xi)$ with $\gamma > \alpha$. For every $\beta \geqslant \gamma$ we have then $\Phi(\beta) \geqslant \gamma > \alpha$ and hence the set $U(\alpha, n) \cap A_n \subset \bigcup \{A_{\beta n}\colon \beta < \gamma\} \subset A_\gamma$ is σ-discrete by the inductive assumption.

Proposition. *Assuming $E(\omega_2)$ there is a decomposition of $B(\aleph_2)$ into $\aleph_1$ disjoint pieces E_α, where $\alpha < \omega_1$, such that*

(i) *the family $\mathscr{E} = \{E_\alpha\colon \alpha < \omega_1\}$ is weakly discrete (i.e. each selector for $\mathscr{E}$ is σ-discrete*; see 1.1),

(ii) *if $U_\alpha \supset E_\alpha$ is open then the intersection $\bigcap \{U_\alpha\colon \alpha < \omega_1\}$ is dense in $B(\aleph_2)$; moreover, the complement of this intersection is $\sigma\mathrm{Lw}(<\aleph_2)$.*

Proof. Let $\Sigma \subset C(\omega_2)$ witnesses $E(\omega_2)$ and let us split the set Σ, using R. Solovay's theorem, into disjoint stationary sets K_α, for $\alpha < \omega_1$ (cf. proof of Theorem 3.3). We assume $E_\alpha = B(K_\alpha)$ for $\alpha \geqslant 1$ and $E_0 = B(\aleph_2) \setminus \bigcup_{\alpha \geqslant 1} E_\alpha$. If $a_\xi \in E_\xi$ for $1 \leqslant \xi < \omega_1$, then there exists an ordinal $\alpha < \omega_2$ such that $\{a_\xi\colon \xi < \omega_1\} \subset A_\alpha$ (see the lemma), as $a_\xi \in B_{\xi'}$ with $\xi' \in K$. This proves (i) by the lemma. For the proof of (ii) let us put $F_\alpha = B(\aleph_2) \setminus U_\alpha$. The space F_α is complete and, by Lemma 3.2, it is totally $\aleph_2$-imperfect; hence each set F_α is $\sigma\mathrm{Lw}(<\aleph_2)$, by A. H. Stone's theorem [20; 2.2 (2)], and so is the union $\bigcup \{F_\alpha\colon \alpha < \omega_1\}$, by [16; Proposition].

Example. Assuming $E(\omega_2)$ there is a weakly discrete, disjoint covering $\mathscr{E}$ of $B(\aleph_2)$ which has not any σ-discrete refinement.

We shall verify that the decomposition $\mathscr{E} = \{E_\alpha\colon \alpha < \omega_1\}$ which we have just constructed in the proposition has not any σ-discrete refinement. Assume on the contrary that $\mathscr{A} = \bigcup_n \mathscr{A}_n$ is a refinement of $\mathscr{E}$ with $\mathscr{A}_n$ discrete. Let

$$E_{\alpha n} = \bigcup \{A \in \mathscr{A}_n\colon A \subset E_\alpha\}\,.$$

Then $E_\alpha = \bigcup_n E_{\alpha n}$ and each family $\{E_{\alpha n}\colon \alpha < \omega_1\}$ is discrete. Let $U_{\alpha n} \supset E_{\alpha n}$ be open sets such that $U_{\alpha n} \cap U_{\beta n} = \emptyset$ for $\alpha \neq \beta$ and let $U_\alpha = \bigcup \{U_{\alpha n}: n \in N\}$. Then $U_\alpha \supset E_\alpha$ and by the proposition (ii) there is an $x \in \bigcap_{\alpha<\omega_1} U_\alpha$. But then there exists an $n \in N$ and two distinct indices α, β with $x \in U_{\alpha n} \cap U_{\beta n}$, a contradiction.

This example shows that the weight restriction in Theorem 1.1 is essential, even if we consider only disjoint families.

Remark 1. The elements of the decomposition $\mathscr{E}$ in the example are irregular from the point of view of Borel theory: they are totally $\aleph_2$-imperfect and they are not open modulo the ideal of sets of first category. It would be interesting to explain whether a family $\mathscr{E}$ satisfying the conditions of the example could consist of absolutely analytic sets.

Remark 2. In the proof of the proposition and in the proof of Theorem 3.3 we have used only a special case of R. Solovay's decomposition theorem, namely we have applied this theorem to subsets of $C(\lambda)$ with λ regular. Using Remark 5 of [16] (cf. also Corollary 2.2) one can give a simple "topological" proof of this particular case. Indeed, by this remark it is enough to verify a simple fact that every metrizable space E which is not $\sigma\mathrm{Lw}(<\mathfrak{t})$ can be split into $\mathfrak{t}$-pieces which are not $\sigma\mathrm{Lw}(<\mathfrak{t})$.

Added in proof. 1. The results of preprint [7] (see Remark 1 on p. 127) were included in a paper of W. G. Fleissner and K. Kunen, *Bairely Baire spaces*, Fund. Math.

2. W. G. Fleissner constructed in a paper *An axiom for nonseparable Borel theory* (preprint) a model for ZFC in which every point-finite and analytic additive family is an arbitrary space is σ-discretely decomposable (cf. the remark on p. 122); see also footnote in [16] and the remark after Lemma 4.6 in the paper of Fleissner.

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