Concerning the disconnection of continua by the omission of pairs of their points [1]).

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Two point sets are mutually separated if they are mutually exclusive and neither contains a limit point of the other. A point set is connected if and only if it is not the sum of any two mutually separated sets. A set is disconnected if it is not connected, i. e., if it is the sum of two mutually separated sets. The point P of a continuum M is a cut point of M provided the point set $M-P$ is not connected. The point P of a continuous curve M is an endpoint of M provided it is true that P is not an interior point of any simple continuous arc which belongs to M [2]). In a paper *Concerning continua in the plane* which I have submitted for publication in the Transactions of the American Mathematical Society, among other results I proved the following theorems which will be used in this paper.

(I). If K, H and N respectively denote the set of all the cut points, endpoints, and simple closed curves of a continuous curve M, then $K+H+N=M$.

(II). In order that the continuous curve M should be the boundary of a connected domain it is necessary and sufficient that if J is any simple closed curve belonging to M then (1) if I and E respectively denote the interior and exterior of J, then M is a subset either of $J+I$ or of $J+E$, and (2) if A and B are any two points of J, then $M-(A+B)$ is not connected.

These results will be referred to by number as here listed.

Theorem 1. *If L is a connected point set, and A and B are two connected subsets of L such that $L-(A+B)$ is the sum of two mutually separated sets S_1 and S_2, then if S_1+A+B is not connected it is the sum of two mutually separated connected point sets.*

Proof. Let $H=S_1+A+B$. Suppose H is not connected. Then it is the sum of two mutually separated sets T_1 and T_2. It remains to show that both T_1 and T_2 are connected point sets. Now $L=T_1+T_2+S_2$. Neither T_1 nor T_2 is a subset of S_1. For suppose one of them, say T_1, is a subset of S_1. Then T_1 and S_2 are mutually separated sets, and L may be expressed as the sum of two mutually separated sets T_1 and T_2+S_2, contrary to hypothesis. Hence each of the sets T_1 and T_2 contains at least one point of $A+B$. Suppose T_1 contains a point of A. Then since A is a connected subset of H, A must be contained in T_1. Hence T_2 must contain a point of, and therefore all of, the set B. Now suppose T_1 is not connected. Then it is the sum of two mutually separated sets N_1 and N_2. One of the sets N_1 and N_2 contains A and the other contains no point of A. Suppose N_1 contains A. Then N_2 is a subset of S_1, and N_2 and S_2 are mutually separated sets. But $L=N_1+N_2+T_2+S_2=N_2+(N_1+T_2+S_2)$, and we thus have L expressed as the sum of two mutually separated point sets. But this is contrary to the hypothesis that L is connected. It follows that T_1 is connected, and a similar proof shows that T_2 is connected. Hence the truth of Theorem 1 is established.

R. L. Moore has shown [1]) that no continuum M contains a subcontinuum K which contains an uncountable set of points T such that if X is any point of T then M but not K is disconnected by the omission of the point X. I shall establish the following related theorem.

Theorem 2. *No continuum M contains a subcontinuum K which contains an uncountable set of points T such that if X and Y are*

[1]) Presented to the American Mathematical Society, under a different title, Feb. 27, 1926.

[2]) In the paper mentioned in the next sentence I have shown that this definition of an endpoint of a continuous curve is equivalent to the one given by R. L. Wilder, cf. R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 358.

[1]) *Concerning the cut points of continuous curves and of other closed and connected point sets*, Proceedings of the National Academy of Sciences, vol. 9 (1923), pp. 101—106, Theorem B^*.

any two points of T then M but not K is disconnected by the omission of $X+Y$.

Proof[1]). Suppose, on the contrary, that some continuum M contains a subcontinuum K which contains an uncountable set of points T having the property stated in the statement of this theorem. There exists an uncountable set H of pairs of points of T such that every two pairs of H are mutually exclusive. Then if X, Y is any pair in H, $M-(X+Y)$ is the sum of two mutually separated point sets. Since $K-(X+Y)$ is connected, one of these point sets contains $K-(X+Y)$ and the other contains no point of $K-(X+Y)$. Let S_{xy} denote the one which contains no point of $K-(X+Y)$. Then if X_1, Y_1 and X_2, Y_2 are two distinct pairs of H, I will show that $S_{x_1y_1}$ and $S_{x_2y_2}$ can have no point in common. Suppose, on the contrary, that these two sets have a point P in common. It follows by Theorem 1 that either $S_{x_1y_1}+X_1+Y_1$ is connected or it is the sum of two mutually separated sets T_1 and T_2 containing X_1 and Y_1 respectively. Either T_1 or T_2, say T_1, must contain the point P. Now T_1 has at most the points X_1 and Y_1 in common with K. Hence, T_1 is a connected subset of $M-(X_2+Y_2)$, and since T_1 contains the point P in common with $S_{x_2y_2}$, it follows that T_1 is a subset of $S_{x_2y_2}$. But T_1 contains the point X_1 of K, and $S_{x_2y_2}$ has no point whatever in common with K. Thus the supposition that $S_{x_1y_1}$ and $S_{x_2y_2}$ have a point in common leads to a contradiction. Now by the Zermelo postulate, there exists a set of points H' such that (1) for each pair X, Y in H there exists, in H', just one point which belongs to S_{xy}, and (2) for each point U in H' there exists, in H, just one pair X, Y such that S_{xy} contains U. Since the set H' is uncountable, it contains a point Z which is a limit point of $H'-Z$. But there exists in H a pair A, B such that Z belongs to S_{ab}. Since no point of $H'-Z$ belongs to S_{ab}, Z is not a limit point of $H'-Z$. Thus the supposition that Theorem 2 is false leads to a contradiction.

R. L. Moore has shown[2]) that in order that a bounded continuum M should be a continuous curve which contains no simple closed curve it is necessary and sufficient that every subcontinuum of M should contain uncountably many cut points of M. I shall prove the following related theorem.

Theorem 3. *In order that a bounded continuum M should be a continuous curve every subcontinuum of which is a continuous curve it is sufficient (but not necessary) that every subcontinuum of M should contain an uncountable set of points T such that M is disconnected by the omission of any two points of T.*

Proof[1]). Let N denote any definite subcontinuum of M, whether N be a proper subcontinuum of M or not. It is sufficient, then, to prove that N is a continuous curve. Suppose N is not a continuous curve. Then by a theorem of R. L. Moore's[2]) it follows that there exist two concentric circles C_1 and C_2 and a countable infinity of continua K, $K_1, K_2, K_3, \ldots$, such that (1) each of these continua is a subset of N and contains at least one point on each of the circles C_1 and C_2 and is a subset of the point set L which is composed of the two circles C_1 and C_2 together with all those points of the plane which lie between these two circles, (2) no two of these continua have a point in common, and, indeed, no one of them, save possibly K, is a proper subset of any connected point set which is common to N and L, and (3) the set K is the sequential limiting set of the sequence of continua $K_1, K_2, K_3, \ldots$ Since K is a subcontinuum of M, by hypothesis K contains an uncountable set of points T such that M is disconnected by the omission of any pair of points of T. It follows by Theorem 2 that T contains an uncountable set of points T' such that N, as well as M, is disconnected by the omission of any pair of points of T'. There exists an uncountable set H of pairs of points of T' such that every two pairs of H are mutually exclusive. If X, Y is any pair of H, $N-(X+Y)$ is the sum of two mutually separated point sets. One of these sets must contain infinitely many of the continua $K_1, K_2, K_3, \ldots$ Denote the one which does by S'_{xy}, and denote the other one of these sets by S_{xy}. Then since every point

[1]) Compare this proof with that given by Moore to establish his Theorem B^*, loc. cit., and also with an argument given by him on page 338 of his paper *Concerning simple continuous curves*, Transactions of the American Mathematical Society, vol. 21 (1920), pp. 333—347.

[2]) *Concerning the cut points of continuous curves and of other closed and connected point sets*, loc. cit.

[1]) Compare this proof with that given by Moore to establish his theorem just mentioned above.

[2]) *Report on continuous curves from the viewpoint of analysis situs*, Bull. Amer. Math. Society, vol. 29 (1923), pp. 296—297.

of K is a limit point of S'_{xy}, it follows that $K-(X+Y)$ is contained in S'_{xy} and therefore contains no point whatever in common with S_{xy}. Then by an argument identical with the latter part of the proof of Theorem 2, starting with the sentence beginning „Then if X_1, Y_1, and X_2, Y_2 are two distinct pairs of H, etc.", it is shown that this situation leads to a contradiction. Thus the supposition that N is not a continuous curve leads to an absurdity. Hence, every subcontinuum of M is a continuous curve, and the theorem is proved.

That the condition of Theorem 3 is not necessary is shown by the following example. Let AB denote the straight line interval from $(-1, 0)$ to $(1, 0)$. For every positive integer i let C_i denote a semicircle constructed on the interval $(-1/i, 0)$ to $(1/i, 0)$ as its diameter. Then let G_1 denote the collection of all the semicircles (C_i) thus constructed. Let $G_2, G_3, G_4, \ldots$ be collections of semicircles which, with respect to the intervals $(-1, 0)$ to $(-1/2, 0)$, $(1, 0)$ to $(1/2, 0)$, $(-1/2, 0)$ to $(-1/3, 0), \ldots$, correspond to the collection (C_i) selected above with respect to the interval AB. This construction may be continued in such a way that we obtain a countable collection G of semicircles such that (1) each semicircle of the collection G is constructed on some interval of AB as its diameter, (2) for every positive number ε there are not more than a finite number of semicircles of G whose diameter is greater than ε, and (3) if P is any point of AB which is not an endpoint of any semicircle of the collection G and if ε is any positive number, then G contains an element X such that the interval I of AB which is the diameter of X contains P as an interior point and is of length less than ε. Let M denote the point set consisting of AB plus all of the point sets of the collection G. Then M is a bounded continuum, and every subcontinuum of M is a continuous curve. But the subcontinuum AB of M contains no uncountable set of points T such that M is disconnected by the omission of any two points of T. Hence, the condition of Theorem 3 is not necessary.

Theorem 4. *In order that the boundary M of a simply connected bounded domain D should be a continuous curve it is necessary and sufficient that every subcontinuum of M should contain an uncountable set of points such that M is disconnected by the omission of any two of them.*

Proof. That the condition is sufficient is a direct consequence of Theorem 3. I will show that it is necessary. Let E denote any subcontinuum of a continuous curve M which is the boundary of a connected domain D. By a theorem of R. L. Wilder's[1]), E is a continuous curve. Hence, if A and B are two points of E, then E contains an arc AB from A to B. The arc AB contains a subarc t which contains neither A nor B. Now by (I), M is the sum of three point sets K, H, and N, where K, H, and N respectively denote the set of all the cut points, endpoints, and simple closed curves of M. Since every point of t is an interior point of AB, clearly no point of t can belong to H. Hence, t must contain an uncountable set of points T which is a subset either of K or of N. If T is a subset of K, then since every point of K is a cut point of M, clearly M is disconnected by the omission of any two points of T. If T is a subset of N, then since by a theorem of R. L. Wilder's[2]), the collection of all the simple closed curves contained in M is countable, it follows that T contains an uncountable subset T' such that every point of T' belongs to a single simple closed curve J of M. In this case it follows immediately by (II) that M is disconnected by the omission of any two points of T'. Hence in any case, E contains an uncountable set of points such that M is disconnected by the omission of any two of them, and the theorem is proved.

[1]) Loc. cit., Theorem 11.
[2]) Loc. cit., Theorem 4.

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