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ON ALGEBRAS WITH A QUASI-INVOLUTION

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1. The results of this note have been announced in Bulletin de l'Académie Polonaise des Sciences [3].

Let P be an associative and commutative ring with a unity element. An operation * defined on the ring P is called an *involution* if it satisfies the conditions

$$(\alpha+\beta)^* = \alpha^*+\beta^*, \quad \alpha^{**} = \alpha, \quad (\alpha\beta)^* = \alpha^*\beta^*$$

for any $\alpha, \beta \in P$. The aim of this paper is to study a class of algebras over the ring P with an involution.

A set A is said to be an *algebra with a quasi-involution* over the ring P if

1° A is a modul over P,

2° A is closed with respect to a product xy such that the two-sided distributive law holds and

(i) $$(\alpha x)y = \alpha(xy),$$

(ii) $$x(\alpha y) = \alpha^*(xy)$$

are true for any $\alpha \in P$ and $x, y \in A$. We remark that the product is not necessarily associative.

3° An operation $^+$ is defined on A and satisfies the conditions

(iii) $$(x+y)^+ = x^+ + y^+,$$

(iv) $$(\alpha x)^+ = \alpha^* x^+,$$

(v) $$x^{++} = x,$$

(vi) $$(xy)^+ = yx,$$

(vii) $$x(yz) = (xz^+)y$$

for any $\alpha \in P$ and $x, y, z \in A$. The operation $^+$ will be called a *quasi-involution*.

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2. Let A_0 be an associative algebra with an involution * over P satisfying in addition to the usual requirements the condition $(ax)^* = a^*x^*$ for any $a \in P$ and $x \in A_0$ (e. g. see [5]). In the sequel we shall denote by $\mathcal{K}(A_0)$ the set A_0 with addition and multiplication by elements of P unchanged and with a new multiplication in A_0 defined as follows

$$xy = y^* \circ x, \tag{1}$$

where $\circ$ is the product operation in A_0.

If A_0 is the algebra of square matrices of fixed order, then $\mathcal{K}(A_0)$ coincides with the algebra of cracovians introduced by T. Banachiewicz (see [1]). Therefore, for arbitrary A_0, $\mathcal{K}(A_0)$ will be called a *cracovian algebra* generated by A_0. The theory of cracovian rings is developed in [2] and [4].

THEOREM 1. *If A_0 is an associative algebra with an involution, then $\mathcal{K}(A_0)$ is an algebra with a quasi-involution, where as a quasi-involution the involution in A_0 is taken.*

Proof. Of course, to prove our theorem it is sufficient to show that in $\mathcal{K}(A_0)$ the two-sided distributive law and equalities (i), (ii), (vi) and (vii) are true. Taking into account definition (1) we have the equalities

$$x(y+z) = (y+z)^* \circ x = y^* \circ x + z^* \circ x = xy + xz,$$

$$(y+z)x = x^* \circ (y+z) = x^* \circ y + x^* \circ z = yx + zx,$$

$$(ax)y = y^* \circ (ax) = a(y^* \circ x) = a(xy),$$

$$x(ay) = (ay)^* \circ x = a^*(y^* \circ x) = a^*(xy),$$

$$(xy)^+ = (y^* \circ x)^* = x^* \circ y = yx,$$

$$x(yz) = (yz)^* x = (z^* \circ y)^* \circ x = (y^* \circ z) \circ x = y \circ (z \circ x)$$
$$= y^* \circ (z^{**} \circ x) = (xz^*)y = (xz^+)y,$$

which completes the proof.

THEOREM 2. *Every algebra with a quasi-involution is equal to the cracovian algebra generated by an associative algebra with an involution.*

Proof. Let A be an algebra with a quasi-involution over a ring P. By A_0 we denote the set A with addition and multiplication by elements of P unchanged, whereas the involution and $\circ$-multiplication are defined by formulae

$$x^* = x^+, \quad x \circ y = yx^+. \tag{2}$$

First we shall prove that A_0 is an associative algebra with an involution. Using (2) we get the distributive laws

$$x \circ (y+z) = (y+z)x^+ = yx^+ + zx^+ = x \circ y + x \circ z,$$

$$(y+z) \circ x = x(y+z)^+ = xy^+ + xz^+ = y \circ x + z \circ x.$$

Further, according to (vii),

$$(x \circ y) \circ z = (yx^+) \circ z = z(yx^+)^+ = z(x^+y) = (zy^+)x^+$$
$$= (y \circ z)x^+ = x \circ (y \circ z).$$

From (i), (ii) and (iv) we obtain the equalities

$$(ax) \circ y = y(ax)^+ = y(a^*x^+) = a(yx^+) = a(x \circ y),$$

$$x \circ (ay) = (ay)x^+ = a(yx^+) = a(x \circ y).$$

Finally to prove that the operation * is an involution it suffices to show that $(x \circ y)^* = y^* \circ x^*$. By (v) and (vi) we infer that

$$(x \circ y)^* = (yx^+)^+ = x^+y = x^+y^{++} = y^* \circ x^*.$$

Thus A_0 is an associative algebra with a involution. Since

$$y^* \circ x = y^+ \circ x = xy^{++} = xy,$$

$A = \mathcal{K}(A_0)$, which completes the proof.

3. In [2] the notion of τ-rings was introduced. An analogous notion can be introduced for algebras. Namely, a set B is said to be a *τ-algebra* over a ring P with an involution if

1° B is a modul over P;

2° B is closed with respect to a product xy such that the two-sided distributive law holds and $(ax)y = a(xy)$, $x(ay) = a^*(xy)$ are true for any $a \in P$ and $x, y \in B$;

3° There exists an element $\tau \in B$ such that for every $x, y, z \in B$ the following equalities hold:

$$x(yz) = \big(x(\tau z)\big)y, \tag{*}$$

$$\tau(\tau x) = x. \tag{**}$$

THEOREM 3. *Every τ-algebra is an algebra with a quasi-involution defined by means of formula $x^+ = \tau x$.*

Proof. To prove our assertation it is sufficient to show that the operation $^+$ is a quasi-involution. From (*) and (**) we get the equalities

$$(x+y)^+ = \tau(x+y) = \tau x + \tau y = x^+ + y^+,$$

$$(ax)^+ = \tau(ax) = a^*(\tau x) = a^* x^+,$$

$$x^{++} = \tau(\tau x) = x,$$

$$(xy)^+ = \tau(xy) = \big(\tau(\tau y)\big)x = yx,$$

$$(xz^+)y = \big(x(\tau z)\big)y = x(yz).$$

Consequently, $^+$ is a quasi-involution, which proves the theorem.

THEOREM 4. *An algebra with a quasi-involution is a τ-algebra with $x^+ = \tau x$ if and only if it is a cracovian algebra generated by an associative algebra with a unity element.*

Proof. Let us suppose that A is a cracovian algebra generated by an associative algebra A_0 with a unity element e. Denoting by $\circ$ the product in A_0 we have the equalities

$$x(yz) = (yz)^* \circ x = (z^* \circ y)^* \circ x = (y^* \circ z) \circ x = y^* \circ (z \circ x) = (z \circ x) y$$
$$= (xz^*) y = \big(x(z^* \circ e)\big) y = \big(x(ez)\big) y,$$
$$e(ex) = (ex)^* \circ e = (ex)^* = (x^* \circ e)^* = x^{**} = x.$$

Consequently, as an element τ satisfying $(*)$ and $(**)$ the unity element e can be taken. In other words, A is a τ-algebra.

Now let us assume that A is a τ-algebra. By theorem 2 it can be represented by a cracovian algebra $A = \mathscr{K}(A_0)$, where A_0 is an associative algebra and the multiplication $\circ$ in A_0 is given by the formula

$$x \circ y = yx^+ = y(\tau x).$$

We shall prove that τ is the unity element of A_0. In fact, using $(**)$, we have the relation

$$x \circ \tau = \tau(\tau x) = x.$$

Setting $x = y = \tau$ in $(*)$ we get the equality $\tau(\tau z) = \big(\tau(\tau z)\big)\tau$ for any, $z \in A$. Hence, according to $(**)$, $z = z\tau$ for any $z \in A$. In other words, τ is a right unity element in A. Hence $\tau \circ x = x\tau^+ = x(\tau\tau) = x$, and, consequently, τ is the unity element of A_0. The theorem is thus proved.

THEOREM 5. *Every algebra with a quasi-involution can be embedded in a τ-algebra.*

Proof. By theorem 2 an algebra A with a quasi-involution is a cracovian algebra generated by an associative algebra A_0 with an involution: $A = \mathscr{K}(A_0)$. Since the algebra A_0 is contained in an associative algebra $\bar{A}_0$ with a unity element, we have $A \subset \mathscr{K}(\bar{A}_0)$. But, according to Theorem 4, $\mathscr{K}(\bar{A}_0)$ is a τ-algebra, which completes the proof of our Theorem.

Remark. As a τ-algebra containing the algebra A with a quasi-involution the algebra of all pairs $\langle a, x\rangle$ ($a \in P$, $x \in A$) can be taken. The operations in this algebra are defined as follows:

$$\langle a, x\rangle + \langle \beta, y\rangle = \langle a+\beta, x+y\rangle; \quad a\langle \beta, x\rangle = \langle a\beta, ax\rangle,$$
$$\langle a, x\rangle \langle \beta, y\rangle = \langle a\beta^*, ay^+ + \beta^* x + xy\rangle.$$

It is easy to verify that the element $\tau = \langle 1, 0\rangle$ (1 is the unity element of P) satisfies conditions $(*)$ and $(**)$. The isomorphic embedding of A into the algebra under consideration is given by formula $x \to \langle 0, x\rangle$.

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