D'une part, on a en vertu de l'inégalité $|\sin x| < |x|$ et de (B) pour $0 \leq h \leq \frac{\pi}{|x|}$

$$I_1 \leq \frac{\pi}{4n} \left( \sum_{k=1}^{n-1} \sum_{k' \geq k} \left| \frac{1}{k'} \right| \right)^{1/p}$$

$$= \frac{\pi}{4n} \left( \sum_{k=1}^{n-1} \left[ \frac{1}{k^{1+1}} \right] \right)^{1/p}$$

$$\leq \frac{\pi}{4n} \left( \sum_{k=1}^{n-1} \frac{1}{k^{1+1}} \right)^{1/p}.$$ 

A cause de $p \leq 1$, le dernier facteur est d'ordre $O(n^{-1+1/p})$ et il vient

$$(7) \quad I_1 \leq C_1 n^{-1} \left( |a_n| + |b_n| \right) n^{-1+1/p} = C_1 \left( |a_n| + |b_n| \right) n^{-1+1/p}.$$ 

D'autre part, on a pour tout $h$

$$I_2 = \left( \sum_{k=1}^{n-1} \left( |a_k| + |b_k| \right)^2 |\sin kh| \right)^{1/p} \leq \left( \sum_{k=1}^{n-1} \left( |a_k| + |b_k| \right)^2 \right)^{1/p}$$

$$= \left( \sum_{k=1}^{n-1} \left[ \frac{1}{k^{1+1}} \right] \right)^{1/p}.$$ 

En vertu de (A), la dernière somme peut être majorée par le produit

$$n^{-1} \left( |a_n| + |b_n| \right) \left( \sum_{k=1}^{n-1} \frac{1}{k^{1+1}} \right)^{1/p},$$

dont le dernier facteur est convergent d'ordre $O(n^{1/p-1})$ à cause de $p > 1$, c'est-à-dire de $s < 1$. On a donc

$$(8) \quad I_2 \leq C_2 \left( |a_n| + |b_n| \right) n^{1/p}.$$ 

Les formules (6), (7) et (8) entraînent directement la première partie de (*).

**Travaux Cités**


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Reçu par la rédaction le 22.7.1961
(iii) \( F \) fulfills the condition \((T_0)\) on \([a, b]\).

(iv) at each point of \((a, b)\), except perhaps those of an enumerable set, \( F \) is monotone.

Proof. In order to prove that (i) implies (ii) it is enough to use Baire’s theorem. Therefore, we shall now show that (ii) implies (iii). For this purpose, let \( K \) be the class of all closed subintervals \( I \) of \([a, b]\) such that \( F \) fulfills the condition \((T_0)\) on \( I \). We shall show that the class \( K \) satisfies the following conditions:

(a) if \([a_0, b_0]\) and \([b_0, c_0]\) belong to \( K \), then \([a_0, c_0]\) also belongs to \( K \);

(b) if \( I \subset K \), then every interval \( J \subset I \) also belongs to \( K \);

(c) if every interval \( J \subset \text{int}(I) \) belongs to \( K \), then the interval \( I \) belongs to \( K \);

(d) if each interval contiguous to a perfect set \( E \) belongs to \( K \), then there exists an interval \( I_0 \) such that \( I_0 \subset K \) and \( E \cap \text{int}(I_0) \neq \emptyset \).

We see at once that (a) and (b) are satisfied. In order to prove (c) let every interval \( J \subset \text{int}(I) \), where \( I_0 = [a_0, b_0] \), belong to \( K \). Then, for sufficiently large positive integer \( n \), the interval \( I_n = [a_0 + 1/n, b_0 - 1/n] \) belongs to \( K \). Further, since

\[
N(F; I_n) \subset \sum_{n} N(F; I_0) + \frac{N(F; a_0)}{N(F; b_0)}
\]

the set \( N(F; I_0) \) is at most enumerable. It is easy to see that for \( y \) belonging to \( N(F; I_0) \) the set \( F^{-1}(y) - \text{int}(F^{-1}(y)) \) is at most enumerable. Thus \( F \) fulfills the condition \((T_0)\) on \( I_0 \), and this completes the proof of (c). Now we shall show that \( K \) satisfies (d). Let \( F \) be a perfect set, and let each interval contiguous to \( E \) belong to \( K \). Since (i) is satisfied, there exists a portion \( E \) of \( E \) such that \( F \) is \( M_0 \) on \( E \). Let \( I_0 \) be the smallest closed interval containing \( E \). We shall show that the interval \( I_0 \) is the required one. In fact, let \( I_0 \) denote the sequence of the intervals contiguous to \( F \), and let \( A \) be the set of all numbers \( y \) such that the set \( F^{-1}(y) \) contains at least two points. We find that

\[
N(F; I_0) \subset A + \sum_{n} N(F; I_0);
\]

hence it follows that the set \( N(F; I_0) \) it at most enumerable. Further, let \( y \) belong to \( N(F; I_0) \). Since \( F \) is \( M_0 \) on \( E \) and each \( I_0 \) belongs to \( K \), it easily follows that the set \( F^{-1}(y) - \text{int}(F^{-1}(y)) \) is at most enumerable. Thus, since also \( E \cap \text{int}(I_0) \neq \emptyset \), the proof of (c) is completed. We have shown that the class \( K \) satisfies the conditions (a)-(c). Hence, by Banach’s lemma (p. 39 in [4]), it follows that the interval \([a, b]\) belongs to \( K \) and therefore \( F \) fulfills the condition \((T_0)\) on \([a, b]\).

In order to prove that (iii) implies (iv), let \( M \) denote the set of the points at which \( F \) assumes a strict extremum. Since \( F \) fulfills the condition \((T_0)\) and the set \( M \) at most enumerable, it is enough to show that \( F \) is monotone at each point \( x \) belonging to \([a, b]\). For this purpose, let us remark that since the value \( F(x) \) is assumed only for a finite number of times, there exists a neighborhood of \( x \), such that for \( x \) belonging to this neighborhood \( F(x) \neq F(x_0), \) provided that \( x \neq x_0 \). Further, since \( F \) does not assume a strict extremum at \( x_0 \) and is continuous, it is monotone at \( x_0 \).

In order to complete the proof of the theorem, it is enough to show that (iv) implies (i). For this purpose, let \( \mathbf{E}_{n, k} \) (resp. \( \mathbf{E}_{n, k} \)), \( k = 1, 2, \ldots, n \), be the set of all \( x \) belonging to \([a, b]\) such that \( |x| < 1/n \). Further, it implies

\[
(x - x_1) (F(x) - F(a)) > 0 \quad (x - x_2) (F(x) - F(a)) \leq 0.
\]

Further, let \( \mathbf{E}_{n, k} \) (resp. \( \mathbf{E}_{n, k} \)), \( k = 0, 1, \ldots, n - 1 \), denote the intersection \( \mathbf{E}_{n, k} \) (resp. \( \mathbf{E}_{n, k} \)) with \([a + k(b - a)/n, a + (k + 1)(b - a)/n] \). We see at once that \( F \) is non-decreasing (resp. non-increasing) in the restricted sense on each \( \mathbf{E}_{n, k} \) (resp. \( \mathbf{E}_{n, k} \)). Further, since the set \([a, b] = \bigcup \mathbf{E}_{n, k} \) is at most enumerable, \( F \) is \( M_0 \) on \([a, b] \). Thus the theorem is proved.

For simplicity of wording, every continuous function which is \( M_0 \) and fulfills condition \( N \) on an interval will be called \( ACMG \) on that interval. By [5], Theorem 8.8, p. 233, and Theorem 6.8, p. 228, it follows that every function which is \( ACMG \) on an interval is \( ACG \) on that interval.

**Theorem 2.** Let \( f \) be a function \( ACMG \) on an interval \([c, d] \), and let \( f \) be a finite function defined on the interval \([c, d] \) \( \neq [c, d] \). Then the following conditions are equivalent:

(i) the function \( f \) is \( D_1 \)-integrable on \([a, b] \),

(ii) the function \( f(x) \psi \) is \( D_1 \)-integrable on \([c, d] \).

Moreover, if one of these conditions is satisfied, then

\[
(D_1) \int_0^d f(x) \psi(x) dx = (D_1) \int_0^d f(x) \psi(x) dt.
\]

Proof. First let (i) be satisfied, and let \( F \) be an indefinite \( D_1 \)-integral of \( f \). We shall show that the function \( F = \int_0^d f(x) \psi \) is \( ACG \) on \([c, d] \). The function \( F \) is \( ACG \) on \([a, b] \), and thus it is \( VBG \); therefore \([a, b] \) is the sum of a sequence of sets \( E_n \) on which each of which \( F \) is \( VBG \). Let us put \( T_n = \sigma^{-1}(E_n) \). Since \( \psi \) is \( M_0 \), we can express each \( T_n \) as the sum of a sequence of sets \( T_{n, k} \) on each of which \( \psi \) is \( M_0 \). Now it is enough
to show that \( F_1 \) is VB, on each \( U_{a,b} \). For this purpose, let \( (I_k) \) be any finite sequence of non-overlapping intervals whose end-points belong to fixed \( U_{a,b} \). Since \( \varphi \) is monotone on the restricted sense on this \( U_{a,b} \), it easily follows that \( O(F_1; I_k) = O(F_1; \varphi[I_k]) \). Now, since the intervals \( \varphi[I_k] \) are non-overlapping and have end-points belonging to fixed \( B \), on which \( F \) is VB, this completes the proof that \( F_1 \) is VB, on each \( U_{a,b} \). We have thus shown that \( F_1 \) is ACG, on \([a, d]\). By Theorem 2 in [2] it follows that the function \( f(\varphi) \) is \( D_2 \)-integrable on \([e, d]\) and (1) holds.

Let us suppose that (ii) is satisfied. We shall prove that \( f \) is \( D_2 \)-integrable on \([a, b]\). For this purpose, let us denote by \( K \) the class of all closed subintervals \( I \) of \([a, d]\) such that \( f \) is \( D_2 \)-integrable on \( \varphi[I] \). We shall show that the class \( K \) satisfies the conditions (a)-(d) from the proof of Theorem 1. The conditions (a) and (b) are obvious. In order to prove (c), let every interval \( I \subseteq (c_0, d_0) \) belong to \( K \), and let us write \( m_n = \inf_{t \in I}(t) \). \( M_n = \sup_{t \in I}(t) \). We may clearly prove that \( \varphi \) does not assume the values \( m_n, M_n \) at \( t(c_n, d_n) \) and that \( m_n = \varphi(a_n) \), \( M_n = \varphi(b_n) \). Let \( \{a_n\}, \{b_n\} \) be any sequences such that \( a_n < m_n, \lim b_n = M_n \) and \( m_n < b_n < M_n \) for \( n = 1, 2, \ldots \) Further, let \( c_n = \inf(t_n), d_n = \sup(t_n) \), \( a_n \leq t_n \leq d_n \) and \( a_k \neq d_k \) for \( n = 1, 2, \ldots \). Hence, in view of our hypothesis, it follows that \( f \) is \( D_2 \)-integrable on each \( [a_n, b_n] \). Therefore, on account of the part which has already been proved, we obtain

\[
(D_n) \int_{a_n}^{b_n} f(x) \, dx = \int_{a_n}^{b_n} \int_{c_n}^{d_n} f(\varphi(t)) \, \varphi'(t) \, dt \, dx.
\]

Hence, the definite \( D_2 \)-integrals of \( f \) over \([a_n, b_n]\) tend to a finite limit as \( n \to \infty \) and therefore \( f \) is \( D_2 \)-integrable on \([m_n, M_n]\). This completes the proof of (c). We shall now show that \( K \) satisfies (d). In fact, on account of (a), it follows that the intervals \( I_n = \varphi(I_k), n = 1, 2, \ldots \), are contiguous to the closed set \( Q = \varphi[F] \). Since each \( I_n, n = 1, 2, \ldots \), belongs to \( K \), it follows that \( f \) is \( D_2 \)-integrable on each \( I_n \). Moreover, by (a) and the part of the theorem which has already been proved, we obtain

\[
O(D_n; f; I_n) = O(D_n; f; \varphi[I_n]) \text{ for positive integer } n.
\]

By the second part of (b), it follows that \( \sum_{n} O(D_n; f; I_n) < \infty \). Further, since \( \varphi \) is clearly monotone and AC on \( [e, d] \), in view of the first part of (b) and the well-known theorem concerning change of variable in the Lebesgue integral, we infer that \( f \) is summable on \( Q \). Now, it is enough to use Theorem 5.1 of [3] (p. 257) to prove that \( f \) is \( D_2 \)-integrable on \( \varphi[I_n] \), and since \( E = \int \varphi[I_n] \neq 0 \), this completes the proof of (d). We have thus shown that the class \( K \) satisfies the conditions (a)-(d), p. 318. Hence, by R. Romano’s lemma ([1], p. 39) it follows that the interval \([e, d]\) belongs to \( K \), and so \( f \) is \( D_2 \)-integrable on \([a, b]\). Thus the theorem is proved.

Theorem 2 generalizes Bartků’s result ([1], p. 414), and gives more than the result of Mitriv ([3], p. 292) applied to the Denjoy-Perron integral. We shall now prove

**Theorem 3.** Let \( \varphi \) be a function defined on an interval \([e, d]\). If, for every function \( F \) increasing and AC on the set \( \varphi \subseteq [e, d] \), the function \( G = \varphi(F) \) is ACG, on \([e, d] \), then \( \varphi \) is ACMG, on \([e, d] \).

Proof. Suppose that \( \varphi \) is not ACMG, on \([e, d] \). Then, since \( \varphi \) is clearly continuous and satisfies conditions (N) on \([e, d] \), it is not MG, on \([e, d] \). Therefore, on account of Theorem 1, there exists a perfect set \( E \subseteq [e, d] \) such that \( \varphi \) is not MG, on any portion of \( E \). Let \( E_1 \) be the set of points \( t \) such that \( t \) is not the end of the interval contiguous to \( E_1 \) (ii) every neighbourhood of \( t \) contains points \( t_n \) such that \( \varphi(t_n) = \varphi(t) \) and \( \varphi(t_n) \neq \varphi(t) \), provided that either \( t_n > t \) or \( t_n < t \). We set \( E_1 \) dense in \( E \), since otherwise there would exist a portion \( P \) of \( E \) such that \( PE_1 = 0 \). We see at once that \( \varphi \) is monotone at each point of \( P \) at which does not assume a strict extremum and which is not the end of the interval contiguous to \( E \). Therefore, by an argument similar to that used in the proof of Theorem 1, it follows that \( \varphi \) is MG, on \( P \) and, hence, by Baire’s theorem, \( M_n \) on any portion of \( P \), and so \( M_n \) on any portion of \( E \). This contradicts the hypothesis. We have thus proved that \( E_1 \) is dense in \( E \). Let \( \{t_n\}_{n=1,2, \ldots} \) be a sequence of points belonging to \( E \) dense in \( E \). We see at once that there exist points \( t_n \) \( n = 1, 2, \ldots \) such that the following conditions are satisfied:

(a) \( t_n, s \in E \) for \( i = 1, 2 \) and \( n, k = 1, 2, \ldots \),
(b) \( t_n \leq t_n < t_n \), \( s_k = s_k \), \( t_n \), \( k = 1, 2, \ldots \),
(c) \( \lim_{n \to \infty} t_n = s_k \) for \( i = 1, 2 \) and \( n, k = 1, 2, \ldots \),
(d) \( \varphi(t_n) = \varphi(s_k) \) for \( n, k = 1, 2, \ldots \),

\( (*) \) of course some of \( t_n \) can reduce to points.
(e) for fixed \( n \), either \( c_{n,k+1} < c_{n,k} \) for \( k = 1, 2, \ldots \) or \( c_{n,k+1} > c_{n,k} \) for \( k = 1, 2, \ldots \).

(f) for fixed \( n \), either \( 1^\circ \varphi(c_{n,k+1}) > \varphi(c_{n,k}) > \varphi(c_{n,k}) \) for \( k = 1, 2, \ldots \) or \( 2^\circ \varphi(c_{n,k+1}) < \varphi(c_{n,k+1}) < \varphi(c_{n,k}) \) for \( k = 1, 2, \ldots \).

Let us define, for every positive integer \( n \), the function \( F_n \) on the interval \([a, b] = [\varphi(c_{n,k}), \varphi(c_{n,k+1})]\) as follows:

\[
F_n(x) = \begin{cases} \\
0 & \text{at } x = \varphi(c_{n,k}) \\
\frac{a_n}{k} & \text{at } x = \varphi(c_{n,k+1}) \quad \text{for } k = 1, 2, \ldots, \\
\text{linear for other } x \text{ so that } F_n \text{ is increasing,} \\
\end{cases}
\]

where \( a_n = +1 \) for \( 1^\circ \) and \( a_n = -1 \) for \( 2^\circ \). Let us put

\[
F(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{n^2 M_n}
\]

where \( M_n = \sup_{a_n \in \mathbb{R}} |F_n(x)| \). The function \( F_n \) is evidently increasing.

We shall show that \( F \) is also AC on \([a, b] \); hence

\[
(L) \int_a^b F'(s) \, ds = \sum_{n=1}^{\infty} \frac{1}{n^3 M_n} (L) \int_a^b F_n(s) \, ds
\]

for every \( x \) belonging to \([a, b] \). Since each \( F_n \) is evidently AC, it follows that

\[
(L) \int_a^b F'(s) \, ds = F_n(x) - F_n(a) \quad \text{for } x \in [a, b] \quad \text{and every positive integer } n.
\]

Therefore, in view of (3) and (4), we obtain

\[
(L) \int_a^b F'(s) \, ds = F(x) - F(a).
\]

We have thus shown that \( F \) is AC on \([a, b] \). We shall now prove that the function \( \Theta = F(\varphi) \) is not AC on \([a, b] \), and this will contradict the hypothesis of the theorem. For this purpose, in view of (5), Theorem 9.1 (p. 233), it suffices to show that \( \Theta \) is not VB on any portion of \( E \).

Let \( P \) be any portion of \( E \). Since the sequence \( (t_k) \) is dense in \( E \), there exists a point \( t_k \in P \). Further, in view of (e), it follows that \( t_{n,k} + P \) for \( i = 1, 2 \) and \( k \gg k_{n,i} \). Now, by (d), (f) and (2), we obtain

\[
\epsilon_n(\varphi(c_{n,k})) - \varphi(c_{n,k}) > \frac{\epsilon_n}{n^2 M_n}(F_n(\varphi(c_{n,k})) - F_n(\varphi(c_{n,k})) = \frac{1}{k_{n}^2 M_n}
\]