Concerning the disconnection of continua by the omission of pairs of their points.

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Two point sets are mutually separated if they are mutually exclusive and neither contains a limit point of the other. A point set is connected if and only if it is not the sum of any two mutually separated sets. A set is disconnected if it is not connected, i.e., if it is the sum of two mutually separated sets. The point $P$ of a continuum $M$ is a cut point of $M$ provided the point set $M - P$ is not connected. The point $P$ of a continuous curve $M$ is an endpoint of $M$ provided it is true that $P$ is not an interior point of any simple continuous arc which belongs to $M$. In a paper Concerning continua in the plane which I have submitted for publication in the Transactions of the American Mathematical Society, among other results I proved the following theorems which will be used in this paper.

(I). If $K$, $H$ and $N$ respectively denote the set of all the cut points, endpoints, and simple closed curves of a continuous curve $M$, then $K + H + N = M$.

(II). In order that the continuous curve $M$ should be the boundary of a connected domain it is necessary and sufficient that if $J$ is any simple closed curve belonging to $M$ then (1) if $I$ and $E$ respectively denote the interior and exterior of $J$, then $M$ is a subset either of $J + I$ or of $J + E$, and (2) if $A$ and $B$ are any two points of $J$, then $M - (A + B)$ is not connected.

These results will be referred to by number as here listed.

Theorem 1. If $L$ is a connected point set, and $A$ and $B$ are two connected subsets of $L$ such that $L - (A + B)$ is the sum of two mutually separated sets $S_1$ and $S_2$, then if $S_1 + A + B$ is not connected it is the sum of two mutually separated connected point sets.

Proof. Let $H = S_1 + A + B$. Suppose $H$ is not connected. Then it is the sum of two mutually separated sets $T_1$ and $T_2$. It remains to show that both $T_1$ and $T_2$ are connected point sets. Now $L = T_1 + T_2 + S_2$. Neither $T_1$ nor $T_2$ is a subset of $S_1$. For suppose one of them, say $T_1$, is a subset of $S_1$. Then $T_1$ and $S_2$ are mutually separated sets, and $L$ may be expressed as the sum of two mutually separated sets $T_1$ and $T_2 + S_2$, contrary to hypothesis. Hence each of the sets $T_1$ and $T_2$ contains at least one point of $A + B$. Suppose $T_1$ contains a point of $A$. Then since $A$ is a connected subset of $H$, $A$ must be contained in $T_1$. Hence $T_2$ must contain a point of, and therefore all of, the set $B$. Now suppose $T_2$ is not connected. Then it is the sum of two mutually separated sets $N_1$ and $N_2$. One of the sets $N_1$ and $N_2$ contains $A$ and the other contains no point of $A$. Suppose $N_2$ contains $A$. Then $N_2$ is a subset of $S_1$, and $N_1$ and $S_2$ are mutually separated sets. But $L = N_1 + N_2 + T_2 + S_2 = N_1 + (N_2 + T_2 + S_2)$, and we thus have $L$ expressed as the sum of two mutually separated point sets. But this is contrary to the hypothesis that $L$ is connected. It follows that $T_1$ is connected, and a similar proof shows that $T_2$ is connected. Hence the truth of Theorem 1 is established.

R. L. Moore has shown that no continuum $M$ contains a subcontinuum $K$ which contains an uncountable set of points $T$ such that if $X$ is any point of $T$ then $M$ but not $K$ is disconnected by the omission of the point $X$. I shall establish the following related theorem.

Theorem 2. No continuum $M$ contains a subcontinuum $K$ which contains an uncountable set of points $T$ such that if $X$ and $Y$ are

any two points of $T$ then $M$ but not $K$ is disconnected by the omission of $X + Y$.

**Proof**. Suppose, on the contrary, that some continuum $M$ contains a subcontinuum $K$ which contains an uncountable set of points $T$ having the property stated in the statement of this theorem. There exists an uncountable set $H$ of pairs of points of $T$ such that every two pairs of $H$ are mutually exclusive. Then if $X, Y$ is any pair in $H$, $M - (X + Y)$ is the sum of two mutually separated point sets. Since $K - (X + Y)$ is connected, one of these point sets contains $K - (X + Y)$ and the other contains no point of $K - (X + Y)$. Let $S_m$ denote the one which contains no point of $K - (X + Y)$. Then if $X_1, Y_1$ and $X_2, Y_2$ are two distinct pairs of $H$, I will show that $S_m$ and $S_m$ can have no point in common. Suppose, on the contrary, that these two sets have a point $P$ in common. It follows by Theorem 1 that either $S_m + X_1 + Y_1$ is connected or it is the sum of two mutually separated sets $T_1$ and $T_2$ containing $X_1$ and $Y_1$ respectively. Either $T_1$ or $T_1$, say $T_1$, must contain the point $P$. Now $T_1$ has at most the points $X_2$ and $Y_1$ in common with $K$. Hence $T_1$ is a connected subset of $M - (X_2 + Y_1)$, and since $T_1$ contains the point $P$ common with $S_m$, it follows that $T_1$ is a subset of $S_m$. But $T_1$ contains the point $X_2$ of $K$, and $S_m$ has no point whatever in common with $K$. Thus the supposition that $S_m$ and $S_m$ have a point in common leads to a contradiction.

Now by the Zermelo postulate, there exists a set of points $H'$ such that (1) for each pair $X, Y$ in $H$ there exists, in $H'$, just one point which belongs to $S_m$, and (2) for each point $U$ in $H'$ there exists, in $H$, just one pair $X, Y$ such that $U$ contains $X$. Since the set $H'$ is uncountable, it contains a point $Z$ which is a limit point of $H' - Z$. But there exists in $H$ a pair $A, B$ such that $Z$ belongs to $S_m$. Since no point of $H' - Z$ belongs to $S_m$, $Z$ is not a limit point of $H' - Z$. Thus the supposition that Theorem 2 is false leads to a contradiction.

R. L. Moore has shown that in order that a bounded con-

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1) Compare this proof with that given by Moore to establish his Theorem B, loc. cit., and also with an argument given by him on page 388 of his paper Concerning simple continuous curves, Transactions of the American Mathematical Society, vol. 21 (1920), pp. 283-247.

2) Concerning the cut points of continuous curves and of other closed and connected point sets, loc. cit.

3) Compare this proof with that given by Moore to establish his theorem just mentioned above.

of $K$ is a limit point of $S_\infty$, it follows that $K - (X + Y)$ is contained in $S_\infty$ and therefore contains no point whatever in common with $S_\infty$. Then by an argument identical with the latter part of the proof of Theorem 2, starting with the sentence beginning "Then if $X_1, Y_1$, and $X_2, Y_2$ are two distinct pairs of $H$, etc." it is shown that this situation leads to a contradiction. Thus the supposition that $N$ is not a continuous curve leads to an absurdity. Hence, every subcontinuum of $M$ is a continuous curve, and the theorem is proved.

The condition of Theorem 3 is not necessary is shown by the following example. Let $AB$ denote the straight line interval from $(-1, 0)$ to $(1, 0)$. For every positive integer $i$ let $G_i$ denote a semicircle constructed on the interval $(-1/i, 0)$ to $(1/i, 0)$ as its diameter. Then let $G_i$ denote the collection of all the semicircles $(C_i)$ thus constructed. Let $G_2, G_3, G_4, \ldots$, be collections of semicircles which, with respect to the intervals $(-1, 0)$ to $(-1/2, 0)$, $(1, 0)$ to $(1/2, 0)$, $(-1/2, 0)$ to $(-1/3, 0)$, $\ldots$, correspond to the collection $(C_i)$ selected above with respect to the interval $AB$. This construction may be continued in such a way that we obtain a countable collection $G$ of semicircles such that (1) each semicircle of the collection $G$ is constructed on some interval of $AB$ as its diameter, (2) for every positive number $\varepsilon$ there are not more than a finite number of semicircles of $G$ whose diameter is greater than $\varepsilon$, and (3) if $P$ is any point of $AB$ which is not an endpoint of any semicircle of the collection $G$ and if $\varepsilon$ is any positive number, then $G$ contains an element $X$ such that the interval $I$ of $AB$ which is the diameter of $X$ contains $P$ as an interior point and is of length less than $\varepsilon$. Let $M$ denote the point set consisting of $AB$ plus all of the point sets of the collection $G$. Then $M$ is a bounded continuum, and every subcontinuum of $M$ is a continuous curve. But the subcontinuum $AB$ of $M$ contains no uncountable set of points $T$ such that $M$ is disconnected by the omission of any two points of $T$. Hence, the condition of Theorem 3 is not necessary.

**Theorem 4.** In order that the boundary $M$ of a simply connected bounded domain $D$ should be a continuous curve it is necessary and sufficient that every subcontinuum of $M$ should contain an uncountable set of points such that $M$ is disconnected by the omission of any two of them.

**Proof.** That the condition is sufficient is a direct consequence of Theorem 3. I will show that it is necessary. Let $E$ denote any subcontinuum of a continuous curve $M$ which is the boundary of a connected domain $D$. By a theorem of R. L. Wilder's, $E$ is a continuous curve. Hence, if $A$ and $B$ are two points of $E$, then $E$ contains an arc $AB$ from $A$ to $B$. The arc $AB$ contains a subarc $t$ which contains neither $A$ nor $B$. Now by (i), $M$ is the sum of three point sets $K$, $H$, and $N$, where $K$, $H$, and $N$ respectively denote the set of all the cut points, endpoints, and simple closed curves of $M$. Since every point of $t$ is an interior point of $AB$, clearly no point of $t$ can belong to $H$. Hence, $t$ must contain an uncountable set of points $T$ which is a subset either of $K$ or of $N$. If $T$ is a subset of $K$, then since every point of $K$ is a cut point of $M$, clearly $M$ is disconnected by the omission of any two points of $T$. If $T$ is a subset of $N$, then since by a theorem of R. L. Wilder's, the collection of all the simple closed curves contained in $M$ is countable, it follows that $T$ contains an uncountable subset $T'$ such that every point of $T'$ belongs to a single simple closed curve $J$ of $M$. In this case it follows immediately by (ii) of $M$ is disconnected by the omission of any two points of $T'$. Hence in any case, $E$ contains an uncountable set of points such that $M$ is disconnected by the omission of any two of them, and the theorem is proved.

1) Loc. cit., Theorem 11.

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