Homology with equationally compact coeffcients

by

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Abstract. In this paper it is proved that an abelian group G is equationally compact if and only if the Čech homology theory with coefficients in G satisfies the exactness axiom for all compact Hausdorff pairs. This result is used to show that the McCord homology theory defined by an ultrapower coincides with Čech homology for all compact metric pairs. Examples are given to show that this theorem cannot be improved.

1. In this paper I first prove a new characterization of equationally compact groups in terms of Čech homology. In subsequent sections I use this result to answer some questions raised by McCord about the relation between Čech homology and his nonstandard homology theory. These theorems were announced in [2].

DEFINITION. An abelian group G is *equationally compact* if and only if for every set T of linear equations with parameters from G, if every finite subset of T has a solution in G then T has a solution in G.

Many equivalent formulations of this concept are known. I will need the following one, which is due to Balcerzyk [5, p. 293].

THEOREM. *An abelian group G is equationally compact if and only if, for every set of equations*

$$T = \{x_0 - a_n = n!x_n \mid a_n \in G, n \geq 1\},$$

if every finite subset of T has a solution in G then T has a solution in G.

I can use this result to prove the following equivalence:

THEOREM 1. *An abelian group G is equationally compact if and only if the Čech homology theory with coefficients G satisfies the exactness axiom on the category of compact Hausdorff pairs.*

Proof. Suppose G is equationally compact and (X, A) is a compact Hausdorff pair. I will prove exactness of the following segment of the homology sequence:

$$\ldots \to \check{H}_n(A) \xrightarrow{j} \check{H}_n(X) \xrightarrow{i} \ldots$$

(The coefficient group G has been omitted to simplify the notation.) The proofs of exactness for the other parts of the sequence are exactly the same.

* The author was supported during the preparation of this paper by an NSF Graduate Fellowship.

It is always true that $ij = 0$ [1, p. 248]. So suppose $x \in \check{H}_n(X)$ and $i(x) = 0$. Let $\mathrm{Cov}(X, A)$ be the finite open coverings of (X, A) ordered by refinement, and for all $\alpha, \beta \in \mathrm{Cov}(X, A)$, $\beta > \alpha$, let (X_α, A_α) be the nerve of α; let $i_\alpha\colon H_n(X_\alpha) \to H_n(X_\alpha, A_\alpha)$, $j_\alpha\colon H_n(A_\alpha) \to H_n(X_\alpha)$ be the homomorphisms induced by inclusion; and let $\Pi^\beta_\alpha\colon H_n(X_\beta) \to H_n(X_\alpha)$, $\sigma^\beta_\alpha\colon H_n(A_\beta) \to H_n(A_\alpha)$ be the homomorphisms induced by any projection $(X_\beta, A_\beta) \to (X_\alpha, A_\alpha)$. Then $\check{H}_n(X) = \varprojlim H_n(X_\alpha)$ and $\check{H}_n(A) = \varprojlim H_n(A_\alpha)$. Let $x = (x_\alpha)\alpha \in \mathrm{Cov}(X, A)$. Then $i_\alpha(x_\alpha) = 0$ and $\Pi^\beta_\alpha(x_\beta) = x_\alpha$ for all $\alpha, \beta \in \mathrm{Cov}(X, A)$, $\beta > \alpha$. In order to prove exactness we must solve the following set of equations:

$$T = \{j_\alpha(y_\alpha) = x_\alpha,\ \sigma^\beta_\alpha(y_\beta) = y_\alpha \mid\ \alpha, \beta \in \mathrm{Cov}(X, A), \beta > \alpha\}\,.$$

The classical algorithm for computing the homology groups of a finite simplicial complex [1, Chapter 3] allows us to find integers n_α, m_α, and matrices P_α, Q_α with integral entries such that

$$H_n(A_\alpha) \simeq \frac{\{\vec{x} \in G^{n_\alpha} \mid\ P_\alpha \vec{x} = \vec{0}\}}{\{Q_\alpha \vec{x} \mid \vec{x} \in G^{m_\alpha}\}}\,.$$

Similar integers and matrices can be found for each $H_n(X_\alpha)$. Furthermore, the homomorphisms i_α, j_α, Π^β_α, σ^β_α can all be regarded as homomorphisms defined by matrices with integer entries. Consequently, one can construct in an obvious way a set T' of linear equations over G which is solvable in G if and only if T is solvable. Now if S is a finite subset of T let $\mu \in \mathrm{Cov}(X, A)$ be a refinement of any α appearing in S. Since $i_\mu(x_\mu) = 0$ and $\ldots \to H_n(A_\mu) \xrightarrow{j_\mu} H_n(X_\mu) \xrightarrow{i_\mu} \ldots$ is exact, there is some $b_\mu \in H_n(A_\mu)$ such that $j_\mu(b_\mu) = x_\mu$. Then it follows easily that $\{\sigma^\mu_\alpha(b_\mu) \mid\ \alpha < \mu\}$ is a solution for all the equations in S. So each finite subset of T is solvable, and so each finite subset of T' is solvable in G. Since G is equationally compact, this implies that T' is solvable in G. Consequently, T is solvable.

Conversely, suppose G is such that the homology sequence is exact for all compact Hausdorff pairs. For each $n \geqslant 1$ let X_n be the space obtained from the closed unit disc in the complex plane by identifying x and y if and only if $|x| = 1 = |y|$ and $x = ye^{2\pi ik/n!}$ for some $k \in N$. Each X_n is a compact metric space and the image of S^1 in X_n under the prescribed identifications is again homeomorphic to S^1. Let $f_n\colon X_n \to X_{n-1}$ be the map induced by the function $z \to z^n$. Then f_n is continuous, surjective, and maps the image of S^1 in X_n homeomorphically onto the image of S^1 in X_{n-1}. Consider the array

$$\begin{array}{ccccc}
\ldots \to \check{H}_1(S^1) & \to & \check{H}_1(X_1) & \to & \check{H}_1(X_1, S^1)\\
\uparrow (f_2)_* & & \uparrow (f_2)_* & & \uparrow (f_2)_*\\
\ldots \to \check{H}_1(S^1) & \to & \check{H}_1(X_2) & \to & \check{H}_1(X_2, S^1)\\
\uparrow (f_3)_* & & \uparrow (f_3)_* & & \uparrow (f_3)_*\\
\ldots \to \check{H}_1(S^1) & \to & \check{H}_1(X_3) & \to & \check{H}_1(X_3, S^1)\\
\uparrow & & \uparrow & & \uparrow\\
\vdots & & \vdots & & \vdots
\end{array}$$

where the horizontal arrows represent homomorphisms induced by inclusions. An elementary calculation shows that this array is equal to the following one:

$$\begin{array}{ccccc}
\ldots \to G & \to & 0 & \to & 0\\
\uparrow 1_G & & \uparrow & & \\
\ldots \to G & \xrightarrow{P_2} & G/2!\,G & \to & 0\\
\uparrow 1_G & & \uparrow q_3 & & \\
\ldots \to G & \xrightarrow{P_3} & G/3!\,G & \to & 0\\
\uparrow 1_G & & \uparrow q_4 & & \\
\vdots & & \vdots & &
\end{array}$$

where $P_n(g) = g + n!\,G$ for all n and $q_n(g + n!\,G) = g + (n-1)!\,G$ for all n.

Now take the inverse limit of this array, and apply the continuity theorem for Čech homology to obtain the following segment of the homology sequence for $\varprojlim(X_n, S^1)$:

$$(*) \qquad \ldots \to G \xrightarrow{q} (\varprojlim G/n!\,G) \to 0\,.$$

It is exact, by hypothesis. Let $T = \{x_0 - a_n = n!\,x_n \mid\ a_n \in G, n \geqslant 1\}$ be finitely solvable. Then for each n there are x_n, x_{n-1}, x_0 such that $x_0 - a_n = n!\,x_n$, $x_0 - a_{n-1} = (n-1)!\,x_{n-1}$. Therefore,

$$\begin{aligned} a_n - a_{n-1} &= (n-1)!\,x_{n-1} - n!\,x_n\\ &= (n-1)!\,(x_{n-1} - nx_n) \in (n-1)!\,G \end{aligned}$$

so

$$q_n(a_n + n!\,G) = a_{n-1} + (n-1)!\,G\,.$$

Consequently, $(a_n + n!\,G)_{n \geqslant 1}$ is an element of $\varprojlim G/n!\,G$. By the exactness of $(*)$ there is some element $x_0 \in G$ such that $q(x_0) = (a_n + n!\,G)_{n \geqslant 1}$, i.e. $x_0 - a_n \in n!\,G$ for all n, which means that T is solvable. Therefore, by Balcerzyk's theorem, G is equationally compact.

Corollary 1. *The Čech homology theory with coefficients G is exact for all compact Hausdorff pairs if and only if it is exact for all compact metric pairs.*

Proof. This follows immediately from the fact that the pair $\lim(X_n, S^1)$ used in Theorem 1 is a compact metric pair.

2. In [3] McCord constructed a homology theory using the techniques of non-standard analysis. He left open the problem of how his theory was related to Čech homology. In this section and the next I give a partial solution to this problem. In the following, $\bar{H}$ denotes McCord homology, $\bar{C}_n(X)$ is the group of infinitesimal n-chains on X, d is the differential,

$$\bar{Z}_n(X) = \{\alpha \in \bar{C}_n(X) \mid\ d\alpha = 0\}\,,$$

$$\bar{B}_n(X) = \{d\alpha \mid\ \alpha \in \bar{C}_{n+1}(X)\}\,.$$

See [3] for the definitions. In the proof of Theorem 2 I omit the star from the names of standard objects to simplify the notation. The V^* which occurs below is defined in [4, p. 316].

One final remark: in [3] McCord actually defines his theory only for enlargements, but his proofs of the axioms remain valid in an arbitrary nonstandard model. Therefore the statement of the following theorem makes sense even when the ultrapower involved does not yield an enlargement.

THEOREM 2. *If G is an abelian group and $^*G$ is an ultrapower of G with respect to an ω-incomplete ultrafilter on an infinite index set, then the McCord homology theory $\overline{H}(\ ;{}^*G)$ is isomorphic to the Čech theory $\check{H}(\ ;{}^*G)$ on the category of pairs of compact metric spaces.*

Proof. The proof essentially follows Spanier's proof of the continuity theorem for Alexander–Spanier cohomology in [4, pp. 316–319]. Embed X in $[0,1]^\omega$. Then you can find a sequence of compact pairs (X_n, A_n) for $n \in \omega$ such that $X_n \supset X_{n+1}$, $A_n \supset A_{n+1}$ for all n, $\bigcap_{n\in\omega}(X_n, A_n) = (X, A)$, and each (X_n, A_n) has the homotopy type of a compact polyhedral pair. There are natural maps

$$i\colon\ \overline{H}_n(X, A) \xrightarrow{i_1} \varprojlim \overline{H}_n(X_m, A_m) \xrightarrow{\simeq} \varprojlim \check{H}_n(X_m, A_m) \xrightarrow{\simeq} \check{H}_n(X, A)\,,$$
$$j\colon\ \overline{H}_n(X) \xrightarrow{j_1} \varprojlim \overline{H}_n(X_m) \xrightarrow{\simeq} \varprojlim \check{H}_n(X_m) \xrightarrow{\simeq} \check{H}_n(X)\,,$$
$$k\colon\ \overline{H}_n(A) \xrightarrow{k_1} \varprojlim \overline{H}_n(A_m) \xrightarrow{\simeq} \varprojlim \check{H}_n(A_m) \xrightarrow{\simeq} \check{H}_n(A)$$

where i_1, j_1, k_1 are induced by inclusion and the other maps are the natural isomorphisms provided by the uniqueness theorem for homology on compact polyhedra and the continuity theorem for Čech homology. I want to prove that i, j, k are isomorphisms. It is sufficient to prove this for j because the proof for k is exactly the same, and the result for i then follows from the Five Lemma applied to the following diagram:

$$\begin{array}{ccccccccc}
\check{H}_n(A) & \to & \check{H}_n(X) & \to & \check{H}_n(X, A) & \to & \check{H}_{n-1}(A) & \to & \check{H}_{n-1}(X) \\
\downarrow{\scriptstyle k} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle j} \\
\overline{H}_n(A) & \to & \overline{H}_n(X) & \to & \overline{H}_n(X, A) & \to & \overline{H}_{n-1}(A) & \to & \overline{H}_{n-1}(X)
\end{array}$$

The bottom sequence is exact because $\overline{H}$ satisfies the exactness axiom for all pairs and coefficient groups [3]. The top sequence is exact by Theorem 1 since $^*G$ is ω_1-saturated and hence, by Balcerzyk's theorem, equationally compact.

So it is sufficient to show that $j_1\colon \overline{H}_n(X) \to \varprojlim \overline{H}_n(X_m)$ is an isomorphism. Let r be some metric on $[0,1]^\omega$ which induces the product topology, and let

$$\alpha = \sum g_i(x_0^i, \ldots, x_n^i)$$

be any (internal) n-chain on $[0,1]^\omega$. Define

$$\operatorname{diam}(\alpha) = \max_i\,[\max_{0\leqslant j,k\leqslant n} r(x_j^i, x_k^i)]\,.$$

For any standard real $\varepsilon > 0$ and any standard $Y \subset [0,1]^\omega$ let $\overline{C}_n(Y, \varepsilon)$ be the set of internal n-chains α on Y with $\operatorname{diam}(\alpha) \leqslant \varepsilon$, $\overline{Z}_n(Y, \varepsilon) = \{\alpha \in \overline{C}_n(Y, \varepsilon) \mid d\alpha = 0\}$, and $\overline{B}_n(Y, \varepsilon) = \{d\alpha \mid \alpha \in \overline{C}_{n+1}(Y, \varepsilon)\}$. These are all internal sets.

Now I first prove that j_1 is surjective. Suppose

$$(\alpha_m)_{m\in\omega} \in \varprojlim \overline{H}_n(X_m)\,.$$

Let $a_m \in \overline{Z}_n(X_m)$ be a representative of the homology class α_m. Consider the set P of formulas

$$\left\{x \in \overline{Z}_n\left(X, \frac{1}{s}\right) \,\middle|\, s \in N\right\} \cup \left\{(\exists y)\left(y \in \overline{C}_{n+1}\left(X_m, \frac{1}{s}\right) \wedge dy = x - a_m\right) \,\middle|\, \begin{matrix} s \in N \\ m \in \omega \end{matrix}\right\}.$$

I will show P is finitely satisfiable. Take any finite subset T of P, and let $X_{m_1}, \ldots, X_{m_k}$ be the sets occurring in T, and let $1/s_1, \ldots, 1/s_k$ be the diameters occurring in T. Pick $s > \max(s_1, \ldots, s_k)$. Let U be a standard finite open cover of $[0,1]^\omega$ by balls of diameter $< 1/s$. By Lemma 1 of [4, p. 316] there is a standard finite open cover W of $[0,1]^\omega$ such that W^* is a refinement of U, and a standard function f and a standard open neighborhood M of X such that

(1) $f(M) \subset X$,

(2) $f(x) = x$ for $x \in X$,

(3) if $V \in W$ then $f(V \cap M) \subset V^*$ (and so $f(V \cap M)$ has diameter $< 1/s$).

Pick $m > \max(m_1, \ldots, m_k)$ large enough so that $X_m \subset M$. Such an m exists by compactness since $\bigcap_{m\in\omega} X_m \cap CM = X \cap CM = \emptyset$. Then (1), (2), and (3) are still valid with M replaced by X_m. Let $a_m = \sum_i g_i(x_0^i, \ldots, x_n^i)$. $\operatorname{diam}(a_m) \cong 0$ so for each i there is some $V \in W$ such that $x_0^i \in V, \ldots, x_n^i \in V$ since W is standard. So by condition (3) $\operatorname{diam}(fx_0^i, \ldots, fx_n^i) < 1/s$. Define $f_\#(y_0, \ldots, y_n) = (fy_0, \ldots, fy_n)$ and extend $f_\#$ linearly. Then we have $\operatorname{diam}(f_\# a_m) < 1/s$ and $d(f_\# a_m) = f_\# da_m = 0$, so $f_\# a_m \in \overline{Z}_n(X, 1/s)$. Define

$$D(y_0, \ldots, y_n) = \sum_{j=0}^{n} (-1)^j (y_0, \ldots, y_j, fy_j, \ldots, fy_n)$$

and extend D linearly. An easy computation proves that

$$d(Da_m) + D(da_m) = f_\# a_m - a_m\,,$$

i.e. $d(Da_m) = f_\# a_m - a_m$. Since $V \subset V^*$, $f(V \cap X_m) \subset V^*$ for all $V \in W$, each $(n+1)$-tuple $(x_0^i, \ldots, x_j^i, fx_j^i, \ldots, fx_n^i)$ is contained in some V^* and hence has diameter $< 1/s$. Consequently, $Da_m \in \overline{C}_{n+1}(X_m, 1/s)$. Then it is clear that $x = f_\# a_m$ satisfies the formulas in T. Since the ultrapower is ω_1-saturated, P can be simultaneously satisfied by some α. Then clearly $\alpha \in \overline{Z}_n(X)$ (since $\operatorname{diam}(\alpha) < 1/s$ for all $s \in N$) and for each $m \in \omega$, $\alpha - a_m \in \overline{B}_n(X_m)$. Therefore $j_1(\alpha + \overline{B}_n(X)) = (\alpha_m)_{m\in\omega}$. Therefore, j_1 is surjective.

Proving that j_1 is injective is very similar. Suppose $\alpha, \beta \in \overline{H}_n(X)$ and $j_1(\alpha) = j_1(\beta)$. Let $a, b \in \overline{Z}_n(X)$ be representatives of the homology classes α and β respectively. Then for each $m \in \omega$ there is some $a_m \in \overline{C}_{n+1}(X_m)$ such that $d(a_m) = a-b$. Consider the set P of formulas:

$$\{x \in \overline{C}_{n+1}(X, 1/s) |\ s \in N\} \cup \{dx = a-b\}.$$

Let T be a finite subset of P and let s be larger than any m_k such that $\overline{C}_{n+1}(X, 1/m_k)$ occurs in T. Let U be a finite cover of $[0,1]^\omega$ by balls of diameter $<1/s$, and choose f, M, W as before, and take m large enough so that $\dot{X}_m \subset M$. Then since $d(a_m) = a-b$ we obtain

$$d(f_\# a_m) = f_\# d(a_m) = f_\# a - f_\# b = a-b$$

since f is the identity on X. As before, $\operatorname{diam}(f_\# a_m) < 1/s$, so $f_\# a_m \in \overline{C}_{n+1}(X, 1/s)$. Therefore $x = f_\# a_m$ satisfies the formulas in T. By ω_1-saturation there is some c satisfying P. Clearly $c \in \overline{C}_{n+1}(X)$ and $d(c) = a-b$. This means that $\alpha = \beta$, i.e. j_1 is injective.

In [1, p. 288] it is proved that there is a unique functorial homomorphism $h\colon \overline{H} \to \check{H}$ extending the identity on $*G$. The definition of h given there coincides exactly with the definition of j given here, so it follows that j is a functorial isomorphism. Q.E.D.

3. It is natural to ask whether Theorem 2 can be extended in any way. It is clear, for example, that if α is a cardinal and we consider an α^+-saturated ultrapower then the proof of Theorem 2 will work for all compact spaces with a base of cardinality $\leqslant \alpha$. Consequently, the question arises whether there is some ultrapower for which $\overline{H}(X, A) = \check{H}(X, A)$ holds for all compact spaces. The following trivial theorem shows that this is never true.

THEOREM 3. *If G is any nonzero abelian group and $*G$ is an ultrapower of G with respect to an ω-incomplete ultrafilter on an infinite index set J, then there is a compact Hausdorff space X such that $\overline{H}_0(X; *G) \neq \check{H}_0(X; *G)$.*

Proof. Let $\alpha = \operatorname{card}(J)$, $\beta = \operatorname{card}(G)$, $\delta = \max(\alpha, \beta)$. Let X be the one-point compactification of a discrete space of cardinality 2^δ. Then

$$\begin{aligned}\operatorname{card}(\check{H}_0(X; *G)) &= \operatorname{card}(*G^{2^\delta})\\ &\geqslant 2^{2^\delta}, \text{ but } \operatorname{card}(\overline{H}_0(X; *G))\\ &\leqslant \operatorname{card}(*(C_0(X; G)))\\ &\leqslant (2^\delta)^\alpha = 2^\delta.\end{aligned}$$

On the other hand, we might try to prove Theorem 2 for arbitrary nonstandard models rather than ultrapowers. The following result shows that this cannot be done even for enlargements.

THEOREM 4. *There is an abelian group G and an enlargement $*G$ of G such that $\overline{H}(\ ; *G)$ is not isomorphic to $\check{H}(\ ; *G)$ on the category of compact pairs of metric spaces.*

Proof. $\overline{H}$ is always exact; consequently, it follows from Corollary 1 that I need only find some G and some enlargement $*G$ such that $*G$ is not equationally compact. Let $G = Z^\omega$ where Z is the group of integers. Construct an enlargement in which $*N-N$ has a countable coinitial sequence $\{a_n\}_{n \in N}$, i.e. $a_i > a_{i+1}$ for all $i \in N$, $a_i \in *N-N$ for all i, and if $x \in *N-N$ then there is some i such that $a_i < x$. It is easy to construct enlargements of this type using the compactness theorem.

If $b \in *(Z^\omega)$ let $b(i)$ be its ith coordinate, for $i \in *\omega$. Define $f\colon *Z \times *N \to *N$ by $f(x, y)$ = the smallest element n of $*N$ such that 2^y divides $x-n$. f is an internal function. Now define a sequence c_n of elements of $*(Z^\omega)$ as follows:

$$c_1(i) = 1 \quad \text{for all } i \in *\omega, \tag{1}$$

$$c_2(i) = \begin{cases} 3 & \text{if } i<2,\\ 3 & \text{if } i>a_1,\\ 1 & \text{if } 2 \leqslant i \leqslant a_1,\end{cases} \tag{2}$$

for $n \geqslant 2$ define

$$c_{n+1}(i) = \begin{cases} c_n(i) & \text{if } i<n \text{ or } i>a_{n-1},\\ 2^n+1 & \text{if } i=n \text{ or } a_n < i \leqslant a_{n-1},\\ 1 & \text{if } n+1 \leqslant i \leqslant a_n.\end{cases} \tag{3}$$

Consider the set of equations $\{x_0 - c_n = 2^n x_n |\ n \geqslant 1\}$. Any finite subset has a solution in $*(Z^\omega)$ since for all i and all $j<i$ $2^j |\ c_i - c_j$. Suppose the entire set had a solution with $x_0 = c$ in $*(Z^\omega)$. Then $c - c_n \in 2^n[*(Z^\omega)]$ for all $n \in N$, so $2^n | c(i) - c_n(i)$ for all $n \in N$, $i \in *\omega$. But $0 < c_n(i) < 2^n$ so $f(c(i), n) = c_n(i)$ for all $n \in N$, $i \in *\omega$. It is clear from the definition of c_j that if i is finite then $c_j(i) = 1$ for all $j \leqslant i$, but if i is infinite and $a_k < i$ then $c_{k+1}(i) > 1$. Consequently

$$\{i \in *\omega |\ f(c(i), j) = 1 \text{ for all } j \leqslant i, j \in *N\} = \omega.$$

But a set with this kind of definition is internal, whereas ω is never internal, so we have reached a contradiction. Consequently, no such $c \in *(Z^\omega)$ can exist, and that means that $*(Z^\omega)$ is not equationally compact.

References

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Accepté par la Rédaction le 22. 3. 1976