\section*{References}

\begin{enumerate}
\item T. Homma, A theorem on continuous functions, Kodai Mathematical Seminar Reports 1 (1958), p. 13-16.
\item J. Mioduszewski, Solution générale d'un problème de Németiši, ibidem 6 (1958), p. 169-172.
\end{enumerate}

\section*{Mathematical Institute of the Wrocław University}

Reçu par la Rédaction le 14.7.1961

\section*{Colloquium Mathematicum}

\section*{1962}

\section*{Fasc. 2}

\section*{On the Dimension of Quasi-Components in Peripherically Compact Spaces}

\textbf{by A. Lelek (Wrocław)}

It is well known that a compact metric space has a dimension at most $n$ provided that all its components have dimensions at most $n$. An example given in 1927 by Mazurkiewicz [4] shows that there exists such a separable metric space of an arbitrary positive dimension that each of its components, even every quasi-component, consists of a single point. Recently, Engelking asked if there exists a space of this kind which simultaneously is peripherally compact (also called semicompact), i.e. satisfies the condition that each point has arbitrarily small neighbourhoods whose boundaries are compact. This question has partially been answered by Duda, who constructed the following

\begin{example}
There is such a separable metric space $X$ that $\dim X = 1$, every component of $X$ consists of a single point and $X$ is peripherally compact. Indeed, denoting by $\mathcal{J}$ the segment $0 \leq t < 1$ and by $\mathcal{F}$ the Cantor ternary set in $\mathcal{J}$, let $A$ be a set which is obtained from a biconnected set by removing its "explosive" point and is contained in $\mathcal{F} \times \mathcal{J}$ in such a way that the intersection $A \cap (\{t\} \times \mathcal{F})$ is a point $p_t$ for every $t \in \mathcal{F}$. Then $\dim A = 1$. Put $X = A \cup (\mathcal{F} \times \mathcal{J})$, where $\mathcal{F}$ is the set of all rational numbers in $\mathcal{J}$. Obviously, $X$ is a 1-dimensional peripherally compact set and contains only degenerate connected subsets.

However, in this example the quasi-components of $X$ are the sets $\{p_t\} \setminus (\{t\} \times \mathcal{F})$ for $t \in \mathcal{F}$. As we show, it is impossible to find any space $X$ possessing all the properties of Duda's example described above with the word "quasi-component" instead of "component". Namely, we have the following

\begin{theorem}
If every quasi-component $Q$ of a peripherally compact separable metric space $X$ is locally compact and has the dimension $\dim Q \leq 0$, then $\dim X \leq 0$.
\end{theorem}

\textbf{Proof.} The space $X$ being peripherally compact, we may assume by a theorem proved in 1942 by Freudenthal [3] that $X$ is a subset of
a compact metric space $Y$ such that
\[ \dim (Y - X) \leq 0. \]

Let $p$ be an arbitrary point of $X$ and $Q$ the quasi-component of $X$, containing $p$. For each open neighbourhood $W$ of $p$ in $X$ there is such an open set $V$ in $Y$ that $W = V \cap X$. Since $Q$ is locally compact, there exists an open neighbourhood $U$ of $p$ in $Y$ such that $U \cap Q \cap U$ is compact (*). Then $p \in U \cap V$ and it follows from the inequality $\dim Q \leq 0$ that there exists an open neighbourhood $T$ of $p$ in $Y$ satisfying
\[ T \subseteq U \cap V \quad \text{and} \quad Q \cap Fr_T(T) = 0, \]
where $Fr_T(T)$ denotes the boundary of $T$ in $Y$ (see [2], p. 164 and 173).

Consequently,
\[ Fr_T(T) \subseteq Y - \overline{U \cap Q \cap U} \]
and since the set $\overline{U \cap Q \cap U}$ is compact, and thus closed in $Y$, it follows from (1) that each point $q \in Fr_T(T)$ has an open neighbourhood $S(q)$ in $Y$ satisfying
\[ \overline{S(q)} \subseteq Y - \overline{U \cap Q \cap U} \quad \text{and} \quad (Y - X) \cap Fr_T[S(q)] = 0 \]
(ibidem), whence $p$ does not belong to $\overline{S(q)}$ and $Fr_T[S(q)] \subseteq X$. According to the compactness of $Fr_T(T)$, let us take a finite cover $S_1, \ldots, S_m$ from the cover of $Fr_T(T)$ consisting of all the sets $S(q)$. So we have
\[ S_i \cup \cdots \cup S_m \subseteq Y - \overline{U \cap Q \cap U}, \quad \text{for} \quad i = 1, \ldots, m \]
and the set $R = T - (S_1 \cup \cdots \cup S_m)$ contains the point $p$ and is open in $Y$. Furthermore, since the open sets $S_1, \ldots, S_m$ form a cover of $Fr_T(T)$, it is easily seen that each point belonging to the boundary of the set $R$ also belongs to the boundary of the union $S_1 \cup \cdots \cup S_m$. Thus
\[ Fr_T(R) \subseteq Fr_T(S_1) \cup \cdots \cup Fr_T(S_m) \subseteq X \]
(see [2], p. 29), according to (3). Moreover, we have
\[ \overline{Q \cap Fr_T(R)} \subseteq \overline{Q \cap Fr_T(R)} \subseteq \overline{Q \cap Fr_T(T)} \subseteq \overline{U \cap V} \subseteq Q, \]
whence
\[ Q \cap Fr_T(R) = (\overline{Q \cap Fr_T(T)} \cap \overline{Q \cap Fr_T(T)}) \cap (\overline{Q \cap Fr_T(T)} \cap \overline{Q \cap Fr_T(T)}) \subseteq (\overline{Q \cap Fr_T(T)} \cap \overline{Q \cap Fr_T(T)}) = 0, \]
(*) always denotes here the closure of a set $A$ in the space $X$.

by virtue of (2) and (4), i.e.
\[ Q \cap Fr_T(R) = 0. \]

Since there is a mapping $f : X \to Y$ of $X$ into the Cantor set $\mathcal{C}$ such that the sets $f^{-1}(y)$, where $y \in f(X)$, coincide with the quasi-components of $X$ (see [3], p. 93), one can find a decreasing sequence
\[ G_1 \supset G_2 \supset \ldots \]

of subsets of $X$ which satisfy
\[ Q = \bigcap_{i=1}^{\infty} G_i \]
and are all both closed and open in $X$. For instance, it is sufficient to represent the point $f(Q)$ as the intersection of a decreasing sequence
\[ [a_1, b_1] \supset [a_2, b_2] \supset \ldots \]
of segments on the real line such that no end point $a_i$, $b_i$ belongs to $\mathcal{C}$, and take
\[ G_i = f^{-1}([a_i, b_i]) \]
for $i = 1, 2, \ldots$ Then, if all the sets $G_i \cap Fr_T(R)$ were non-empty ($i = 1, 2, \ldots$), they would form a decreasing sequence of non-empty closed subsets of $Fr_T(R)$, according to (4), and the compactness of the set $Fr_T(R)$ would imply that
\[ 0 \neq \bigcap_{i=1}^{\infty} G_i \cap Fr_T(R) = Q \cap Fr_T(R), \]
contrary to (5). Consequently, a positive integer $j$ exists such that
\[ G_j = f^{-1}([a_i, b_i]) = 0. \]

Now, consider the set $G_j \cap R$. It is an open neighbourhood of $p$ in $X$ as $p \in G_j \cap R$, and $p \in R$. Further, since $G_j \subseteq X$ and $R \subseteq Y$, we have
\[ G_j \cap R \subseteq X \cap V = W. \]

Moreover, the set $G_j$ being both closed and open in $X$, we have
\[ Fr_X(G_j \cap R) \subseteq Fr_X(R) \cap X \cap G_j \cap X = G_j \]
and the boundary $Fr_X(G_j)$ of $G_j$ in $X$ is empty; thus
\[ Fr_X(G_j \cap R) = Fr_X(G_j \cap (X \cap R)) = Fr_X(G_j) \cup Fr_X(R \cap X) = Fr_X(R \cap X) = Fr_X(R) \]
(see [2], p. 29). It follows that
\[ Fr_X(G_j \cap R) \subseteq G_j \cap Fr_X(R) = 0, \]

according to (6). Hence we conclude that
\[ \dim_p X \leq 0, \]
since \( W \) has been an arbitrarily taken open neighbourhood of \( p \) in \( X \), and the proof of Theorem 1 is complete.

It is seen by Duda's example given at the beginning of this note that the local compactness of quasi-components is a necessary hypothesis in Theorem 1. In fact, the example shows that a peripherically compact space can be 1-dimensional and have only 0-dimensional quasi-components. However, the difference between the dimension of a peripherically compact metric space and the maximal dimension of its quasi-components cannot be greater than 1. This is a consequence of the following

**Theorem 2.** If every component \( C \) of a peripherically compact metric space \( X \) has the dimension \( \dim C \leq n \) (where \( n = 0, 1, \ldots \)), then \( \dim X \leq n+1 \).

**Proof.** For any point \( p \) of \( X \) there is an arbitrarily small open neighbourhood \( V \) of \( p \) in \( X \) such that the boundary \( \partial X(V) \) is compact. Since each component \( K \) of this boundary is contained in a component \( C \) of \( X \), we have
\[ \dim K \leq \dim C \leq n, \]
whence \( \dim \partial X(V) \leq n \) [see [3], p. 106]. It follows that \( \dim_p X \leq n+1 \), and Theorem 2 is proved.

At last, the following question concerning some generalization of Theorem 1 on higher dimensions remains open:

**P 37.** Is it true that if every quasi-component \( Q \) of a peripherically compact separable metric space \( X \) is locally compact and has the dimension \( \dim Q \leq n \) (where \( n = 1, 2, \ldots \)), then \( \dim X \leq n \)?

**REFERENCES**


**MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES**

*Received for publication on 26.9.1961*