A Condition that every Subcontinu of a Continuous Curve be a Continuous Curve.

By

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It is well known that every two points of a continuous curve M can be joined by a simple continuous arc[1]), which belongs to M. The converse, that every closed connected set having the property that every pair of points of the set lie on a simple continuous arc belonging to the set, is a continuous curve, is not true as may easily be seen by an example. Indeed it is not necessary that a continu, in which every irreducible subcontinu is an arc, be a continuous curve[2]). However, we shall show in this note that if every irreducible continu joining two points of a continuous curve is an arc, then every subcontinu of the continuous curve is itself a continuous curve[3]). While the proof is carried out for two dimensions, a similar proof holds for any number of dimensions.

[1]) Cf. R. L. Moore, *A Theorem Concerning Continuous Curves*, Bulletin of the American Mathematical Society vol. 24 (1916—17).

[2]) Let us consider the point set P_i, where for every positive integer P_i is the point whose polar coordinates are $(1, \pi/2^i)$. Let O and P be the points whose polar coordinates are $(0,0)$ and $(1,0)$ respectively. Then let M be the continu composed of l, the straight line interval from $(0,0)$ to $(1,0)$ plus the infinite collection of intervals $l_1, l_2, l_3, \ldots$ where l_i is the straight line interval from O to P_i. It is clear that if Q is a point of l different from O, then M is not connected im kleinen at Q.

[3]) For a study of conditions, necessary and sufficient that every subcontinu of a continuous curve be a continuous curve see H. M. Gehman, Concerning the Subsets of a Continuous Curve, Annals of Mathematics vol. 26 (1925).

Theorem A. *A necessary and sufficient condition that every subcontinu of a continuous curve be a continuous curve is that every irreducible continu be an arc*[1]).

Proof: That the condition is *necessary* follows as follows. Suppose M is a bounded continuous curve such that every subcontinu is a continuous curve and suppose further that there exists an irreducible continu k from A to B that is not an arc. Now as k is a continuous curve, by assumption, A and B can be joined by an arc h belonging entirely to k. Hence as h is a continu from A to B and as k is an irreducible continu between the same two points, k and the arc h must be identical, contrary to assumption.

The condition is *sufficient*. Suppose that M contains as a subset a continu H which is not a continuous curve. Then there exist two concentric circles K_1 and K_2 whose radii are r_1 and r_2, respectively and a sequence of subcontinua of H, H_∞, H_1, $H_2,\ldots$ such that (1) each of these subcontinua contains at least one point on K_1 and at least one point on K_2 but no point exterior to K_1 or interior to K_2, (2) no two of these subcontinua have a point in common and no two of them contain points of any connected subset of M which lies wholly in $K_1 + K_2 + I$ (where I is the annular domain bounded by K_1 and K_2) (3) H_∞ is the sequential limiting set of the sequence H_1, H_2, $H_3,\ldots$ (4) if K is the maximal subcontinu of H containing H_∞ and lying wholly in the set $K_1 + K_2 + I$, then all the continua H_1, H_2, $H_3,\ldots$ lie in a connected set of $M - K$[2]). For every i, H_i is a closed connected set containing a point P_i on K_1 and a point Q_i on K_2. Now there is an irreducible continu I_i of H_i from P_i to Q_i[3]) and by hypothesis P_iQ_i is an arc from P_i to Q_i. Let B_i be the first point on the arc

[1]) If A and B are distinct points, then an *irreducible continu* from A to B is a continu containing both A and B, which contains no proper subcontinu containing both A and B. If A and B are distinct points, then a *simple continuous arc* from A to B is a continu containing both A and B, which contains no proper connected subset containing both A and B. Throughout this paper „arc" and „simple continuous arc" will be used interchangeably. If AB is an arc, then the symbol $\overline{AB}$ will denote the point set $AB - A - B$.

[2]) Cf. R. L. Moore, *Report on Continuous Curves from the Viewpoint of Analysis Situs*, Bull. Amer. Math. Soc. vol. 29 (1923) p. 296. See also R. L. Wilder, *Concerning Continuous Curves*, Fund. Math. vol. 7 p. 371.

[3]) Cf. S. Janiszewski, *Sur les continus irréductibles entre deux points*, Journal de l'Ecole Polytechnique 2-me série vol. 16 p. 109.

$P_i Q_i$ which is on K_2 while A_i is the last point of $P_i B_i$ on K_1. The point set $A_1, A_2, A_3, \dots$ has a limit point A which belongs to H_∞. Select a sequence $A_{1,1}, A_{1,2} \dots$ approaching A as the sequential limiting point.

Put about A as center a circle C_1 of diameter ε where ε is less than onetenth of r_1 minus r_2. As M is connected im kleinen, it is possible to find an integer α_1 such that if $i \neq j$, $i \geqslant \alpha_1$ and $j \geqslant \alpha_1$, then $A_{1,i}$ and $A_{1,j}$ can be joined by an arc lying entirely within C_1. Consider the point set $B_{1,\alpha_1+1}, B_{1,\alpha_1+2} \dots$ where $B_{1,i}$ is the endpoint of $A_{1,i} B_{1,i}$ on K_2 and $\underset{\smile}{A_{1,i} B_{1,i}}$ is free of points of $K_1 + K_2$. These points have a limiting point B on H_∞ and we can pick out a subset of $B_{1,\alpha_1+1} B_{1,\alpha_1+2}, \dots$ having B as its sequential limiting point. Let this sequence be $B_{1,n_1}, B_{1,n_2}, \dots$ Put about B a circle C_2 of radius $\leqslant \varepsilon/2$ such that no point of the arc $A_{1,\alpha_1} B_{1,\alpha_1}$ is within or on C_2. By the connectivity im kleinen of M at B, it is possible to find an integer β_1 such that if $i \neq j$, $i \geqslant \beta_1$ and $j \geqslant \beta_1$, then B_{1,n_i} and B_{1,n_j} are the endpoints of an arc of M which lies entirely within C_2. Join A_{1,α_1} to $A_{1,n_{\beta_1}}$ by an arc of M lying wholly within C_1. This arc will contain as a subset an arc $\overline{A}_1 \overline{A}_2$ where $\overline{A}_1$ is on $A_{1,\alpha_1} B_{1,\alpha_1}$ and $\overline{A}_2$ is on $A_{1,n_{\beta_1}} B_{1,n_{\beta_1}}$ while $\underset{\smile}{\overline{A}_1 \overline{A}_2}$ is free of points of $A_{1,\alpha_1} B_{1,\alpha_1} + A_{1,n_{\beta_1}} B_{1,n_{\beta_1}}$.

Draw a circle K_3, concentric with K_1 and K_2 and having as its radius $r_1 - \varepsilon$. Let A_{2,n_i} be the first point of $B_{1,n_i} A_{1,n_i}$ $(i \geqslant \beta_1)$ going from B_{1,n_i} to A_{1,n_i} which is on K_3 so that $\underset{\smile}{B_{1,n_i} A_{2,n_i}}$ is a subset of the annular domain I_{23} between K_2 and K_3. Consider the point set $A_{2,n_{\beta_1+1}}, A_{2,n_{\beta_1+2}} \dots$ This set will have as a limit point a point A_2 of H_∞ on K_3 and again we can select from it a subsequence $A_{2,m_1}, A_{2,m_2}, \dots$ having A_2 as its sequential limiting point. Now put about A_2 as center a circle C_3 of diameter $\leqslant \varepsilon/2^2$ such that there are within or on C_3 no points of the arcs $A_{1,\alpha_1} B_{1,\alpha_1}$ and $A_{1,n_{\beta_1}} B_{1,n_{\beta_1}}$. It is clear that no point of $\overline{A}_1 \overline{A}_2$ is within this circle. By the connectivity im kleinen, there will exist an integer α_2 such that if $i \neq j$, $i \geqslant \alpha_2$ and $j \geqslant \alpha_2$, then A_{2,m_i} and A_{2,m_j} can be joined by an arc of M which lies wholly within C_3. Join $B_{1,n_{\beta_1}}$ to $B_{2,m_{\alpha_2}}$ by an arc of M lying wholly in C_2. This arc will contain as a subarc, an arc $\overline{B}_2 \overline{B}_3$ where $\overline{B}_2$ is on $A_{1,n_{\beta_1}} B_{1,n_{\beta_1}}$ and $\overline{B}_3$ is on $A_{2,n_{\alpha_2}} B_{2,n_{\alpha_2}}$ while $\underset{\smile}{\overline{B}_2 \overline{B}_3}$ is free of points of both these arcs and of course of the arc $A_{1,\alpha_1} B_{1,\alpha_1}$ which has previously been excluded.

Now draw a circle C_4 concentric with K_1 and K_2, having as its radius $r_2 + \varepsilon/2$ and proceed as before by means of first picking out a sequence of the B's and then getling an arc $\overline{A}_3 \overline{A}_4$ lying entirely within C_3, which has no point except $\overline{A}_3$ in common with $B_{\alpha_1} \overline{A}_1$ (on $A_{\alpha_1} B_{\alpha_1}$) $+ \overline{A}_1 \overline{A}_2 + \overline{A}_2 \overline{B}_2$ (on $A_{1,n_{\beta_1}} B_{1,n_{\beta_1}}$) $+ \overline{B}_2 \overline{B}_3 + \overline{B}_3 \overline{A}_3$ (on $A_{2,m_{\alpha_2}} B_{2,m_{\alpha_2}}$).

Continue this process[1]). Consider the set $W = (B_{\alpha_1} \overline{A}_1 + \overline{A}_1 \overline{A}_2 + \overline{A}_2 \overline{B}_2 + \overline{B}_2 \overline{B}_3 + \dots)$ and let $(\overline{W} - W)$ denote the limit points of W not in W. Clearly all points of $(\overline{W} - W)$ which are limit points of $B_{\alpha_1} \overline{A}_1 + \overline{A}_2 \overline{B}_2 + \overline{A}_3 \overline{B}_3 + \dots$ lie in H_∞ and also between or on circles K and $\overline{K}$, concentric with K_1 and K_2 and of radii $(r_1 - 4\varepsilon/3)$ and $(r_2 + 2\varepsilon/3)$ respectively. Clearly while the sets $\overline{A}_{2i-1} \overline{A}_{2i}$ and $\overline{B}_{2j} \overline{B}_{2j+1}$ may have points in common with H_∞, they can, by our method of construction, have no points in common with those points of H_∞ which lie on or between K and $\overline{K}$ and the only limit points of $\overline{A}_1 \overline{A}_2 + \overline{A}_3 \overline{A}_4 + \dots$ not in the set, are limit points of $\overline{A}_1 + \overline{A}_2 + \overline{A}_3 + \dots$ The set $\overline{W}$ is clearly an irreducible continu between B_{α_1} and any point of $\overline{W} - W$. It is not an arc as it contains infinitely many mutually exclusive subarcs of diameter greater than $\frac{1}{2}(r_1 - r_2)$. Thus we are led to a contradiction if we suppose our condition is not sufficient.

[1]) The process described above could have been simplified very much had one been interested merely in a proof for the subcontinua of a two dimensional continuous curve. In two dimensions we are aided by properties following rather directly from the Jordan curve separation theorem.

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