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Direct sums of stratifiable spaces

by

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Abstract. We prove that any direct sum of stratifiable spaces, with the box topology, is stratifiable. This answers a question of E. K. van Douwen.

Our main task is to prove that a direct sum of stratifiable spaces, with the box topology, is stratifiable. This answers a recent question of E. K. van Douwen [2].

1. Introduction. If $\{X_\alpha\}_{\alpha\in A}$ is a family of spaces, we let $X = \prod_{\alpha\in A} X_\alpha$. For each cardinal m, we let $P(m)$ be the topology on X for which the set $\{\prod_{\alpha\in A} U_\alpha | U_\alpha$ is open in X_α and $\operatorname{card}(\{\alpha | U_\alpha \neq X_\alpha\}) < m\}$ is a base. We let $X(m)$ denote X with the topology $P(m)$. In case $m > \operatorname{card} A$, we denote $P(m)$ and $X(m)$ by $P(\infty)$ and $X(\infty)$, respectively. Clearly $P(\aleph_0)$ is the Tychonoff product topology and $P(\infty)$ is the box product topology.

For each $p\in X$, let $\Sigma_p = \{x\in X | \ x_\alpha \neq P_\alpha$ for at most finitely many $\alpha\}$. Let $\Sigma_p(m)$ denote Σ_p as a subspace of $X(m)$.

Recently M. E. Rudin [4] proved that, assuming the *continuum hypothesis*, the box product of countably many σ-compact metrizable spaces is paracompact. Subsequently, E. K. van Douwen [2] proved that the box product of countably many copies of the *irrationals* is not a normal space, while $\Sigma_p(\infty)$ is stratifiable for *any* family of *metrizable* spaces.

2. The main result. Before answering E. K. van Douwen's question, we need a new characterization of stratifiable spaces, which generalizes the concept of Nagata spaces of Ceder [1].

LEMMA 2.1. *A T_1-space X is a stratifiable space if and only if to each $x\in X$ one can assign neighborhood bases $\{S_{\alpha nx}\}$, $\{U_{\alpha nx}\}$ where α runs over a set D_x which satisfy*

(i) $S_{\alpha nx} \cap S_{\beta ny} \neq \emptyset \Rightarrow x\in U_{\beta ny}$ *and* $y \in U_{\alpha nx}$,

(ii) $U_{\alpha nx} \supset U_{\alpha(n+1)x}$, $S_{\alpha nx} \supset S_{\alpha(n+1)x}$, *for each n and $x\in X$.*

Proof. The "only if" part: For each $x\in X$, let $\mathcal{N}_x = \{N_{\alpha x}\}_{\alpha\in D_x}$ be a neighborhood base for x. Also, let $U \to \{U_n\}$ be an increasing stratification for X. Then, let

$$U_{\alpha nx} = \bigcap \{V | \ V \text{ is open and } N_{\alpha x} \subset V_n\},$$

$S'_{\alpha nx} = \bigcap \{V_n|\ N_{\alpha x} \subset V_n\} - \bigcup \{U_n|\ x \notin U,\ U \text{ is open}\}$, $S_{\alpha nx} = \bigcap_{k=1}^{n} S'_{\alpha kx}$, for each n. It is clear that each $S_{\alpha nx} \subset S'_{\alpha nx} \subset U_{\alpha nx}$, $S_{\alpha nx} \supset S_{\alpha(n+1)x}$ and $U_{\alpha nx} \supset U_{\alpha(n+1)x}$ for each $\alpha \in D_x$ and n. (It is not necessarily true that $S'_{\alpha nx} \supset S'_{\alpha(n+1)x}$.)

It is also clear that $x \in (S'_{\alpha nx})^0$, and therefore $x \in S^0_{\alpha nx}$, for each $\alpha \in D_x$ and n, because the U_n's are cushioned in the U's. This makes it easy to show that $\{S_{\alpha nx}\}$ and $\{U_{\alpha nx}\}$ are neighborhood bases of each $x \in X$: Let $x \in W$, with W open. Now pick $N_{\alpha x} \subset$ some W_n. Then $x \in U_{\alpha nx} \subset W$, which does the trick.

Next we show that the neighborhood bases $\{S_{\alpha nx}\}$, $\{U_{\alpha nx}\}$ satisfy condition (i): Say $y \notin U_{\alpha nx}$. Then there exists open V such that $N_{\alpha x} \subset V_n$ but $y \notin V$. Then $S'_{\alpha nx} \subset V_n \subset \bigcup \{U_n|\ y \notin U,\ U \text{ is open}\}$ and therefore $S_{\alpha nx} \cap S_{\beta ny} = \emptyset$, for all $\beta \in D_y$, (since $S'_{\alpha nx} \cap S'_{\beta ny} = \emptyset$). The "only if" part is thus proved.

The "if" part: For each open $U \subset X$ and n, let $U_n = \bigcup \{S_{\alpha nx}|\ x \in U \text{ and } U_{\alpha nx} \subset U\}$. It is easy to see that $U \to \{U_n\}$ is a stratification of X, which completes the proof.

We will also need another characterization of stratifiable spaces, which is due to R. Heath [3]:

LEMMA 2.2. *A T_1-space X is stratifiable if and only if there is a function $g: X \times N \to \{\text{open subsets of } X\}$ such that* (a) *$x \in g(x, n)$, for $n = 1, 2, \ldots$, and* (b) *given any closed subset M of X and any point $q \in X - M$, there exists $n \in N$ such that $q \notin (\bigcup \{g(x, n)|\ x \in M\})^-$.*

THEOREM 2.2. *If $\{X_v\}_{v \in A}$ is a family of stratifiable spaces, then $\Sigma_p(\infty)$ is a stratifiable space, for each $p \in \prod_{v \in A} X_v$.*

Proof. For each $x \in \Sigma_p$, let $A(x) = \{v \in A|\ p_v \neq x_v\}$ and pick positive integer $n(x)$ such that $p_v \notin$ some $U_{\beta_v n(x) x_v}$, for each $v \in A(x)$. Then, for each positive integer n, let $g(x, n) = \Sigma_p \cap \prod_{v \in A} S_{\beta_v(n(x)+n)x_v}$ with $p_v \notin U_{\beta_v(n(x)+n)x_v}$ for $v \in A(x)$ and $S_{\beta_v(n(x)+n)x_v}$ being arbitrarily chosen for each $v \notin A(x)$.

Clearly $\{g(x, n)\}$ is a sequence of neighborhoods of x, for each $x \in X$. We will now show that, given $x = (x_v) \notin \prod_{v \in A} U_{\mu_v n_v q_v}$, with $q = (q_v)$ and $n \geqslant \max\{n_v|\ v \in A(q)\}$, then $\prod_{v \in A} S_{\mu_v n_v q_v} \cap g(x, n) = \emptyset$ (note that this is equivalent to showing that the second condition of Lemma 2.2 is satisfied): Clearly some $x_\gamma \notin U_{\mu_\gamma n_\gamma q_\gamma}$. We consider two cases.

Case 1. $\gamma \in A(q)$. If $z \in \prod_{\alpha \in A} S_{\mu_\alpha n_\alpha q_\alpha} \cap g(x, n)$ then $S_{\mu_\gamma n_\gamma q_\gamma} \cap S_{\beta_\gamma n x_\gamma} \neq \emptyset$ which implies that $S_{\mu_\gamma n_\gamma q_\gamma} \cap S_{\beta_\gamma n_\gamma x_\gamma} \neq \emptyset$, because $n \geqslant n_\gamma$, which implies that $x_\gamma \in U_{\mu_\gamma n_\gamma q_\gamma}$, a contradiction.

Case 2. $\gamma \notin A(q)$. Then $\gamma \in A(x)$, since $q_\gamma = p_\gamma$ and $x_\gamma \neq q_\gamma$. Therefore $z \in \prod_{\alpha \in A} S_{\mu_\alpha n_\alpha q_\alpha} \cap g(x, n)$ implies that $S_{\mu_\gamma n_\gamma p_\gamma} \cap S_{\beta_\gamma n x_\gamma} \neq \emptyset$ which implies that $x_\gamma \in U_{\mu_\gamma n_\gamma p_\gamma}$ or $p_\gamma \in U_{\beta_\gamma n x_\gamma}$ (this follows easily from Lemma 2.1(a), since either $n_\gamma \geqslant n$ or $n \geqslant n_\gamma$). But $p_\gamma \notin U_{\beta_\gamma n x_\gamma}$ by the definition of $g(x, n)$. Therefore $x_\gamma \in U_{\mu_\gamma n_\gamma p_\gamma}$, a contradiction. Because of Lemma 2.2, the proof is complete.

3. Concluding remarks. It is quite easily seen that $\Sigma_p(\aleph_0)$ is stratifiable if and only if Σ_p is a direct sum of countably many stratifiable spaces (clearly, if $X = \prod_{\alpha \in A} X_\alpha$ is a product of uncountably many spaces then each point of $\Sigma_p(\aleph_0)$ is not a G_δ-subset of $\Sigma_p(\aleph_0)$). Indeed, the preceding argument shows that $\Sigma_p(m)$ is not stratifiable whenever Σ_p is a direct sum of a collection $\mathscr{C}$ of spaces with card $\mathscr{C} > m \geqslant \aleph_0$, or card $\mathscr{C} \geqslant m > \aleph_0$.

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