Stable graphs

by

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Abstract. We show for a simple class of graphs that there is no definable ordering of an infinite set of n-tupels of vertices. This class contains all planar graphs and all graphs of finite valency. A major step is the proof of the equivalence of two graph-theoretical notions.

Introduction. A graph has a definable ordering if there is a graph-theoretical formula φ, an infinite set $\bar{A}$ of k-tupels of vertices and a linear ordering $<$ of $\bar{A}$ s.t. two elements $\bar{a}, \bar{b} \in \bar{A}$ satisfy φ iff $\bar{a} < \bar{b}$.

In [1] is shown that no tree, no n-separated graph and no graph of finite valency has an ordering of 1-tupels. By a refinement of the method in [1] we prove that no graph has a definable ordering which satisfies the following property:

(∗) For every infinite set U of vertices and every natural number m there is a finite set S of vertices and an infinite $U' \subset U$ s.t. all pathes connecting two elements of U' of length smaller m contain an element of S.

We do not see any reasonable weakening of (∗) from which we can derive the same result. So it is surprising that (∗) holds exactly in those graphs which contain no bounded subdivision of edges of a complete infinite graph, which is a simple and easy to handle property. We call such graphs flat. Since every subdivision of an infinite complete graph is neither a tree nor of finite valency, n-separated, planar or embeddable in a surface of finite genus, all such graphs are flat. It seems difficult to find a reasonable graph-theoretical property which extends flatness and implies the nonexistence of a definable ordering.

A model-theoretic property which is connected with definable orderings is stability [3]. The following sharpening of flatness implies stability. For each natural number m there is a natural number n s.t. no subdivision — by fewer than m many points on each edge — of the complete graph with n vertices is contained in the graph. This graphs are called superflat. Since every tree, every graph with bounded valency, every n-separated graph, every planar graph and every graph which is embeddable in a surface of finite genus is superflat, they are all stable.

Flat graphs. A *graph* is a structure (E, K), where K is a binary irreflexive and symmetric relation on E. A graph (F, L) is called a *subgraph* of (E, K) if $F \subset E$

and $K\subset L$. If $S\subset E$, we denote by (E,K)-S the largest subgraph of (E,K) which contains no elements of S.

Let n be a natural number, then ${}^{n}A$ is the set of all sequences of length n of elements of A. Such a sequence is a function from $\{0,1,\dots,n-1\}$ to A. A subgraph (Q,W) of (E,K) is said so be a *path of length* n from a to b (in (E,K)) if there is an injective sequence $\bar a$ of length $n+1$ s.t. $\bar a(0)=a, \bar a(n)=b, Q=\{\bar a(i)|\ i\leqslant n\}$ and

$$W=\bigcup_{i<n}\{(\bar a(i),\bar a(i+1)),(\bar a(i+1),\bar a(i))\}.$$

Two different paths $(Q_i,W_i)_i$, $i=1,2$ from a_i to b_i are called *disjoint* if

$$Q_1\cap Q_2=\{a_1,b_1\}\cap\{a_2,b_2\}.$$

Let S be a subset of E and let $a,b\in E$. We define $d_s(a,b)$ to be the minimum of the lengths of paths from a to b in (E,K)-S, if there is such a path, $d_s(a,b)=\infty$ otherwise. Note that $d_s(a,b)=\infty$ if $a\in S$ or $b\in S$. For $\bar a,\bar b\in {}^{n}E$ we define

$$d_s(\bar a,\bar b):=\min\{d_s(a(i),b(k))|\ i,k<n\}.$$

Let m be a natural number and λ a cardinal, then K_λ^m is the subdivision of the complete graph with λ many vertices obtained by inserting m new vertices on each edge.

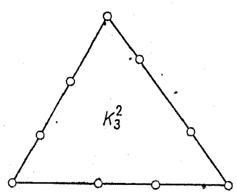

Fig. 1

The following property (*) of graphs will be important for later discussions.

(*) For every infinite $U\subset E$ and every natural number m there is a finite $S\subset E$ and an infinite $U'\subset U$ s.t. for all different $a,b\in U'$:

$$d_s(a,b)>m.$$

This will be equivalent to the following property.

1. DEFINITION. A graph (E,K) is called *flat* if no subgraph is isomorphic to K_ω^m for any natural number m.

(E,K) is called *superflat* if for every natural number m there is a natural number n such that no subgraph of (E,K) is isomorphic to K_n^m.

2. THEOREM. *A graph has the property* (*) *iff it is flat.*

Proof. Since no K_ω^n has the property (*), we have that (*) implies flatness. To prove the other direction assume that there is an infinite $U\subset E$ and a natural number m s.t.:

For every infinite $U'\subset U$ and finite $S\subset E$ there are two distinct $a,b\in U'$ with $d_s(a,b)\leqslant m$.

We first observe:

LEMMA. *For every infinite $U'\subset U$ and finite $S\subset E$ there is $c\in E\setminus S$, an infinite $U''\subset U'$ and for every $a\in U''$ a path (Q_a,W_a) from c to a of length $\leqslant m$ such that (Q_a,W_a) and (Q_b,W_b) are disjoint for $a\neq b$.*

Proof. Since there is no infinite $U^*\subset U'$ such that $d_s(a,b)>m$ for all distinct $a,b\in U^*$, Ramsey's theorem yields an infinite $U^*\subset U'$ s.t.:

$$d_s(a,b)\leqslant m \quad \text{for all } a,b\in U^*.$$

Let $a\in U^*$ and let $F'=\{b\in E|\ d_s(a,b)\leqslant m\}$. For every $b\in F'$, $b\neq a$, we choose an $c_b\in F'$ s.t. $d_s(a,c_b)<d_s(a,b)$ and $(c_b,a)\in K$. Let

$$L'=\bigcup_{a\neq b\in F'}\{(c_b,b),(b,c_b)\}.$$

Then (F',L') is a tree. Let (F,L) the largest subtree whose endpoints are elements of U^*.

Since the distance in (F,L) of two vertices is smaller than $2m+1$ there must exist a $c\in F$ s.t. $x:=\{d|\ c=c_d\}$ is infinite. For every $d\in x$ we choose a path (Q_{a_d},W_{a_d}) in (F,L) from c to an endpoint a_d s.t. $d\in Q_{a_d}$. Let $U''=\{a_d|\ d\in x\}$. Then c and U'' have the desired properties.

Now we can continue with the proof of the theorem. Using the preceding lemma we choose vertices $c_n\in E\setminus\{c_0,\dots,c_{n-1}\}$ infinite set $U_n\subset U_{n-1}$ ($U_0\subset U$) and for every $a\in U_n$ a path (Q_a^n,W_a^n) of length $\leqslant m$, s.t. (Q_a^n,W_a^n), (Q_b^n,W_b^n) are disjoint for $a\neq b\in U_n$. Then we construct subgraphs (F_n,L_n) s.t.

a) (F_n,L_n) is a subdivision of the complete graph with the vertices $c_{i_1},\dots,c_{i_n}$ by fewer than $2m+1$ vertices.

b) (F_n,L_n) is a subgraph of (F_{n+1},L_{n+1}) as follows:

Let $F_0=\emptyset, L_0=\emptyset$. Suppose (F_{n-1},L_{n-1}) is already choosen. Let i_n be greater then all i with $c_i\in F_{n-1}$. To connect c_{i_n} with c_{i_k}, $k<n$, we choose subgraphs (F_n^k,L_n^k), $k<n$, in the following manner:

Let $F_n^0=F_{n-1}\cup\{c_{i_n}\}$ and let $L_n^0=L_{n-1}$. Assume (F_n^{k-1},L_n^{k-1}) is choosen. Then there exists $a,b\in U_{i_n}$ s.t.

$$Q_b^{i_n}\cap F_n^{k-1}=\{c_{i_n}\} \quad\text{and}\quad Q_b^{i_k}\cap F_n^{k-1}=\{c_{i_k}\}.$$

Let (Q,W) the path from c_{i_k} to c_{i_n} s.t. $W\subset W_b^{i_n}\cup W_b^{i_k}$, and define

$$F_n^k=F_n^{k-1}\cup Q,\qquad L_n^k=L_n^{k-1}\cup W.$$

Let $F_n=F_n^{n-1}$ and $L_n=L_n^{n-1}$. Then the graph (F_n,L_n) has properties a) and b). To finish the construction define

$$F'=\bigcup_{n=1}^{\infty}F_n \quad\text{and}\quad L'=\bigcup_{n=1}^{\infty}L_n.$$

An easy application of Ramsey's theorem shows that there is an $e<2m$ and a subgraph (F,L) of (F',L') which is isomorphic to K_ω^e. This proves the theorem.

3. Corollary. *Let (E,K) be a flat graph, m a natural number and $\bar{A}\subset{}^nE$ infinite. Then there is an infinite $\bar{B}\subset\bar{A}$ and a finite $S\subset E$ s.t. for all distinct $\bar{a},\bar{b}\in\bar{B}$:*

$$d_s(\bar{a},\bar{b})>m\,.$$

Proof. By recursion on $e\leqslant n$ we choose infinite $\bar{A}_e\subset\bar{A}$ and finite $S_e\subset E$ as follows:

Let $\bar{A}_0=\bar{A}$ and $S_0=\emptyset$. Suppose that $\bar{A}_e$ and S_e are choosen. If

$$U=\{\bar{a}(e)|\ \bar{a}\in\bar{A}_e\}$$

is finite, let $S_{e+1}=S_e\cup U$ and $\bar{A}_{e+1}=\bar{A}_e$. Otherwise, since (E,K) is flat, there is an infinite $U'\subset U$ and a finite $S_{e+1}\subset E$, s.t. for all distinct $a,b\in U'$, $d_{S_{e+1}}(a,b)>2m$. Let $\bar{A}_{e+1}$ be an infinite subset of $\bar{A}_e$ s.t. for all distinct $\bar{a},\bar{b}\in\bar{A}_{e+1}$, $\bar{a}(e)\neq\bar{b}(e)\in U'$. If $S=\bigcup_{e\leqslant n}S_e$ we have for all distinct $\bar{a},\bar{b}\in\bar{A}_n$ and for all $k<n$:

$$d_s(\bar{a}(k),\bar{b}(k))>2m\,.$$

By induction we choose elements $\bar{a}_i\in\bar{A}_n$ as follows:

Let $\bar{a}_0\in\bar{A}_n$. Suppose that $\bar{a}_j, j\leqslant i$, is already choosen. Then we choose $\bar{a}_{i+1}\in\bar{A}_n$ such that

$$d_s(\bar{a}_{i+1},\bar{a}_j)\leqslant m\quad\text{ for }\quad j\leqslant i\,.$$

If such an element does not exist, the infinite $\bar{A}_n$ must contain $\bar{a}$, $\bar{b}$ s.t. for some $j\leqslant i$; $l,k<n$ we have:

$$d_s(\bar{a}_j(k),\bar{a}(e))\leqslant m,\qquad d_s(\bar{a}_j(k),\bar{b}(e))\leqslant m\,.$$

This implies $d_s(\bar{a}(e),\bar{b}(e))\leqslant 2m$, which is a contradiction to the construction of $\bar{A}_n$.

Graphs with definable orderings. The (first order) language $\mathscr{L}$ of the theory of graphs contains besides the logical symbols a binary relation symbol P. Let X be a set, then $\mathscr{L}_X$ denotes the language $\mathscr{L}$ extended by using the elements of X as constant symbols. Let V be the set of variables, let φ be a formula from $\mathscr{L}_X$ and let $\bar{V}_i\in{}^{n_i}V$ s.t. $\bar{V}_i(m)\neq\bar{V}_j(l)$ for all $i\neq j$ $(m\neq l)$. If every free variable of φ is equal to some $\bar{V}_i(l)$, $i\leqslant k$, $l<n_i$; we write $\varphi(\bar{V}_1,\ldots,\bar{V}_k)$. This notion indicates how to substitute constants:

Let $\bar{a}_i\in{}^{n_i}X$ then $\varphi(\bar{a}_1,\ldots,\bar{a}_k)$ denotes the sentence from $\mathscr{L}_X$, which is obtained from $\varphi(\bar{V}_1,\ldots,\bar{V}_n)$ by substituting $\bar{V}_i(l)$ by $\bar{a}_i(l)$.

If (E,K) is a graph and f a function from X to E, then $(E,K,f(x))_{x\in X}$ is a structure for $\mathscr{L}_x$. For $\varphi(\bar{V}_1,\ldots,\bar{V}_k)$ from $\mathscr{L}_X$ and $\bar{a}_i\in{}^{n_i}E$ let

$$(E,K,f(x))_{x\in X}\models\varphi[\bar{a}_1,\ldots,\bar{a}_k]$$

express that φ holds in $(E,K,f(x))_{x\in X}$ if $\bar{V}_i(l)$ is interpreted by $\bar{a}_i(l)$. $\mathrm{Th}(E,K,f(x))_{x\in X}$ is the set of all sentences of $\mathscr{L}_X$ which hold in $(E,K,f(x))_{x\in X}$.

For example let S be a finite set, $\bar{V}_1$, $\bar{V}_2$ two n-tupels of variables. Define

$$\varphi_s^m(\bar{V}_1,\bar{V}_2)$$
$$:=\bigwedge_{l,k<n}\ \bigwedge_{i\leqslant m}\forall_{w_0}\ldots\forall_{w_i}\Big(w_0=\bar{V}_1(l)\wedge\bigwedge_{j<i}P(w_j,w_{j+1})\wedge w_i=\bar{V}_2(k)\to\bigvee_{x\in s}\ \bigvee_{j\leqslant i}x=w_j\Big).$$

Then

$$(E,K,f(x))_{x\in S}\models\varphi_s^m[\bar{a},\bar{b}]\quad\text{ iff }\quad d_{f[S]}(\bar{a},\bar{b})>m\text{ in }(E,K)\,.$$

Similarly we find for all natural numbers n, m a sentence ψ_n^m, s.t. $(E,K)\models\psi_n^m$ iff (E,K) contains no isomorphic copy of K_n^m.

From this we can derive

4. Lemma. *(E,K) is superflat iff all graphs (F,L) which are elementary equivalent to (E,K) (i.e. $\mathrm{Th}(E,K)=\mathrm{Th}(F,L)$) are flat.*

Proof. *If* (E,K) is superflat there is for every m an n s.t. $(E,K)\models\psi_n^m$. This holds also in every (F,L) elementary equivalent to (E,K). So clearly for every m K_ω^m is not embeddable in (F,L).

The other direction is shown by an easy application of the compactness theorem.

The following notion is important in model theory [3].

5. Definition. A formula $\varphi(\bar{V},\bar{U})$ is said to *define an ordering* of the graph (E,K) if there are an infinite $\bar{A}\subset{}^nE$ and a linear ordering $<$ on $\bar{A}$ s.t. for all $\bar{a},\bar{b}\in\bar{A}$

$$(E,K)\models\varphi[\bar{a},\bar{b}]\quad\text{ iff }\quad\bar{a}<\bar{b}$$

(E,K) is called *stable* if there is no definable ordering in any (F,L) elementary equivalent to (E,K).

It is quite useful to make the following definition:

6. Definition. A formula $\psi(\bar{V},\bar{W})$ is called *large* in a graph (E,K) if there is an infinite $\bar{A}\subset{}^nE$ s.t. for every infinite $\bar{B}\subset\bar{A}$ there are $\bar{a},\bar{b}\in\bar{B}$ s.t. $(E,K)\models\psi[\bar{a},\bar{b}]$. For example, if $\varphi(\bar{V},\bar{W})$ defines an ordering in (E,K), then $\varphi(\bar{V},\bar{W})$ and $\varphi(\bar{V},\bar{W})\wedge\neg\varphi(\bar{W},\bar{V})$ are large. A major step to prove that every flat graph has no definable ordering, is the following theorem.

7. Theorem. *Let $\psi(\bar{V},\bar{W})$ be a large formula in a flat graph (E,K). Then there are an extension (F,L) of (E,K), an automorphism h of (F,L) and $\bar{a},\bar{b}\in{}^nE$ s.t.*

1. $h\circ\bar{a}=\bar{b}$ *and* $h\supset\mathrm{id}_E$,
2. $d_E(\bar{a},\bar{b})=\infty$,
3. $(F,L)\models\psi[\bar{a},\bar{b}]$.

Proof. If there is an $\bar{a}\in{}^nE$ s.t. $(E,K)\models\psi[\bar{a},\bar{a}]$, then let $\bar{b}=\bar{a}$ and $(F,L)=(E,K)$. Otherwise, since $\psi(\bar{V},\bar{W})$ is large in (E,K) there is an infinite $\bar{A}\subset{}^nE$ s.t. every infinite $\bar{B}\subset\bar{A}$ contains two different elements $\bar{a},\bar{b}$ such that $(E,K)\models\psi[\bar{a},\bar{b}]$. Since (E,K) is flat, we have by Corollary 3 that for every natural number m there is a finite $S_m\subset E$ and an infinite $\bar{A}_m\subset\bar{A}$ s.t. $d_{S_m}(\bar{a},\bar{b})>m$ for all distinct $\bar{a},\bar{b}\in\bar{A}_m$. Clearly we can assume that $S_m\subset S_{m+1}$ and $\bar{A}_m\supset\bar{A}_{m+1}$.

Now we extend $\mathscr{L}_E$ to $\mathscr{L}_X$ by $2n$ new constant symbols which we can arrange in two sequences $\bar{d}$, $\bar{e}$ of length n. We define sets T_0, T_1, T_2, T_3 of sentences of $\mathscr{L}_X$:

$$T_0 = \mathrm{Th}(E, K, x)_{x \in E},$$

$$T_1 = \{\varphi^m_{S_m}(\bar{d}, \bar{e}) | \ m \text{ a natural number}\},$$

where $\varphi^m_{S_m}(\bar{V}, \bar{W})$ is the formula defined above, which expresses "$d_{S_m}(\bar{V}, \bar{W}) > m$"

$$T_2 = \{\sigma(\bar{d}) \leftrightarrow \sigma(\bar{e}) | \ \sigma(\bar{w}) \text{ from } \mathscr{L}_E\},$$

$$T_3 = \{\psi(\bar{d}, \bar{e})\}.$$

Finally let $T = T_0 \cup T_1 \cup T_2 \cup T_3$. First we prove, that T is consistent:

Let Δ be a finite set of formulas $\sigma(\bar{w})$ from $\mathscr{L}_E$ and let m be a natural number. Define

$$\bar{T}_1 = \{\varphi^r_{S_r}(\bar{d}, \bar{e}) | \ r \leqslant m\},$$

$$\bar{T}_2 = \{\sigma(\bar{d}) \leftrightarrow \sigma(\bar{e}) | \ \sigma(\bar{w}) \in \Delta\}.$$

By compactness it suffices to show that $\bar{T} = T_0 \cup \bar{T}_1 \cup \bar{T}_2 \cup T_3$ has a model. Since $\bar{A}_m$ is infinite, we get by an easy application of Ramsey's theorem an infinite $\bar{B} \subset \bar{A}_m \subset \bar{A}$ s.t.

$$(E, K, e)_{e \in E} \models \sigma[\bar{a}] \quad \text{iff} \quad (E, K, e)_{e \in E} \models \sigma[\bar{b}]$$

for all $\bar{a}, \bar{b} \in \bar{B}$ and all $\sigma \in \Delta$. Choose two different sequences $\bar{a}, \bar{b} \in \bar{B}$ such that $(E, K) \models \psi[\bar{a}, \bar{b}]$ and let f be the map from X to E which satisfies $\mathrm{id}_E \subset f$, $f \circ \bar{d} = \bar{a}$ and $f \circ \bar{e} = \bar{b}$. Then $(E, K, f(x))_{x \in X}$ is a model of $\bar{T}$ and therefore T is consistent.

Let $(F', L', g(x))_{x \in X}$ be a model of T. Since $T_0 \subset T$ we can assume that (E, K) is an (elementary) subgraph of (F', L') and $g \supset \mathrm{id}_E$. Let $\bar{a} = g \circ \bar{d}$ and $\bar{b} = g \circ \bar{e}$. Since $T_2 \subset T$, $\bar{a}$ and $\bar{b}$ satisfy the same formulas of $\mathscr{L}_E$ in $(F', L', e)_{e \in E}$. Therefore using a result of [2, p. 49] we find an elementary extension (F, L) of (F', L') and an automorphism h of (F, L) s.t. $h \supset \mathrm{id}_E$ and $h \circ \bar{a} = \bar{b}$. Since $T_1 \subset T$, $d_E(\bar{a}, \bar{b}) = \infty$ in (F, L). Finally $T_3 \subset T$ implies $(F, L) \models \psi[\bar{a}, \bar{b}]$.

By a similar argument as in [1, p. 178] one can show:

8. Lemma. *Let (F, L) be a graph, $E \subset F$ and h an automorphism of (F, L) with $h \supset \mathrm{id}_E$. If $\bar{a}, \bar{b} \in {}^nF$ such that $d_E(\bar{a}, \bar{b}) = \infty$ and $h \circ \bar{a} = \bar{b}$, then there is an automorphism f s.t. $f \circ \bar{a} = \bar{b}$, $f \circ \bar{b} = \bar{a}$ and $f \supset \mathrm{id}_E$.*

9. Corollary. *No flat graph has a definable ordering.*

Proof. If $\varphi(\bar{V}, \bar{W})$ defines an ordering in (E, K), then

$$\psi(\bar{V}, \bar{W}) = \varphi(\bar{V}, \bar{W}) \wedge \neg\varphi(\bar{W}, \bar{V})$$

is large. If (E, K) is flat, by Theorem 7 and Lemma 8 there are a graph (F, L) extending (E, K), $\bar{a} \in {}^nF$, $\bar{b} \in {}^nF$ and an automorphism h of (F, L) s.t. $h \circ \bar{a} = \bar{b}$, $h \circ \bar{b} = \bar{a}$ and $(F, L) \models \psi[\bar{a}, \bar{b}]$. This implies $(F, L) \models \psi[\bar{b}, \bar{a}]$, which is impossible by the special form of ψ.

A immediate consequence of Corollary 9 and Lemma 4 is

10. Corollary. *Every superflat graph is stable.*

Remark. Let $\mathfrak{A} = (A, U_i, R_j, f_k)_{i \in I, j \in J, k \in K}$ be a structure, where U_i, $i \in I$, are unary relations, R_j, $j \in J$, are binary relations and f_k, $k \in K$, are unary functions.

We define

$$K_{\mathfrak{A}} = \Big(\bigcup_{j \in J} (R_j \cup R_j^{-1}) \cup \bigcup_{k \in K} (f_k \cup f_k^{-1})\Big) \setminus \mathrm{id}_{\mathfrak{A}}.$$

Then $(A, K_{\mathfrak{A}})$ is a graph and by a similar argument as before one can show:

a) If $(A, K_{\mathfrak{A}})$ is flat, then $\mathfrak{A}$ has no definable ordering.

b) If $(A, K_{\mathfrak{A}})$ is superflat, then $\mathfrak{A}$ is stable. For example, if $J = \emptyset$, then $\mathfrak{A}$ is stable.

Moreover we can extend a) as in [1]:

c) If $(A, K_{\mathfrak{A}})$ is flat, then any (in $\mathfrak{A}$) definable e-ary relation of ${}^nA$ is almost symmetric.

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