Stable graphs

by

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Abstract. We show for a simple class of graphs that there is no definable ordering of an infinite set of $n$-tupels of vertices. This class contains all planar graphs and all graphs of finite valency. A major step is the proof of the equivalence of two graph-theoretical notions.

Introduction. A graph has a definable ordering if there is a graph-theoretical formula $\varphi$, an infinite set $\mathbb{A}$ of $k$-tupels of vertices and a linear ordering $<$ of $\mathbb{A}$ s.t. two elements $\bar{a}, \bar{b} \in \mathbb{A}$ satisfy $\varphi$ iff $\bar{a} < \bar{b}$.

In [1] it is shown that no tree, no $n$-separated graph and no graph of finite valency has an ordering of $1$-tupels. By a refinement of the method in [1] we prove that no graph has a definable ordering which satisfies the following property:

(*) For every infinite set $U$ of vertices and every natural number $m$ there is a finite set $S$ of vertices and an infinite $U' \subset U$ s.t. all paths connecting two elements of $U'$ of length smaller than $m$ contain an element of $S$.

We do not see any reasonable weakening of (*) from which we can derive the same result. So it is surprising that (*) holds exactly in those graphs which contain no bounded subdivision of edges of a complete infinite graph, which is a simple and easy to handle property. We call such graphs flat. Since every subdivision of an infinite complete graph is neither a tree nor of finite valency, $n$-separated, planar or embeddable in a surface of finite genus, all such graphs are flat. It seems difficult to find a reasonable graph-theoretical property which extends flatness and implies the nonexistence of a definable ordering.

A model-theoretic property which is connected with definable orderings is stability [3]. The following sharpening of flatness implies stability. For each natural number $m$ there is a natural number $n$ s.t. no subdivision — by fewer than $m$ many points on each edge — of the complete graph with $n$ vertices is contained in the graph. This graphs are called superflat. Since every tree, every graph with bounded valency, every $n$-separated graph, every planar graph and every graph which is embeddable in a surface of finite genus is superflat, they are all stable.

Flat graphs. A graph is a structure $(E, K)$, where $K$ is a binary irreflexive and symmetric relation on $E$. A graph $(F, L)$ is called a subgraph of $(E, K)$ if $F \subseteq E$
and $K \subseteq L$. If $S \subseteq E$, we denote by $(E, K) - S$ the largest subgraph of $(E, K)$ which contains no elements of $S$.

Let $n$ be a natural number, then "a" is the set of all sequences of length $n$ of elements of $A$. Such a sequence is a function from $\{0, 1, ..., n-1\}$ to $A$. A subgraph $(\mathcal{Q}, W)$ of $(E, K)$ is said to be a path of length $n$ from $a$ to $b$ (in $(E, K)$) if there is an injective sequence $\bar{a}$ of length $n + 1$ s.t. $\bar{a}(0) = a, \bar{a}(n) = b, \mathcal{Q} = \{\bar{a}(i) \mid i \leq n\}$ and

$$W = \bigcup_{i+1}^{n} \{(\bar{a}(i), \bar{a}(i+1)), (\bar{a}(i+1), \bar{a}(i))\}.$$ Two different paths $(\mathcal{Q}_i, W_i), i = 1, 2$ from $a_i$ to $b_i$ are called disjoint if $\mathcal{Q}_1 \cap \mathcal{Q}_2 = \{a_1, b_1, c_1, a_2, b_2\}$.

Let $S$ be a subset of $E$ and let $a, b \in E$. We define $d_\mathcal{Q}(a, b)$ to be the minimum of the lengths of paths from $a$ to $b$ in $(E, K) - S$, if there is such a path, $d_\mathcal{Q}(a, b) = \infty$ otherwise. Note that $d_{\mathcal{Q}}(a, b) = \infty$ if $a \in S$ or $b \in S$. For $\bar{a}, E \subseteq E$ we define

$$d_{\mathcal{Q}}(\bar{a}, b) = \min \{d(\bar{a}(i), b) \mid i, k < n\}.$$ Let $m$ be a natural number and $\lambda$ a cardinal, then $K^m_\lambda$ is the subdivision of the complete graph with $\lambda$ many vertices obtained by inserting $m$ new vertices on each edge.

![Fig. 1](image)

The following property (b) of graphs will be important for later discussions.

(b) For every infinite $U \subseteq E$ and every natural number $m$ there is a finite $S \subseteq E$ and an infinite $U' \subseteq U$ s.t. for all different $a, b \in U'$:

$$d_{\mathcal{Q}}(a, b) = m.$$ This will be equivalent to the following property.

1. DEFINITION. A graph $(E, K)$ is called flat if no subgraph is isomorphic to $K^m_\lambda$ for any natural number $m$.

$(E, K)$ is called superflat if for every natural number $m$ there is a natural number $n$ such that no subgraph of $(E, K)$ is isomorphic to $K^m_\lambda$.

2. THEOREM. A graph has the property (b) iff it is flat.

Proof. Since no $K^m_\lambda$ has the property (b), we have that (b) implies flatness. To prove the other direction assume that there is an infinite $U \subseteq E$ and a natural number $m$ s.t.

For every infinite $U' \subseteq U$ and finite $S \subseteq E$ there are two distinct $a, b \in U'$ with $d_{\mathcal{Q}}(a, b) = m$.

We first observe:

LEMMA. For every infinite $U \subseteq U$ and finite $S \subseteq E$ there is $c \in E \subseteq S$, an infinite $U'' \subseteq U'$ and for every $a, b \in U''$ a path $(\mathcal{Q}_a, W_a)$ from $c$ to $a$ of length $\leq m$ such that $(\mathcal{Q}_a, W_a)$ and $(\mathcal{Q}_b, W_b)$ are disjoint for $a \neq b$.

Proof. Since there is no infinite $U' \subseteq U'$ such that $d_{\mathcal{Q}}(a, b) > m$ for all distinct $a, b \in U'$, Ramsey's theorem yields an infinite $U'' \subseteq U''$ s.t.

$$d_{\mathcal{Q}}(a, b) = m$$ for all $a, b \in U''$.

Let $a \in U''$ and let $F'' = \{b \in E \mid d_{\mathcal{Q}}(a, b) = m\}$. For every $b \in F''$, $b \neq a$, we choose an $c_k \in F''$ s.t. $d_{\mathcal{Q}}(a, c_k) < d_{\mathcal{Q}}(a, b)$ and $(c_k, a) \in L$. Let

$$L'' = \bigcup_{a \in F''} \{(c_k, a), (b, c_k)\}.$$ Then $(F', L')$ is a tree. Let $(F, L)$ the largest subtree whose endpoints are elements of $U''$.

Since the distance in $(F', L')$ of two vertices is smaller than $2m+1$ there must exist a $c_0 \in F$ s.t. $x := \{d \mid c = c_d\}$ is infinite. For every $d \in x$ we choose a path $(\mathcal{Q}_d, W_d)$ in $(F, L)$ from $c$ to an endpoint $a_d$ s.t. $d \in \mathcal{Q}_d$. Let $U'' = \{a_d \mid d \in x\}$. Then $c$ and $U''$ have the desired properties.

Now we can continue with the proof of the theorem. Using the preceding lemma we choose vertices $c_{i_0}, c_{i_1}, ..., c_{i_{m-1}}$ infinite set $U_i \subseteq U_{i-1}$ $(U_0 \subseteq U)$ and for every $a \in U_i$ a path $(\mathcal{Q}_a, W_a)$ of length $\leq m$ s.t. $(\mathcal{Q}_a, W_a)$ and $(Q_b, W_b)$ are disjoint for $a \neq b \in U_i$. Then we construct subgraphs $(F_{i_n}, L_{i_n})$ s.t.

a) $(F_n, L_n)$ is a subdivision of the complete graph with the vertices $c_{i_0}, ..., c_{i_n}$ by fewer than $2m+1$ vertices.

b) $(F_{i_n}, L_{i_n})$ is a subgraph of $(F_{i-1}, L_{i-1})$ as follows:

Let $F_{i-1} = \emptyset, L_{i-1} = \emptyset$. Suppose $(F_{i-1}, L_{i-1})$ is already chosen. Let $t$ be greater then all $i$ with $c_{i_{t-1}}, c_{i_{t+1}}$. To connect $c_{i_t}$ with $c_{i_k}$, $k < n$, we choose subgraphs $(F_{i_t}^1, L_{i_t}^1), k < n$, in the following manner:

Let $F_{i_t}^1 = F_{i_{t-1}} \cup \{c_{i_t}\}$ and let $L_{i_t}^1 = L_{i_{t-1}}$. Assume $(F_{i_t}^1, L_{i_t}^1)$ is chosen. Then there exists $a, b \in U_{i_t}^1$ s.t.

$$\mathcal{Q}_{i_t}^1 \cap F_{i_t}^1 = \{c_{i_t}\} \quad \text{and} \quad \mathcal{Q}_{i_t}^1 \cap F_{i_t}^1 = \{c_{i_t}\}.$$ Let $(\mathcal{Q}, W)$ the path from $c_{i_t}$ to $c_{i_k}$ s.t. $W \subseteq W_{i_{t-1}} \cup W_{i_{t+1}}$, and define

$$F_{i_t}^2 = F_{i_{t-1}} \cup Q, \quad L_{i_t}^2 = L_{i_{t-1}} \cup W.$$ Let $F_t = F_{i_t}^2$ and $L_{i_t} = L_{i_t}^2$. Then the graph $(F_t, L_t)$ has properties a) and b).

To finish the construction define

$$F' = \bigcup_{n=1}^{m} \{F_n\} \quad \text{and} \quad L' = \bigcup_{n=1}^{m} \{L_n\}.$$
An easy application of Ramsey's theorem shows that there is an \( e < 2m \) and a subgraph \((F,L)\) of \((F',L')\) which is isomorphic to \( K^*_m \). This proves the theorem.

3. **Corollary.** Let \((E,K)\) be a flat graph, \( m \) a natural number and \( A \in E \) infinite. Then there is an infinite \( B \in A \) and a finite \( S \in E \) s.t. for all distinct \( \bar{a}, \bar{b} \in B \):

\[
d_d(\bar{a}, \bar{b}) > m.
\]

**Proof.** By recursion on \( e \leq n \) we choose infinite \( A_e \in A \) and finite \( S_e \in E \) as follows:

Let \( A_0 = A \) and \( S_0 = \emptyset \). Suppose that \( A_e \) and \( S_e \) are chosen. If

\[
U = \{(\bar{a}, e) \mid \bar{a} \in A_e\}
\]

is finite, let \( S_{e+1} = S_e \cup U \) and \( A_{e+1} = A_e \). Otherwise, since \((E,K)\) is flat, there is an infinite \( U' \subset U \) and a finite \( S_{e+1} \in E \) s.t. for all distinct \( \bar{a}, \bar{b} \in U', \bar{a} \neq \bar{b} \in U' \) we have \( d_{d}(\bar{a}, \bar{b}) > m \).

Let \( A_{e+1} \) be an infinite subset of \( A_e \) s.t. for all distinct \( \bar{a}, \bar{b} \in A_{e+1} \) we have \( d_{d}(\bar{a}, \bar{b}) > m \). Then

\[
\text{if } S = \bigcup_{\bar{a} \in A} \text{ we have for all distinct } \bar{a}, \bar{b} \in A_{e+1} \text{ and for all } k < n:
\]

\[
d_d(\bar{a}(k), \bar{b}(k)) > m.
\]

By induction we choose elements \( \bar{a}_e \in A_e \), as follows:

Let \( \bar{a}_0 \in A_0 \). Suppose that \( \bar{a}_j, j < i \), is already chosen. Then we choose \( \bar{a}_{i+1} \in A_i \) such that

\[
d_d(\bar{a}_{i+1}, \bar{a}_j) < m \quad \text{for } j < i.
\]

If such an element does not exist, the infinite \( A_e \) must contain \( \bar{a}, \bar{b} \) s.t. for some \( j < i, j < k < n \) we have:

\[
d_d(\bar{a}(k), \bar{a}(j)) < m, \quad d_d(\bar{a}(k), \bar{b}(j)) < m.
\]

This implies \( d_d(\bar{a}(j), \bar{b}(j)) < 2m \), which is a contradiction to the construction of \( A_e \).

**Graphs with definable orderings.** The (first order) language \( L' \) of the theory of graphs contains besides the logical symbols a binary relation symbol \( P \). Let \( X' \) be a set, then \( L' \) denotes the language \( L' \) extended by using the elements of \( X \) as constant symbols. Let \( V \) be the set of variables, let \( \varphi \) be a formula from \( L' \) and let \( V_i \in \text{"} V \text{"} \) s.t. \( \varphi[V_i] \in L \) for all \( i \neq j \) \( (m \neq i) \). If every free variable of \( \varphi \) is equal to some \( V_i \), \( i < k, k < n \), we write \( \varphi(V_1, ..., V_k) \). This notion indicates how to substitute constants:

Let \( \bar{a} \in \epsilon \text{"} X \text{"} \), then \( \varphi(\bar{a}_1, ..., \bar{a}_k) \) denotes the sentence from \( L' \), which is obtained from \( \varphi(\bar{a}_1, ..., \bar{a}_k) \) by substituting \( V_i \) by \( \bar{a}_i \).

If \((E,K)\) is a graph and \( f \) a function from \( X \) to \( E \), then \((E,K,f(\bar{x}))_{\text{ex}} \) is a structure for \( L' \). For \( \varphi(V_1, ..., V_k) \) from \( L' \) and \( \bar{a} \in \epsilon \text{"} X \text{"} \) let

\[
(E,K,f(\bar{x}))_{\text{ex}} \models \varphi(\bar{a}_1, ..., \bar{a}_k)
\]

express that \( \varphi \) holds in \((E,K,f(\bar{x}))_{\text{ex}} \) if \( f(\bar{a}) \) is interpreted by \( \bar{a} \). Then \((E,K,f(\bar{x}))_{\text{ex}} \) is the set of all sentences of \( L' \) which hold in \((E,K,f(\bar{x}))_{\text{ex}} \).

For example let \( S \) be a finite set, \( V_1, V_2 \) two \( n \)-tuples of variables. Define

\[
\varphi_{\text{ex}}(V_1, V_2)
\]

\[
:= \bigwedge_{l < m} \bigvee_{i < j} (\varphi(V_1(l), V_2(j)), \varphi(V_1(l), V_2(i))) \wedge \varphi(V_2(i), V_1(j)) \wedge \varphi(V_2(j), V_1(i)) \wedge \varphi(V_2(i), V_2(j)) \wedge \varphi(V_2(j), V_2(i))
\]

Then

\[
(E,K,f(\bar{x}))_{\text{ex}} \models \varphi_{\text{ex}}(\bar{a}, \bar{b}) \iff d_{d}(\bar{a}, \bar{b}) > m \text{ in } (E,K).
\]

Similarly we find for all natural numbers \( n, m \) a sentence \( \psi_{\text{ex}} \), s.t. \((E,K) \models \psi_{\text{ex}} \) iff \((E,K)_w \) contains no isomorphic copy of \( K^*_m \).

From this we can derive

4. **Lemma.** \((E,K)\) is superflat iff all graphs \((F,L)\) which are elementary equivalent to \((E,K)\) (i.e., \( \text{Th}(E,K) = \text{Th}(F,L) \)) are flat.

**Proof.** If \((E,K)\) is superflat there is for every \( m \) an \( n \) s.t. \((E,K) \not\models \psi_{\text{ex}} \). This holds also in every \((F,L)\) elementary equivalent to \((E,K)\). So clearly for every \( n \) \( K^*_n \) is not embeddable in \((F,L)\).

The other direction is shown by an easy application of the compactness theorem.

The following notion is important in model theory [3].

5. **Definition.** A formula \( \varphi(V, U) \) is said to define an ordering of the graph \((E,K)\) if there are an infinite \( \bar{A} \in E \) and a linear ordering \( < \) on \( \bar{A} \) s.t. for all \( \bar{a}, \bar{b} \in \bar{A} \):

\[
(E,K) \models \varphi(\bar{a}, \bar{b}) \iff \bar{a} < \bar{b}
\]

\((E,K)\) is called stable if there is no definable ordering in any \((F,L)\) elementary equivalent to \((E,K)\).

It is quite useful to make the following definition:

6. **Definition.** A formula \( \varphi(V, W) \) is called large in a graph \((E,K)\) if there is an infinite \( \bar{A} \in E \) s.t. for every infinite \( B \subset \bar{A} \) there are \( \bar{a}, \bar{b} \in B \) s.t. \((E,K) \not\models \psi(\bar{a}, \bar{b}) \).

For example, if \( \varphi(V, W) \) defines an ordering in \((E,K)\), then \( \varphi(V, W) \wedge \neg \varphi(W, V) \) are large. A major step to prove that every flat graph has no definable ordering, is the following theorem:

7. **Theorem.** Let \( \varphi(V, W) \) be a large formula in a flat graph \((E,K)\). Then there is an extension \( (F,L) \) of \((E,K)\), an automorphism \( h \) of \((F,L)\) and \( \bar{a}, \bar{b} \in \epsilon \text{"} E \text{"} \) s.t.

\[
1. \ h \bar{a} = \bar{b} \text{ and } h \bar{a} = \bar{a}
\]

\[
2. \ d_d(\bar{a}, \bar{b}) = \infty
\]

\[
3. \ (F,L) \not\models \varphi(\bar{a}, \bar{b})
\]

**Proof.** If there is an \( \bar{a} \in \epsilon \text{"} E \text{"} \) s.t. \((E,K) \not\models \psi(\bar{a}, \bar{a})\), then let \( \bar{b} = \bar{a} \) and \((F,L) = (E,K)\). Otherwise, since \( \varphi(V, W) \) is large in \((E,K)\) there is an infinite \( \bar{A} \in E \) s.t. every infinite \( B \subset \bar{A} \) contains two different elements \( \bar{a}, \bar{b} \) s.t. such that

\[
(E,K) \not\models \psi(\bar{a}, \bar{b}) \text{.}
\]

Since \((E,K)\) is flat, we have by Corollary 3 that for every natural number \( m \) there is a finite \( S_{m} \in E \) and an infinite \( A_{m} \in A \) s.t. \( d_{d}(\bar{a}, \bar{b}) > m \) for all distinct \( \bar{a}, \bar{b} \in A_{m} \). Clearly we can assume that \( S_{m} = S_{m+1} \) and \( A_{m} = A_{m+1} \).
Now we extend \( \mathcal{L}_\alpha \) to \( \mathcal{L}_b \) by \( 2n \) new constant symbols which we can arrange in two sequences \( \bar{a}, \bar{b} \) of length \( n \). We define sets \( T_0, T_1, T_2, T_3 \) of sentences of \( \mathcal{L}_b \):

\[
\begin{align*}
T_0 &= \text{Th}(E, K, \varepsilon)_{\text{res}}, \\
T_1 &= \{ \varphi^g_a(\bar{a}, \bar{b}) \mid m \text{ a natural number} \}, \\
T_2 &= \{ \sigma(\bar{a}) \mapsto \sigma(\bar{b}) \mid \sigma \in \mathcal{L}_b \}, \\
T_3 &= \{ \psi(\bar{a}, \bar{b}) \}.
\end{align*}
\]

where \( \varphi^g_a(V, W) \) is the formula defined above, which expresses \( "d_{\text{res}}(V, W) > m" \)

Finally let \( T = T_0 \cup T_1 \cup T_2 \cup T_3 \). First we prove, that \( T \) is consistent:

Let \( \bar{d} \) be a finite set of formulas \( \sigma(\bar{w}) \) from \( \mathcal{L}_b \) and let \( m \) be a natural number. Define

\[
\begin{align*}
T_1 &= \{ \varphi^g_a(\bar{a}, \bar{b}) \mid \bar{r} \in \mathcal{L}_b \}, \\
T_2 &= \{ \sigma(\bar{a}) \mapsto \sigma(\bar{b}) \mid \sigma \in \mathcal{L}_b \}.
\end{align*}
\]

By compactness it suffices to show that \( T = T_0 \cup T_1 \cup T_2 \cup T_3 \) has a model. Since \( A_\alpha \) is infinite, we get by an easy application of Ramsey's theorem an infinite \( B \subseteq A_\alpha \). If

\[
(E, K, \varepsilon) \models \sigma[\bar{a}] \iff (E, K, \varepsilon) \models \sigma[\bar{b}]
\]

for all \( \bar{a}, \bar{b} \in B \) and all \( \sigma \in \mathcal{E} \). Choose two different sequences \( \bar{a}, \bar{b} \in B \) such that \( (E, K) \not\models \psi[\bar{a}, \bar{b}] \) and let \( f \) be the map from \( X \) to \( E \) which satisfies \( \varepsilon \models \alpha \). Then \( (E, K, f(\bar{a})) \) is a model of \( T \) and therefore \( T \) is consistent.

Let \( F, L, \varepsilon(x) \) be a model of \( T \). Since \( T_1 \subseteq T \) we can assume that \( (E, K) \) is an (elementary) subgraph of \( (F, L) \) and \( g \models \text{id}_E \). Let \( \bar{a} = g \circ \bar{a} \) and \( \bar{b} = g \circ \bar{b} \). Since \( T_2 \subseteq T \) and \( \bar{a}, \bar{b} \) satisfy the same formulas of \( \mathcal{L}_b \) in \( (F, L, \varepsilon(x)) \). Therefore using a result of [2, p. 49] we find an elementary extension \( (F', L', \varepsilon(x)) \) of \( (F, L) \) and an automorphism \( h \) of \( (F, L) \) such that \( h \models \text{id}_L \) and \( h \circ \bar{a} = \bar{b} \). Since \( T_3 \subseteq T \) implies \( (F, L) \models \psi[\bar{a}, \bar{b}] \).

**References**


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