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ON REARRANGEMENT OF SERIES, II

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1. Let $N_1, N_2, \ldots$ be a permutation of the positive integers. Then the series $a_{N_1}+a_{N_2}+\ldots$ is a rearrangement in the order of its terms of the series $a_1+a_2+\ldots$ (hereafter called series A) of complex terms.

In a recent note [1] we generalized a theorem of Jasek [2] to

THEOREM 1. *Suppose that*

$$na_n \to 0 \quad \text{as} \quad n \to \infty, \tag{1}$$

and $na_{N_n} \to 0$ *as* $n \to \infty$. *Then*

$$(a_{N_1}+\ldots+a_{N_n})-(a_1+\ldots+a_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{2}$$

Jasek's theorem follows as a special case if the series A is convergent. In [1], using Theorem 1, we obtained a solution of Jasek's problem P300 [2].

In this note we shall prove

THEOREM 2. *For* (2) *to be true for every series* A *satisfying* (1), *a necessary and sufficient condition* (*NSC*) *is*

$$\sum_{r>n\geqslant N_r} \frac{1}{N_r} = O(1) \quad \text{as} \quad n \to \infty. \tag{3}$$

Jasek's problem P 301 [2] may be restated as

Find an NSC that (2) *be true for every convergent series* A *satisfying* (1).

Theorem 2, while not a solution of the above, solves a related problem.

In what follows we shall, for brevity, write $\sum'$ for $\sum\limits_{r>n\geqslant N_r}$ and $\sum''$ for $\sum\limits_{r\leqslant n<N_r}$.

2. To prove Theorem 2, we note first that

$$(a_{N_1}+\ldots+a_{N_n})-(a_1+\ldots+a_n) = \sum{}'' a_{N_r} - \sum{}' a_{N_r}. \tag{4}$$

From (1), it easily follows that

(5) $$\sum{}'' a_{N_r} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, from (4), we have

THEOREM 3. *For (2) to be true for the series A satisfying (1), an NSC is*

(6) $$\sum{}' a_{N_r} \to 0 \quad as \quad n \to \infty.$$

Theorem 2 follows from Theorem 3 and the following lemmas:

LEMMA 1. (1) *and* (3) $\Rightarrow$ (6).

LEMMA 2. *Let* (3) *be false and* $a_n = 1/ns_n$, *where*

$$s_\nu = 1 + \max_{n \leqslant \nu} \sum{}' \frac{1}{N_r}.$$

Then (1) *is true and* (6) *false.*

The proof of Lemma 1 becomes obvious if we note that $N_n \to \infty$ with n.

To prove Lemma 2, we note first that $s_{N_r} \leqslant s_n$ for $r > n \geqslant N_r$. Hence

$$\sum{}' a_{N_r} = \sum{}' \frac{1}{N_r s_{N_r}} \geqslant \sum{}' \frac{1}{N_r s_n}.$$

But $s_n \to \infty$ with n, since (3) is false. Hence (1) is true and

$$\limsup_{n\to\infty} \sum{}' a_{N_r} \geqslant \limsup_{n\to\infty} \sum{}' \frac{1}{N_r s_n} = 1.$$

Thus (6) is false. This concludes the proof.

Remark 1. Theorem 2 remains true if the word "series" is replaced by the phrase "divergent series". This follows since

$$s_n \leqslant 1 + \left(\frac{1}{1} + \ldots + \frac{1}{n}\right) \sim \log n$$

so that the series A, where $a_n = 1/ns_n$, is divergent.

Remark 2. In each of Theorems 1 and 3 condition (1) may be replaced by condition (5).

3. The following results are easily proved:

THEOREM 4. *Condition* (3) *is an NSC for*

(7) $$(a_{N_1} + \ldots + a_{N_n}) - (a_1 + \ldots + a_n) = O(1) \quad as \quad n \to \infty$$

to be true for every series A *satisfying the condition*

(8) $$na_n = O(1) \quad as \quad n \to \infty.$$

THEOREM 5. *If*

(9) $$\sum{}'' \frac{1}{N_r} \to 0 \quad as \quad n \to \infty,$$

for (2) *to be true for every series* A *satisfying* (8), *an NSC is*

(10) $$\sum{}' \frac{1}{N_r} \to 0 \quad as \quad n \to \infty.$$

Note. The sums in (9) and (10) have equal numbers of terms. Consequently (10) implies (9). Thus (8) and (10) imply (2).

Remark 3. Theorems 4 and 5 are also true with "series" replaced by "divergent series".

The following related problems are of interest:

P 377. *Find results corresponding to Theorems* 4 *and* 5 *for "convergent series".*

P 378. (a) *Find an NSC that* (7) *be true for every* (i) *series*, (ii) *divergent series*, (iii) *convergent series* A *satisfying* (5) *and* (8). (b) *Solve* (a) *with* (7) *replaced by* (2). (c) *Solve* (a) *with* (5) *and* (8) *replaced by* (1).

Addendum (1 November 1961). An NSC that $a_{N_1} + a_{N_2} + \ldots$ converges for every convergent series A has been found by Kronrod [3]. I am indebted to B. Jasek for informing me of this.

Addendum (in proof). The same NSC has also been found by Levi [4]. I am indebted to U. C. Guha for drawing my attention to [4].

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