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# A fixed point theorem for plane homeomorphisms

by

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**Abstract.** Every homeomorphism of the plane into itself that leaves a non-separating continuum invariant has a fixed point in this continuum.

In [3] Brouwer proved that if $h$ is an orientation preserving homeomorphism of the plane onto itself and the iterates of some point $x$, $x$, $h(x)$, $h(h(x))$, ... has a cluster point, then $h$ has a fixed point. In [4] Cartwright and Littlewood proved that every orientation preserving homeomorphism of the plane onto itself that leaves a non-separating continuum $M$ invariant has a fixed point in $M$. In this paper it shall be shown that every homeomorphism of the plane into itself that leaves a non-separating continuum $M$ invariant has a fixed point in $M$.

BASIC ASSUMPTION. It shall be assumed throughout this paper that $h$ is a fixed point free homeomorphism of the plane into itself that leaves a continuum $M$ invariant. It is also assumed, without loss of generality, that $M$ is a non-separating plane continuum and that $M$ does not contain a proper non-separating invariant subcontinuum.

All set will be assumed to be subsets of the plane unless otherwise is indicated.

In section I the theorem is proven for two special cases. The first (1.2) is a direct generalization of the Brouwer fixed point theorem for two-cells and the second (1.3) is designed to illustrate a type of proof in a setting that yields geometric intuition while minimizing formal constructions. In section II continua $Y$ (2.9) and $Y'$ (2.10) are constructed such that $M \subset Y \subset Y'$. In section III it is shown that if $Y$ is a two-cell (1.2) guarantees a fixed point in $Y$. In section IV it is shown that if $Y$ is not a two-cell, then $Y'$ resembles the continuum $N$ in (1.3) well enough to employ the technique used in the proof of (1.3).

**Section I. Two special cases.** In this section the theorem is proven for two special cases.

(1.1) DEFINITION. *The operator* $T$. For each bounded set $A$ let $T(A)$ be the smallest compact set that contains $A$ and has a connected complement. It is handy to notice that $T(A)$ is the complement of the unbounded component of the complement of $\bar{A}$.

(1.2) *Let* $f\colon D\to R^2$ *be a map defined on a simple closed curve* $D$ *and let* $x_0, x_1, \ldots, x_n = x_0$ *be a partition of* $D$. *If for each arc in* $D$ $x_{i-1}x_i$, *there is an arc* $A_i$ *in* $T(D)$, *joining* $f(x_{i-1})$ *to* $f(x_i)$, *such that* $x_{i-1}x_i \cap T(A_i \cup f(x_{i-1}x_i)) = \emptyset$ *then every extension of* $f$ *to a map defined on* $T(D)$ *has a fixed point.*

Proof. Suppose by way of contradiction there is a fixed point free extension of $f$ to a map $g$ defined on $T(D)$. For each $i$ there is a simple closed curve $D_i$ such that

$$T(f(x_{i-1}x_i) \cup A_i) \subset (T(D_i))^0 \quad \text{and} \quad x_i x_{i-1} \cap T(D_i) = \emptyset$$

[1, Lemma 3]. For each $i$, let $K_i$ be an arc in $T(D)$ that joins $x_i$ to $x_{i-1}$ such that $g(K_i) \subset T(D_i)$, $K_i \cap T(D_i) = \emptyset$, and no two distinct $K_i$ intersect except possibly at a common endpoint. A new map $g'\colon T(D)\to R^2$ may then be defined for which $g'(z) = g(z)$ if $z \notin \bigcup \{T(K_i \cup x_{i-1}x_i)\colon i = 1, 2, \ldots, n\}$, $g'(x_{i-1}x_i) \subset A_i$, and $g'(T(K_i \cup x_{i-1}x_i)) \subset T(D_i)$. The map $g'$ is a fixed point free map defined on $T(D)$ for which $g'(D) \subset T(D)$, contradicting the Brouwer fixed point theorem for two-cells.

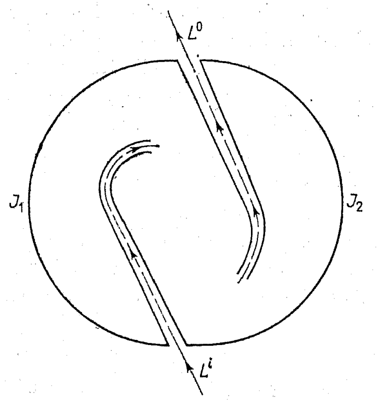


Fig. 1.1

In Figure 1.1 we mean to portray a plane continuum $N$ that is similar to the "Lakes of Wada" [5]. In the figure two infinitely long "channels", an "in channel", and an "out channel" have been dug in a cellular island. The center lines of the channels are labelled $L^0$ and $L^i$ respectively and have the property $\overline{L^0} - L^0 = \text{bdry}(N) = \overline{L^i} - L^i$. The set of accessible points of $N$ has two arc components $J_1$ and $J_2$.

(1.3) Theorem. *There does not exist a homeomorphism* $k$ *of the plane into itself that leaves* $N$ *invariant and is such that the image of small arcs that cut across the out channel have images that cut across the out channel further out and small arcs that cut across the in channel have images that cut across the in channel further in* (see Fig. 1.2).



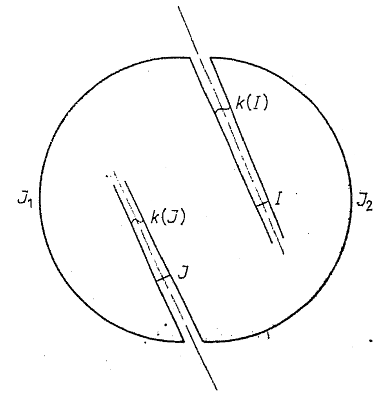


Fig. 1.2

Proof. Suppose such a homeomorphism exists. Then either the situation depicted in Figure 1.3 or the situation depicted in Figure 1.4 must occur. In each figure a simple closed curve $D$, that surrounds $N$, is constructed using a small arc $A^0$ that cuts across the out channel, a small arc $A^i$ that cuts across the in channel, and parts of $J_1$ and $J_2$. A small arc $P$ with one endpoint on the out channel and the other endpoint in $N$ is chosen so that $P$ separates $N$ and no subarc of $P$, $k(P) = P_1$, or $k(P_1) = P_2$ cuts across the out channel up to $A^0$ or the in channel up to $A^i$. Then either $k(J_1) = J_1$, $k(J_2) = J_2$ and Figure 1.3 is accurate or $k(J_1) = J_2$, $k(J_2) = J_1$ and Figure 1.4 is accurate.

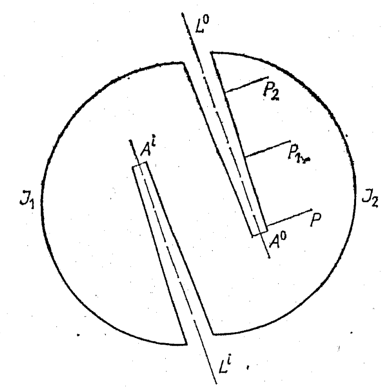


Fig. 1.3

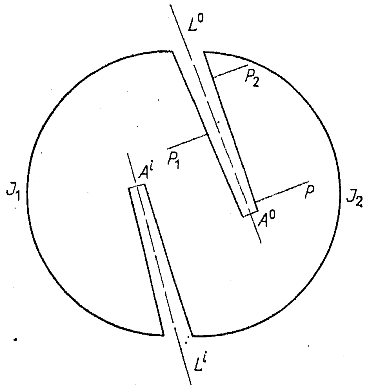


Fig. 1.4

Now since $P$, $P_1$, and $P_2$ separate $N$, $P-N$, $P_1-N$, and $P_2-N$ must each contain infinitely many components that cut across the in channel. Let $C_1$ be the first component of $P_1-N$ that cuts across the in channel. Clearly, $C = k^{-1}(C_1)$ is a component of $P-N$ that cuts across the in channel before $C_1$ does. Notice that there are no components of $P_2-N$ that cut across the in channel before $C_1$ does.

It follows that either Figure 1.5 or Figure 1.6 is accurate. In either case the Jordan Curve Theorem dictates that the out channel must pass through $P$ on its way to $P_2$. Let $C^0$ be the first component of $P-N$ that cuts across the out channel. Then $k(C^0)$ is a component of $P_1-N$ that cuts across the out channel before $C^0$ does and

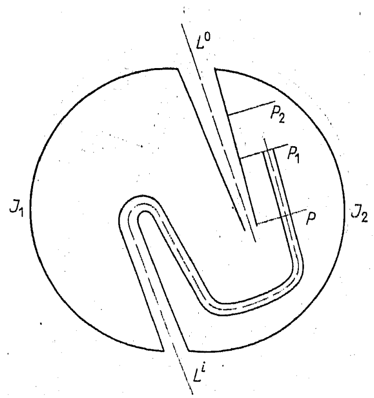


Fig. 1.5



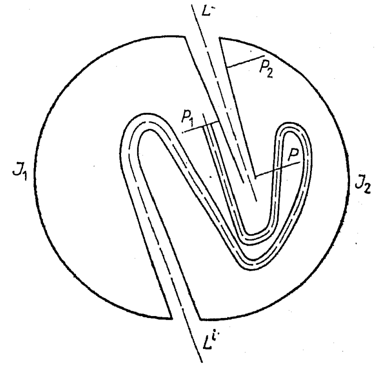


Fig. 1.6

$k(k(C^0))$ is a component of $P_2-N$ that cuts across the out channel before $k(C^0)$ does. That is, the out channel must pass through $P$ before it can pass through $P_2$, it must pass through $P_1$ before it can pass through $P$, (thus eliminating Figures 1.3 and 1.5 as possibilities), and it must pass through $P_3$ before it can pass through $P_2$, clearly an impossible situation.

**Section II. Some constructions and definitions.**

(2.1) *The number* $r_0$. Let $r_0>0$ be chosen such that if $A$ is any set of diameter $\leqslant 2r_0$, then the distance from $A$ to the convex hull of $h(A)$ is at least $r_0$.

(2.2) *Topological ray.* A set $L$ will be called a *topological ray* if there is a homeomorphism $r$ of $[0, \infty)$ onto $L$ such that $\lim_{t\to\infty}|r(t)| = \infty$. The point $r(0)$ will be called the *endpoint of the topological ray*. Sometimes the homeomorphism $r$ shall also be referred to as a ray.

(2.3) The distance from a point $p$ to a non-empty set $A$,

$$d(p, A) = \inf\{|p-a|:\ a\in A\}.$$

(2.4) $S(p, A)$. Let $S(p, A) = \{x:\ d(x, A) = |x-p|\}$.

(2.5) *The sets* $E(M)$ *and* $E'(M)$. A point $e$ is in $E(M)$ if $M\cap S(e, M)$ has at least two points, $e$ is in $E'(M)$ if $M\cap S(e, M)$ has at least three points.

(2.6) *The sets of chords* $\mathscr{C}_e(M)$ *and* $\mathscr{C}(M)$. If $e\in E(M)$, then $\mathscr{C}_e(M)$ is the set of open intervals $(a, b)$ for which there is a component of $S(e, M)-M$ with endpoints $a$ and $b$. An open interval $(a, b)$ is in $\mathscr{C}(M)$ if $(a, b)\in\mathscr{C}_e(M)$ for some $e\in E(M)$ or there is a sequence of such open intervals $(a_n, b_n)$ with $\lim a_n = a$ and $\lim b_n = b$.

(2.7) *The number* $r'$. Let $r'>0$ be chosen so that

$$T(\{z:\ d(z, M)\leqslant r'\})\subset\{z:\ d(z, M)\leqslant r_0\}.$$

(2.8) DEFINITION OF $\mathscr{C}(n)$. For each positive integer $n$, let $\mathscr{C}(n) = \{I: I \in \mathscr{C}_e(M)$ for some $e$ with $d(e, M) \leqslant r'/n\}$

(2.9) DEFINITION OF $Y$. Let $Y = T(\bigcup \{I \in \mathscr{C}(1): I \cap T(h(I) \cup M) = \emptyset\} \cup M$.

(2.10) DEFINITION OF $Y'$. Let $Y' = T(\bigcup \{I \in \mathscr{C}(1): I \cap T(h(I) \cup M) = \emptyset$ and $I \cap T(h^{-1}(I) \cup M) = \emptyset\} \cup M)$.

**Section III. The set $Y$ is not a two-cell.**

(3.1) *Let $I_n = (a_n, b_n) \in \mathscr{C}(1)$ for $n = 1, 2, 3, \ldots$ be such that $a = \lim a_n$ and $b = \lim b_n$ exist. If $a \neq b$ then*

(i) $(a, b) \in \mathscr{C}(1)$.

(ii) *If $I_n \cap T[h(I_n) \cup M] = \emptyset$ for each $n$, then*

$$(a, b) \cap T[h([(a, b)] \cup M)] = \emptyset .$$

(iii) *If $I_n \subset T[h(I_n) \cup M]$ for each $n$, then $(a, b) \subset T[h[(a, b)] \cup M]$.*

Proof. (i) was proven in [2, Corollary 2.4.1]. (ii) and (iii) are straightforward.

(3.2) *The $\{I: I \in \mathscr{C}(1)$ and $I \cap T(h(I) \cup M) = \emptyset\} \cup M$ is closed.*

Proof. In [2, Lemma 2.3] it was shown that $\bigcup \{I: I \in \mathscr{C}(1)\} \cup M$ is closed. It follows that if

$$p \in \overline{\bigcup \{I: I \in \mathscr{C}(1) \text{ and } I \cap T(h(I) \cup M) = \emptyset\} \cup M},$$

then $p \in M$ or there is a sequence of $I_n = (a_n, b_n)$ in $\mathscr{C}(1)$ where

$$I_n \cap T(h(I_n) \cup M) = \emptyset$$

for each $n$ and $p \in (a, b) = (\lim a_n, \lim b_n) \in \mathscr{C}(1)$. According to (3.1),

$$(a, b) \cap T(h((a, b)) \cup M) = \emptyset .$$

It follows that $(a, b) \subset Y$.

(3.3) *The continuum $Y$ is not a two-cell.*

Proof. Suppose $Y$ is a two-cell. Then the boundary of $Y$, $D$, is a simple closed curve. For each $p \in D \cap M$ there is a ray $L$ with endpoint $p$ such that $L \cap Y = \{p\}$. Since $h(p) \notin L$ there is an open set $U$ that contains $p$ such that $h(U) \cap L = \emptyset$. Let $C_p$ be the component of $U \cap D$ that contains $p$. If $ab$ is any arc in $C_p$ with endpoints $a, b \in M$ and $A_{ab}$ is any arc in $Y$ that joins $h(a)$ to $h(b)$ and does not intersect $ab$, then $T(A \cup h(A_{ab})) \cap ab = \emptyset$. It follows from the compactness of $D$ that there is a partition of $D$ $\{x_0, x_1, \ldots, x_n = x_0\} \subset M$ such that each arc $x_{i-1}x_i$ is either contained in some $C_p$ with $p \in D \cap M$ or $x_{i-1}x_i = (x_{i-1}, x_i) \in \mathscr{C}(1)$. According to (1.1) $h$ must have a fixed point in $Y$ if for each $i$ there is an arc $A_i \subset Y$ that joins $h(x_{i-1})$ to $h(x_i)$ such that $T(A_i \cup h(x_{i-1}x_i)) \cap x_{i-1}x_i = \emptyset$. If $x_{i-1}x_i \subset C_p$ for some $p \in D \cap M$, then let $A_{x_{i-1}x_i} = A_i$. If on the other hand,

$$x_{i-1}x_i = (x_{i-1}, x_i) \in \mathscr{C}(1)$$



then $x_{i-1}x_i \cap T(M \cup h(x_{i-1}x_i)) = \emptyset$. Let $J$ be a ray that intersects $x_{i-1}x_i$ but does not intersect $T(M \cup h(x_{i-1}x_i))$. Let $Q$ be a simple closed curve such that $M \subset T(Q) \subset Y$ and $(J \cup x_{i-1}x_i) \cap Q = \emptyset$. Let $A_i$ be an arc in $T(Q)$ that joins $h(x_{i-1})$ to $h(x_i)$ and does not contain $x_{i-1}$ or $x_i$. Then

$$x_{i-1}x_i \cap T(h(x_{i-1}x_i) \cup A_i) \subset (L \cup x_{i-1}x_i) \cap T(h(x_{i-1}x_i) \cup Q) .$$

Since $L \cup x_{i-1}x_i$ is unbounded, connected and disjoint from $h(x_{i-1}x_i) \cup Q$ it follows that $(L \cup x_{i-1}x_i) \cap T(h(x_{i-1}x_i) \cup Q) = \emptyset$. Since $h$ is assumed to be fixed point free in $Y$, it follows that $Y$ is not a two-cell.

**Section IV. There is an out channel for $Y$, an in channel for $Y'$ and a fixed point for $h$.**

(4.1) *The continuum $M$ has no cutpoints.*

Proof. In [1] it was shown the boundary of $M$ must be an indecomposable continuum from which it follows easily that $M$ has no cutpoints.

(4.2) DEFINITION. $\{D(e): e \in E'(M)\}$. For $e \in E'(M)$, let $D(e)$ be the interior of the convex hull of $M \cap S(e, M)$.

(4.3) It was shown [2, Theorem 2.2] that the convex hull of $M$ is the disjoint union of $M$, the open intervals in $\mathscr{C}(M)$, and the open two-cells $D(e)$, $e \in E'(M)$.

(4.4) DEFINITION OF $D_n$ AND $D'_n$. For each positive integer $n$, let $D_n$ be the boundary of $T(\bigcup \{I: I \in \mathscr{C}(n)\} \cup M)$. Let $D'_n$ be the boundary of $T(Y \cup D_n)$.

(4.5) *Each $D_n$ and $D'_n$ is a simple closed curve.*

Proof. In [2, Theorem 4.4] it was shown if $M$ has no cut points and $A$ is a compact set for which $M \subset (T(A))^0$ then the boundary of $T(\bigcup \{I: I \in \mathscr{C}_e(M)$ for some $e \in A\} \cup M)$ is a simple closed curve. If $A_n = \{x: d(x, M) \leqslant r'/n\}$, then it follows from the definition of $\mathscr{C}(n)$ that $D_n$ is a simple closed curve. It was also shown [2, Theorem 4.3] that if $\mathscr{C} \subset \mathscr{C}(M)$ and $\bigcup \{I: I \in \mathscr{C}\} \cup M$ contains a simple closed curve $D$ such that $M \subset T(D)$, then the boundary of $T(\bigcup \{I: I \in \mathscr{C}\} \cup M)$ is a simple closed curve. It follows that $D'_n$ is a simple closed curve.

(4.6) *There is an $I'_1 = (a', b') \in \mathscr{C}(1)$ such that $I'_1 \subset D'_1 - Y$. Fix $e'_1 \in E(M)$ such that $d(e'_1, M) \leqslant r'$ and $I'_1 \in \mathscr{C}_{e'_1}(M)$.*

Proof. Combine (3.3) and (4.5).

(4.7) *Let $e \in E'(M)$ with $d(e, M) \leqslant r_0$. Then $Y$ contains all but at most two $I \in \mathscr{C}_e(M)$.*

Proof. Given any three intervals in $\mathscr{C}_e(M)$ there will be two of them say $I$ and $J$ for which $[T(I \cup M) - M] \cap [T(J \cup M) - M] = \emptyset$. Since $h$ is a homeomorphism it follows that

$$[T(h(I) \cup M) - M] \cap [T(h(J) \cup M) - M] = \emptyset .$$

If $I \not\subset Y$ then $I \subset T(h(I) \cup M)$. Since $I \cup D(e) \cup J$ is connected and $h(I) \cap \overline{D(e)} = \emptyset$, it follows that

$$J \subset I \cup D(e) \cup J \subset T(h(I) \cup M) - M .$$

Therefore,

$$\emptyset = J \cap T(h(J) \cup M) - M = J \cap T(h(J) \cup M).$$

It follows that $J \subset Y$.

(4.8) *Let $e \in E'(M)$ with $D(e) \subset T(D_1)$. Then either $Y$ contains every $I \in \mathscr{C}_e(M)$ or there are exactly two intervals in $\mathscr{C}_e(M)$ that are not contained in $Y$.*

Proof. According to (4.7) there are at most two $I \in \mathscr{C}_e(M)$ not contained in $Y$. Suppose $I \in \mathscr{C}_e(M)$ and $I \not\subset Y$. Then $I \subset T(h(I) \cup M)$. Write the boundary of $D(e)$ as $I \cup K$ where $K$ is an arc with the same endpoints as $I$. A straightforward application of the Jordan curve theorem will show that $I \subset T(h(K) \cup M)$. Then, according to [1, Lemma 7] there is a component of $h(K)-M$, $P$, such that $I \subset T(P \cup M)$. Since $h$ is a homeomorphism it follows that $P = h(J)$ for some component $J$ of $K-M$. Clearly, $J \in \mathscr{C}_e(M)$ and $J \subset T(h(J) \cup M) = T(P \cup M)$. Therefore $J \not\subset Y$.

(4.9) Definition. *A partial order on $E(M)$ and $\mathscr{C}(M)$.* For $e, f \in E(M)$ define $e \leqslant f$ if there are $a, b \in M \cap S(f, M)$ such that $e \in T[(a, f] \cup [f, b) \cup M]$. If $I, J \in \mathscr{C}(M)$ define $I \leqslant J$ to mean $T(I \cup M) \subset T(J \cup M)$.

(4.10) Definition. *The topological rays $L_e$.* For $e \leqslant E(M)$ let

$$L_e = \{f \in E(M) \colon e \leqslant f\}.$$

From [2, Theorem 3.1] we have

(4.11) *Each $L_e$ is a topological ray with endpoint $e$.*

From [2, Lemma 3.4] we have

(4.12) *Let $e, f \in E(M)$. If $e \leqslant f$ then there exists an $I \in \mathscr{C}_e(M)$ and a $J \in \mathscr{C}_f(M)$ such that $I \leqslant J$. If there is an $I \in \mathscr{C}_e(M)$ and a $J \in \mathscr{C}_f(M)$ such that $I < J$, then $e \leqslant f$.*

(4.13) *The $\{I \in \mathscr{C}(1) \colon I \leqslant I_1' \text{ and } I \not\subset Y\}$ is totally ordered by $\leqslant$.*

Proof. Suppose by way of contradiction that there exist $I, J \in \mathscr{C}(1)$ such that $I, J \leqslant I_1'$, $I, J \not\subset Y$, $I \not\leqslant J$ and $J \not\leqslant I$. Let $e, f \in E(M)$ such that $I \in \mathscr{C}_e(M)$ and $J \in \mathscr{C}_f(M)$. According to (4.12), $e_1' \in L_e \cap L_f$. Let $e'$ be the least element in $L_e \cap L_f$. Clearly, $e' \in E'(M)$ and there are $K_1$, $K_2$, and $K_3 \in \mathscr{C}_{e'}(M)$ such that $I \leqslant K_1 \leqslant K_3$ and $J \leqslant K_2 \leqslant K_3 \leqslant I_1'$. It follows that none of the $K_i$ are contained in $Y$, which contradicts (4.7).

(4.14) *For each positive integer $n$ there is a unique $I_n' \in \mathscr{C}(1)$ such that $I_n' \subset D_n'$, $I_n' \leqslant I_1'$, and $I_n' \not\subset Y$.*

Proof. Let $\mathscr{C} = \{I \in \mathscr{C}(1) \colon I \leqslant I_1',\ I \not\subset Y \text{ and } I \not\subset T(D_n')\}$. By (4.13) $\mathscr{C}$ is totally ordered by $\leqslant$. A straightforward application of (3.1) shows that $\mathscr{C}$ has a greatest lower bound, say $J$. If $J \subset D_n'$, let $I_n' = J$. If not (4.3) implies that $J \in \mathscr{C}_e(M)$ for some $e \in E'(M)$. In this case (4.7) indicates that there is a $K \in \mathscr{C}_e(M)$ such that $K < J$ and $K \not\subset Y$. Clearly $K \subset D_n'$. Let $I_n' = K$. Since no two $I \in \mathscr{C}(1)$ contained in the simple closed curve $D_n'$ are comparable it follows that $I_n'$ is unique.

(4.15) Definition. Let $I_n'$ be as in (4.13) and for each $i$ let $e_1'$ be such that $I_n' \in \mathscr{C}_{e_i'}(M)$.



(4.16) Definition. *The center line $L^0$.* Let $L^0 = \bigcup \{L_{e_i'} \colon i = 1, 2, \ldots\}$.

(4.17) *$L^0$ is homeomorphic to the set of real numbers.*

Proof. By (4.12) the $\{e_i' \colon i = 1, 2, \ldots\}$ is totally ordered by $\leqslant$. It follows that $L^0$ is either a topological ray or is homeomorphic to the set of real numbers. Since $\lim_{i \to \infty} d(e_i', M) = 0$ it follows that $L^0$ is not a topological ray.

(4.18) Definition. *The out channel.* Let $U$ be the union of those $I \in \mathscr{C}(M)$ and those $D(e)$, $e \in E'(M)$ that intersect $L^0$. An arc $ab$ contained in $R^2 - M$, except for the endpoints $a, b \in M$ will be said to cut across $L^0$ if $T(ab \cup M)$ contains $\{(e \in L^0 \colon e \leqslant f\}$ for some $f \in L^0$.

(4.19) *$\overline{L^0} - L^0$ is an invariant subcontinuum of* bdry$(M)$. *Therefore, $\overline{L^0} - L^0$ is the boundary of $M$.*

Proof. $\overline{L^0} - L^0$ is clearly a subcontinuum of bdry$(M)$. Let $p \in \overline{L^0} - L^0$ and let $V$ be an open neighborhood of $h(p)$. Since $h^{-1}(V)$ is a neighborhood of $p$, $h^{-1}(V)$ must contain an $I \in \mathscr{C}(1)$ that cuts across $L^0$. Then $I \cup L^0$ is unbounded, connected, and does not intersect $M$. Since $I \subset T(h(I) \cup M)$ it follows that $h(I) \cap L^0 \neq \emptyset$. Therefore $h(h^{-1}(V)) \cap L^0 = V \cap L^0 \neq \emptyset$. Therefore, $T(\overline{L^0} - L^0)$ is an invariant non-separating subcontinuum of $M$. Since $M$ is a minimal invariant non-separating continuum it follows that $M = T(\overline{L^0} - L^0)$, from which it follows that $\overline{L^0} - L^0 = \text{bdry}(M)$.

It seems clear that

(4.20) *If $A$ is an arc of diameter $\leqslant r'$ that cuts across $L^0$, then*

$$(A - M) \subset T(h(A) \cup M).$$

(4.21) *There is an "in channel". That is, there is a subset of $E(M)$ $L^i$ such that $L^i$ is homeomorphic to the set of real numbers, if $A$ is any arc of diameter $\leqslant r'$ that cuts across $L^i$, then $h(A) \subset T(A \cup M)$ and $\overline{L^i} - L^i =$* bdry$(M)$.

Proof. We have shown that if $h$ has no fixed point, then there must be an "out channel" for $h$ and $M$. If $h$ has no fixed point then neither does $h^{-1}$. Since $M$ is also a minimal invariant non-separating continuum for $h$ it is also a minimal invariant non-separating continuum for $h^{-1}$. It follows that $h^{-1}$ must have an "out channel". Clearly, an "out channel" for $h^{-1}$ is an "in channel" for $h$.

(4.22) *There is a fixed point in $M$.*

Proof. Let $Y'' = T(\bigcup \{I \in \mathscr{C}(1) \colon I \cap (L^0 \cup L^i) = \emptyset\} \cup M)$. Then $Y''$ pretty much resembles the continuum $N$ of (1.3). That is, $Y''$ has the required "in channel" and "out channel". Although $Y''$ is not an invariant continuum and $\overline{L^0} - L^0$ is not all of $Y''$, the procedure used to prove (1.3) is still operative. For $A^0$ choose an $I_n'$ for which $h(h(I_n')) \subset T(I_1' \cup M)$. Let $A^i$ be the largest $I \in \mathscr{C}(1)$ that intersects $L^i$. Finally, choose the arc $P$ so that $P$ separates $M$, so that it shares an endpoint with $A^0$ and so that $P$, $h(P)$, and $h(h(P))$ do not intersect $(L^0 \cup L^i) \cap (R^2 - T(D))$. The proof proceeds as before.

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# Note on decompositions of metrizable spaces II

by

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**Abstract.** This paper is a continuation of the author's paper [16]. We improve some results from [16] and investigate the special decompositions of metrizable spaces introduced in [16] which establish close relations between A. H. Stone's [20] property $\sigma\mathrm{Lw}(<\mathfrak{k})$ and stationary sets of ordinals. On this ground we construct decompositions of Baire spaces $B(\mathfrak{k})$ which yield results on absolutely $\mathfrak{k}$-analytic spaces (considered by A. H. Stone [19]) and give, under an additional set theoretic axiom, the negative answer to a question raised in [16]. Connections between these topics and non-separable theory of Borel sets are also investigated.

This paper is a continuation of our paper [16]. In the first section we prove a theorem on $\sigma$-discrete reduction which improves a result from [16] and a proposition on completely additive-Borel families which extends an important R. W. Hansell's theorem [9]; these results together give a reduction theorem in non-separable theory of Borel sets which yields a selection theorem.

In the second section we investigate the special decompositions of metrizable spaces introduced in [16] (we call them "natural") which allow to establish close relations between $\sigma\mathrm{Lw}(<\mathfrak{k})$ property (considered by A. H. Stone [20]) and the notion of stationary sets of ordinals and we consider the class of mappings preserving $\sigma$-discretness which is closely related to these topics.

In the third section we apply some of results of Section 2 to obtain special decompositions of $B(\mathfrak{k})$ (i.e. the countable product of discrete spaces of cardinality $\mathfrak{k}$) which generalize the classical F. Bernstein's decompositions of irrationals $B(\aleph_0)$ into totally imperfect sets. These decompositions yield a theorem on absolutely $\mathfrak{k}$-analytic spaces (introduced by A. H. Stone [19]) and, under an additional set theoretic axiom, provide an example which settles a problem raised by the author in [16].

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**Notation and terminology.** Our topological terminology follows [3] and [12]; set theoretic terminology is taken from [13] — with the only exception — a regular cardinal is always understanding to be uncountable. *By a space we shall mean in this paper always a metrizable space.* Given a space $X$ we denote by $\varrho$ a metric agreeing