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RELATIVE IDEALS IN SEMIGROUPS, I

(FAUCETT'S THEOREM)

BY

A. D. WALLACE (NEW ORLEANS, LA.)

0. In this note we show how Faucett's Theorem [2] on cut-points of the minimal ideal of a compact connected semigroup may be relativized. We also extend some results of Clifford's [1].

1. A semigroup is a non-void Hausdorff space together with a continuous associative multiplication, denoted by juxtaposition, $(x, y) \to xy$. In all that follows S will denote a semigroup. A subset T of S will be called a *subsemigroup* provided T is not empty and $T^2 \subset T$. We shall generally not distinguish between x and $\{x\}$ if confusion of meaning is unlikely so that we write xA in place of $\{x\}A$, $x \cup A$ in place of $\{x\} \cup A$ and $A \setminus x$ in place of $A \setminus \{x\}$. Moreover, we omit inclusive quantifiers relative to S if doing so will result in no misunderstanding.

We first define some set-valued set-functions which will depend on a subset T of S.

In all instances T will be at least a closed subsemigroup of S.

For $A \subset S$ let

$$L(A) = A \cup TA, \quad J(A) = A \cup TA \cup AT \cup TAT, \quad R(A) = A \cup AT,$$

$$H(A) = L(A) \cap R(A).$$

If it is desirable to call attention to T we write $L(A; T)$ for $L(A)$, and so on.

The set $A \subset S$ is termed a *left T-ideal* if $L(A) \subset A$ (i. e., if $TA \subset A$) and if A is not empty. Similarly we define a *right T-ideal* using R and a *T-ideal* using J. The set A being non-void we see that A is a T-ideal if and only if $TA \subset A \supset AT$.

It will be observed at once that the union and intersection (if non-void) of T-ideals of any category is again a T-ideal of the same category. If $A \neq \square$ (the empty set) then $L(A)$, $J(A)$ and $R(A)$ are T-ideals of the appropriate category.

In order to avoid excessive wordage we shall frequently give definitions and state propositions for only one sort of T-ideal and, further, if $T = S$, we shall speak of *left ideal* rather than *left S-ideal*, and so on.

The set $A \subseteq S$ is a *minimal left-T-ideal* if A is a left T-ideal but if no proper subset of A is a left T-ideal.

(1.1) *If L is a minimal left T-ideal then La is a minimal left T-ideal, for any $a \in S$.*

Proof. Suppose that M is a left T-ideal contained in La and let

$$N = \{x \mid xa \in M,\ x \in L\}.$$

Clearly N is non-void (since $M \neq \square$) and it is a left T-ideal. In fact, $TNa \subset TM \subset M$ so that, by $TL \subset L$, $TN \subset N$. Thus, since L is minimal, $N = L$, i. e. $M \supset Na \supset La$ and thus $La = M$. This completes the proof.

(1.2) *If there is at least one minimal left T-ideal then M, the union of all such, satisfies $TM = M \supset MS$ so that M is a left T-ideal and a right ideal. If also there is a minimal right T-ideal and if N is the union of all such then N satisfies $NT = N \supset SN$ and we have*

$$MN \subset M \cap N,$$

each of these sets being a subsemigroup of S.

Proof. Assume that there is at least one minimal left T-ideal L. We have at once that $TM \subset M$ and by (1.1) we have $MS \subset M$. If $x \in M$ then $x \in L$ for some minimal T-ideal L and thus $Tx \subset TL \subset L$. Since $T^2 \subset T$ we have $T(Tx) \subset T^2 x \subset Tx$ so that Tx is a left T-ideal contained in the minimal left T-ideal L and hence $Tx = L$. Since $x \in L$ we have $x \in Tx \subset TM$ and therefore $M \subset TM$.

As above it follows that $NT = N \supset SN$. Now the intersection and product (in this order) of a right ideal and a left ideal is a subsemigroup and the product is contained in the intersection. It is clear that an ideal of any category is a subsemigroup.

It may be that T (as a semigroup in its own right) has a minimal left ideal, say L. Then clearly L is a minimal left T-ideal. There may, however, be minimal left T-ideals which do not intersect T. If S is the closed unit interval with its usual multiplication and if $T = \{1\}$, then any subset of S is a left T-ideal and any element of S is a minimal left T-ideal.

(1.3) *With the notation of* (1.2) *suppose that T has a minimal left ideal L_0 and a minimal right ideal R_0; then $M = L_0 M = L_0 S$ and $N = R_0 N = R_0 S$. Moreover, $M \cap N$ is the union of all minimal T-ideals and if $a \in M \cap N$, then TaT is that minimal T-ideal which contains a and $TaT = L_0 a R_0$. Finally, $MN = M \cap N$.*

Proof. Let I be a minimal T-ideal and let $a \in I$. Then $L_0 a \subset L_0 I \subset TI \subset I$. Right-multiplication of $L_0 a \subset I$ by R_0 gives $L_0 a R_0 \subset I$ and from this we get $L_0 a R_0 = I$ since $L_0 a R_0$ is a T-ideal contained in the minimal T-ideal I. But clearly

$$L_0 a R_0 = U\{L_0 ar \mid r \in R_0\}$$

is a union of minimal left T-ideals by (1.1) and is thus a subset of M. Equally, by the left-right dual of (1.1) we see that I is a subset of N. Hence any minimal T-ideal is contained in $M \cap N$.

Suppose now that $a \in M \cap N$; we will show that TaT is a minimal T-ideal. If L is the minimal left T-ideal containing a, then it follows readily that $L = L_0 a$ since $L_0 a \subset L_0 L \subset TL \subset L$ and $T(L_0 a) \subset (TL_0) \subset L_0 a$ and $L_0 a = L$ by the minimality of L. Clearly, $a \in L_0 a$ and, in the same fasion, $a \in aR_0$. Thus, $a \in aR_0 \subset L_0 a R_0$ and bilateral multiplication by T gives $TaT \subset L_0 a R_0$ since $TL_0 \subset L_0$ and $R_0 T \subset R_0$.

Suppose that I is a T-ideal contained in TaT and that $x \in I$, so that $x = paq$ with $p \in L_0$ and $q \in R_0$. It follows readily that $L_0 p = L_0$ and that $qR_0 = R_0$ so that

$$L_0 a R_0 = L_0 paq R_0 \subset L_0 I R_0 \subset TIT \subset I.$$

From this we conclude that TaT is a minimal T-ideal and that $TaT = L_0 a R_0$.

This is immediate: for if $x \in M \cap N$, then $x \in L_0 x R_0$ as in the above argument, $L_0 x \subset M$ and $R_0 \subset N$ so that $x \in MN$.

(1.4) *If T is connected (compact) then minimal T-ideals of all categories are also connected (compact). If T and S are connected and if T has a minimal left T-ideal and a minimal right T-ideal then M, N and $MN = M \cap N$ are connected. If T is compact then it has minimal ideals of all categories and if S is compact then M, N and MN are closed.*

Proof. The first statement is readily disposed of since, for example, if L is a minimal left T-ideal and if $a \in L$, then $Ta = L$.

Suppose that the hypotheses in the second assertion hold — then $M = L_0 S$ by (1.3), where L_0 is a minimal left T-ideal of T. Since L_0 is connected (by the first part) and since S is connected we see that M is connected. Similarly, N is connected and therefore MN is connected.

The preceding results relativize propositions of A. H. Clifford [1].

It is well-known (Numakura [4]) that any compact semigroup contains minimal ideals of all three categories. To prove this most quickly one takes the intersection of a maximal tower (under inclusion, by the Hausdorff Maximality Principle) of, say, closed left ideals and it turns out that this is a minimal left ideal. The proof then proceeds as in the last argument.

We write

$$L_0(A) = \{x | L(x) \subset A\}$$

and similarly for $J_0(A)$ and $R_0(A)$. Just as $L(A)$ is the smallest left T-ideal containing A, $L_0(A)$ is the largest left T-ideal contained in A, assuming of course that $L(A)$ and $L_0(A)$ are non-void.

As in [3] or [5] the following may be proved:

(1.5) *If A is closed, then $L_0(A)$ is closed while if A is open and if T is compact, then $L_0(A)$ is open.*

(1.6) *If T is compact, if A is compact and if there is a left T-ideal not containing A, then there is a left T-ideal maximal among left T-ideals that do not contain A; moreover, each such is open.*

In particular, if S is compact and if S properly contains a left T-ideal then there is at least one maximal proper left T-ideal and each of these is open.

This result has a "dual":

(1.7) *If T is compact, if A is closed and if some left T-ideal intersects A then there is a minimal such and each of them is closed. In particular, there exists a minimal left T-ideal.*

2. Certain proofs will be simplified if we introduce the notion of a semigroup acting on a space.

An *act* is such a continuous function

$$T \times X \to X$$

that, employing juxtaposition to denote functional values $((t, x) \to tx)$,

(i) T is a semigroup

(ii) X is a non-void Hausdorff space

(iii) For any elements $t_1, t_2 \in T$ and $x \in X$ we have

$$t_1(t_2 x) = (t_1 t_2)x.$$

Example I. If S is a semigroup and if T is a subsemigroup, then T acts on S by left multiplication.

Example II. Let S be a semigroup and let T be a subsemigroup. With the multiplication $(x, y)(z, w) = (xz, wy)$ the space $T \times T$ is a semigroup and $T \times T$ acts on S in the following way:

$$(x, y)z = xzy.$$

Throughout this section we assume that T acts on X as stipulated in the definition.

It is convenient to write, for $A \subset T$ and $B \subset X$,

$$AB = \{tx | t \in A \text{ and } x \in B\},$$

and

$$A^{[-1]}B = \{x | Ax \subset B\}.$$

A subset $I \subset X$ is *subvariant* if $I \neq \square$ and if $TI \subset I$. It is clear that I is subvariant if it is not empty and if $I \subset T^{[-1]}I$.

It is readily shown that if A is a compact space and if B is any space then the projection of $A \times B$ onto B is a *closed* function. Employing this fact we easily prove

(2.1) *If A is a compact subset of T and if B is an open subset of X then $A^{[-1]}B$ is open.*

(2.2) *If T is connected, if I is subvariant, if B is a subvariant connected subset of I and if C is the component of I containing B then C is subvariant.*

Proof. We have

$$TB \subset B \subset C \subset I$$

so that

$$TB \subset TC \subset TI \subset I$$

and thus TC is a connected subset of I which intersects C. Hence $C \cup TC$ is a connected subset of I and thus $TC \subset C$.

We denote by A^*, A^0 and $F(A)$ the closure, interior and boundary of the set A.

(2.3) *Suppose that X is connected and either locally connected or compact, that A is such a subset of X that $A \cap T^{[-1]}A$ is a non-void proper subset of X and that C is a subvariant component of $A \cap T^{[-1]}A$; if T is compact then the closure C^* intersects the boundary of A.*

Proof. If $C^* \cap F(A) = \square$, then $C^* \subset A^0$ and thus, since $TC \subset C$ implies $TC^* \subset C^*$, we have $TC^* \subset A^0$ and hence $C^* \subset T^{[-1]}A^0$. Thus $C^* \subset A^0 \cap T^{[-1]}A^0$, the latter being open by (2.1). Since $A \cap T^{[-1]}A \supset A^0 \cap T^{[-1]}A^0$ it is clear that $C^* = C$ and that C is a component of the latter set. But since X is connected and either compact or locally connected, no component of a non-void open proper subset of X can have its closure in the subset. Thus $C^* \cap F(A) \neq \square$.

(2.4) *Let X be connected and either compact or locally connected and let T be compact and connected. If $z \in X$ separates two subvariant sets in X then z is subvariant, i. e., $Tz = z$.*

Proof. Suppose that $X \setminus z = U \cup V$, where U and V are disjoint open sets and that $A \subset U$ and $B \subset V$ are subvariant. If $a \in A$ and $b \in B$, then Ta and Tb are subvariant connected sets contained in U and V. We have $T^2a \subset Ta \subset U$ and thus $Ta \subset U \cap T^{[-1]}U$, the latter being a non-void proper subvariant subset of X. By (2.2) the component C of $U \cap T^{[-1]}U$ which contains Ta is subvariant and by (2.3) we know that C^* intersects the boundary of U, i. e., $z \in C^*$ and $Tz \subset C^*$. If we argue similarly concerning D, the component of $V \cap T^{[-1]}V$ which contains Tb, then $z \in D^*$ and $Tz \subset D^*$. It follows at once that $Tz = z$.

3. In this section is the principal result of this paper, a generalization to T-ideals of a result due to Faucett [2]. His theorem is to the effect that if the minimal ideal of a compact connected semigroup has a cutpoint, then every element of the minimal ideal is a left zero or a right zero.

THEOREM. *If S is a compact connected semigroup, if T is a compact connected subsemigroup of S and if I is a minimal T-ideal of S then if z is a cutpoint of I we have either $Tz = z$ or $zT = z$.*

Proof. Since $I \subset MN$ (section 1) it is readily seen that I is the union of minimal left T-ideals and, as well, the union of minimal right T-ideals. Moreover, if I contains just one minimal left T-ideal L then $I = L$ and similarly, if I contains just one minimal right T-ideal R, then $I = R$.

Suppose first that I contains precisely one minimal left T-ideal and one minimal right T-ideal so that $L = I = R$ and let L_0 and R_0 be, respectively, minimal left and right ideals of T, or, equally, minimal left and right T-ideals contained in T, (1.7). Let $G = R_0L_0$ so that G is a group, a result of Clifford's, [1, 2.1]. If $x \in I = L$ then $L_0x = L = I$, by (1.1), and thus

$$Gx = R_0L_0x = R_0I \subset I.$$

On the other hand, $Gx = R_0L_0x = R_0R$ since $L_0x = L = R$ and thus Gx is the union of minimal right T-ideals by the "dual" of (1.1). But I is a minimal right T-ideal and thus $Gx = R$, or, $Gs = I$. We now claim that I is homogeneous. For if e is the unit of G we have

$$eG = G \quad \text{and} \quad eGx = Gx = I$$

so that, since $e^2 = e$, we have $ex = x$ for each $x \in I$. It follows easily that if $g \in G$, then $x \to gx$ takes I homeomorphically onto I and if $x, y \in I$ then from $Gx = I$ we infer that $y = gx$ for some $g \in G$. Thus I is homogeneous. But then, since one point of I is a cutpoint, it follows that every point of I is a cutpoint contrary to a well-known result of R. L. Moore, see [6] or [7, p. 37].

Suppose then that I contains more than one minimal left T-ideal and let L be one of them which does not contain the cutpoint z of I. We observe that T acts on I by left multiplication and we use the relative topology of I from now on. It should be noted that for any $a \in I$ we have $TaT = I$ since I is minimal and thus I is a continuum.

We have $I \setminus z = U \cup V$, where U and V are non-void disjoint open sets and $L \subset U$. Since TL L we have $L \subset U \cap T^{[-1]}U$ and this latter set is open, see section 2. By (2.2) and (2.3) the component C of $U \cap T^{[-1]}U$ has the property that its closure intersects the boundary of U, i. e., $z \in C^*$. But $Tz \subset TC^* \subset C^* \subset U^*$ and Tz is a minimal left T-ideal (see (1.1)). Thus there is a minimal left T-ideal L' which intersects V because $V \subset I \subset M \cap N = MN$. Moreover, L' cannot intersect Tz because $Tz \subset U^*$ and cannot intersect U because it is connected and would then have to contain $z \in Tz$. Accordingly, z separates L and L' in I and thus, by (2.4), we have $Tz = z$.

It is not difficult to show that if also $z \in T$ then $Tx = x$ for each $x \in I$ and thus obtain Faucett's result in toto.

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THE TULANE UNIVERSITY OF LOUISIANA

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