ABSTRACT ALGEBRAS IN WHICH ALL ELEMENTS ARE INDEPENDENT

BY

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1. Introduction. Let \( \mathcal{A} = (A; \mathcal{F}) \) be an abstract algebra, i.e., a set \( A \) of elements and a class \( \mathcal{F} \) of fundamental operations. Every \( f \) from \( \mathcal{F} \) is a function of several variables which associates with each system \( x_1, x_2, \ldots, x_n \), of elements of \( A \) an element \( f(x_1, x_2, \ldots, x_n) \) belonging to \( A \). If \( F = \{f_1, f_2, \ldots, f_m\} \) we often write \( (A; f_1, f_2, \ldots, f_m) \) instead of \( (A; \mathcal{F}) \). We denote by \( A^{(n)}(\mathcal{B}) \) the class of all algebraic operations of \( n \) variables, i.e., the smallest class of operations containing all identity operations

\[
e^{(n)}(x_1, x_2, \ldots, x_m) = x_k \quad (k = 1, 2, \ldots, n)
\]

and closed under the composition with the fundamental operations. The class of all algebraic operations will be denoted by \( A(\mathcal{B}) \). Two algebras \( \mathcal{A} = (A; \mathcal{F}) \) and \( \mathcal{B} = (A; \mathcal{F}) \) having the same class of algebraic operations will be treated here as identical. Further, if \( A(\mathcal{B}) \subseteq A(\mathcal{B}) \), we say that \( \mathcal{B} \) is a subsystem of \( \mathcal{A} \).

We shall call the identity operations (1) also trivial operations. If all algebraic operations are trivial, then the algebra will be called trivial.

Let us consider an \( n \)-element subset \( I = \{a_1, a_2, \ldots, a_n\} \) of \( A \). We say that \( I \) is a set of independent elements, or else that all elements of \( I \) are independent if for any \( f, g \in A^{(n)}(\mathcal{B}) \) the equality \( f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n) \) implies the identity of \( f \) and \( g \) in \( A \). The properties of the notion of independence are given in [1] and [2].

All elements of a trivial algebra are obviously independent. S. Świerczkowski has proved that for algebras having at least three elements the converse implication is true: if all elements are independent, then the algebra is trivial ([5], p. 501, and [6], Theorem 1). In a recent note [3], using a complete characterization of two-element algebras given by Post [4], we described non-trivial two-element algebras in which all elements are independent. Namely, denoting by \( \mathcal{S} \) the class of all such algebras, with fixed elements 0 and 1, we proved that \( \mathcal{S} \) consists of three
algebras, $\mathcal{P}$, $\mathcal{P}_*$, and $\mathcal{P}^*$ (defined below). In this paper we present a complete proof of the result quoted. The proof involves some of Post's ideas as presented in [4]; however, no particular result of [4] is explicitly used. We hope that this will be appreciated by the reader, the more so as the reasons in [4] are rather complicated.

In what follows we shall consider algebras possessing only the elements 0 and 1. The set $T = \{0, 1\}$ can be regarded as a Boolean algebra with 0 as a neutral element. By $\bigcup$, $\bigcap$, and $\setminus$ we denote the elementary Boolean operations: join, meet, complementation and symmetric subtraction. A T-valued operation $f$ of $n$ variables running over $T$ is said to be homogeneous if for every mapping $\tau$ of $T$ into itself the equality

$$f(\tau(a_1), \tau(a_2), \ldots, \tau(a_n)) = \tau(f(a_1, a_2, \ldots, a_n))$$

holds. Since each mapping of $T$ into itself is a composition of two mappings $x \to x'$ and $x \to 0$ ($x \in T$), an operation $f$ is homogeneous if and only if

$$f(a_1, a_2, \ldots, a_n) = f(a'_1, a'_2, \ldots, a'_n)$$

and

$$f(0, 0, \ldots, 0) = 0.$$  

For an arbitrary abstract algebra it was proved in [1] (p. 733) that $I$ is a set of independent elements if and only if each mapping of $I$ into $A$ can be extended to a homomorphism of the subalgebra generated by $I$ into $A$. Applying this result to the algebras in question we get the following statement:

The elements 0 and 1 of an algebra $\mathcal{A}$ are independent if and only if all algebraic operations from $\mathcal{A}(\mathbb{S})$ are homogeneous.

It is very easy to verify that the only operations of two variables satisfying conditions (2) and (3) are trivial ones. Furthermore, it is obvious that the triviality of all operations of two variables implies the independence of the elements 0 and 1. Consequently, the elements 0 and 1 are independent if and only if each operation from $\mathcal{A}(\mathbb{S})$ is trivial.

2. The algebras $\mathcal{P}$, $\mathcal{P}^*$ and $\mathcal{P}_*$. We define three $T$-valued non-trivial operations $p_*, p^*$ and $p$ of three variables running over $T$ by the following conditions:

$$p_*(x, y, z) = p_*(x, y, z) = p_*(y, x, z) = y,$$

$$p^*(x, y, z) = p^*(x, y, z) = p^*(y, x, z) = x,$$

$$p(x, y, z) = x, \quad p(x, y, z) = p(y, x, z) = y.$$  

The three algebras

$$\mathcal{P}_* = (T; p_*), \quad \mathcal{P}^* = (T; p^*), \quad \mathcal{P} = (T; p)$$

were considered by Post [4]. It follows directly from (4), (5), (6) and from the definition of algebraic operations of two variables that all these operations in the three algebras considered are trivial. Thus $\mathcal{P}_*$, $\mathcal{P}^*$, $\mathcal{P}$ further, from the formulas

$$p_*(x, y, z) = p(p(x, y, z), p(x, y, z), z),$$

$$p^*(x, y, z) = p(x, y, z), \quad p(x, y, z) = p_*(x, y, z),$$

we get the equality $\mathcal{P} = (T; p_*, p^*)$. Consequently, the algebras $\mathcal{P}_*$ and $\mathcal{P}^*$ are subsystems of the algebra $\mathcal{P}$.

We shall now describe the structure of the algebras $\mathcal{P}_*$, $\mathcal{P}^*$ and $\mathcal{P}$.

Algebra $\mathcal{P}_*$. The fundamental operation $p_*$ can be expressed in terms of the symmetric difference as follows:

$$p_*(x, y, z) = x \triangle y \triangle z.$$  

Hence we infer that the class $\mathcal{A}(\mathcal{P}_*)$ consists of all operations $f$ of the form

$$f(a_1, a_2, \ldots, a_n) = a_{i_1} \triangle a_{i_2} \triangle \ldots \triangle a_{i_k},$$

where $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ and $k$ is an odd integer.

Algebra $\mathcal{P}^*$. The fundamental operation $p^*$ can be defined by means of the join and the meet operations as follows:

$$p^*(x, y, z) = (x \land y) \lor (x \lor y) \land (y \lor z).$$  

Hence we infer that all algebraic operations from $\mathcal{A}(\mathcal{P}^*)$ can be expressed in terms of the operations $\land$ and $\lor$. Now we shall prove that the class $\mathcal{A}(\mathcal{P}^*)$ consists of all homogeneous operations which can be expressed in terms of the operations $\land$, $\lor$ and $\setminus$.

We present the proof which is an adaptation of a part of paper [4] by Post. In order to prove our statement it is sufficient to show that every homogeneous operation which can be expressed in terms of the operations $\land$, $\lor$ and $\setminus$ belongs to $\mathcal{A}(\mathcal{P}^*)$. Let $f$ be such an operation of $n$ variables. First we shall prove that for every pair $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ of systems of elements of $T$ satisfying the condition

$$f(a_1, a_2, \ldots, a_n) = 1 = f(b_1, b_2, \ldots, b_n)$$

there exists an index $k$ $(1 \leq k \leq n)$ such that $a_k = b_k = 1$. Contrary to this, let us suppose that $a_k \land b_k = 0$ for all indices $j$ $(1 \leq j \leq n)$. In other words, we have $a_j = 1$ whenever $b_j = 1$. Since the operation $f$ can be expressed in terms of $\land$, $\lor$ and $\setminus$, we have, according to (10), the equality $f(a_1, a_2, \ldots, a_n) = 1$, which, by virtue of (2), implies $f(a_1, a_2, \ldots, a_n) = 0$. But this contradicts (10).
we have \( p(1, y, z) = y \lor z \). Hence we infer that for every \( k \)-tuple \((j_1, j_2, \ldots, j_k) \neq (0, 0, \ldots, 0)\) the class \( \mathcal{A}(\mathcal{P}) \) contains an operation \( g_{j_1, j_2, \ldots, j_k} \) of \( k+1 \) variables such that

\[
g_{j_1, j_2, \ldots, j_k}(1, x_1, \ldots, x_k) = \bigcup_{x_{j_1}} \bigcup_{x_{j_2}} \ldots \bigcup_{x_{j_k}} .
\]

Thus for every \((n-1)\)-tuple \((k_1, k_2, \ldots, k_{n-1}) \neq (1, 1, \ldots, 1)\) there exists an operation \( h_{k_1, k_2, \ldots, k_{n-1}} \) in \( \mathcal{A}^{(n)}(\mathcal{P}) \) such that

\[
h_{k_1, k_2, \ldots, k_{n-1}}(1, x_1, x_2, \ldots, x_{n-1}) = \bigcap_{n} x_i^0 .
\]

Further, from the equality \( p(y, z, 1) = y \lor z \) it follows that there exists an operation \( h \in \mathcal{A}^{(0)}(\mathcal{P}) \) such that

\[
h(1, x_1, \ldots, x_k) = \bigcap_{k} x_i^0 .
\]

Let \( J \) be the set appearing in (12). By a suitable composition of the operations \( g_{j_1, j_2, \ldots, j_k}, h_{k_1, k_2, \ldots, k_{n-1}} \) and \( h \) we get an operation \( f_{J} \in \mathcal{A}^{(0)}(\mathcal{P}) \) such that

\[
f_{J}(1, x_1, \ldots, x_k) = \bigcup_{(j_1, j_2, \ldots, j_k) \in J} \bigcup_{n} x_i^0 .
\]

Hence and from (12) we obtain the equality \( f(1, x_1, \ldots, x_k) = f_{J}(1, x_1, \ldots, x_k) \), whence, by virtue of the homogeneity of \( f \) and \( f_{J} \), the equality \( f = f_{J} \) follows. Thus \( f \in \mathcal{A}(\mathcal{P}) \), which completes the proof.

3. Characterization of the class \( \mathcal{J} \). We have seen that all the algebras \( \mathcal{T}_s, \mathcal{T}_s^* \) and \( \mathcal{P} \) belong to \( \mathcal{J} \). The complete description of this class is given by the following

**Theorem.** \( \mathcal{J} = (\mathcal{T}_s, \mathcal{T}_s^*, \mathcal{P}) \).

Before proving the theorem we shall prove some lemmas. In what follows the algebra \( \mathfrak{B} \) is supposed to belong to \( \mathcal{J} \).

**Lemma 1. One of the relations**

\[
\mathcal{A}^{(0)}(\mathfrak{B}) = \mathcal{A}^{(0)}(\mathcal{P}), \quad \mathcal{A}^{(0)}(\mathfrak{B}) = \mathcal{A}^{(0)}(\mathcal{T}_s^*), \quad \mathcal{A}^{(0)}(\mathfrak{B}) \subseteq \mathcal{A}^{(0)}(\mathcal{T}_s),
\]

holds.

**Proof.** Since \( \mathcal{A}(\mathcal{P}) \) consists of all homogeneous operations, we have the inclusion \( \mathcal{A}(\mathfrak{B}) \subseteq \mathcal{A}(\mathcal{P}) \) and, consequently,

\[
\mathcal{A}^{(0)}(\mathfrak{B}) \subseteq \mathcal{A}^{(0)}(\mathcal{P}) .
\]

First let us suppose that \( \mathcal{A}^{(0)}(\mathfrak{B}) \) contains a non-symmetric operation \( f \) depending on every variable. Since operations of two variables are trivial, each of the operations \( f(x, z, y), f(z, x, y) \) and \( f(y, z, x) \) is equal to either \( x \) or \( y \). Further, by the non-symmetry of \( f \), among these opera-
tions there exist two different ones. Without loss of generality we may suppose that

\[(14) \quad f(x, x, y) = x, \quad f(y, x, x) = y.\]

Since the operation \(f\) depends on the last variable, there exists a system \(a_1, a_2, a_3\) such that \(f(a_1, a_2, a_3) \neq f(a_1, a_2, a_4)\). Consequently, \(f(y, x, x) = y\). Taking into account formula \((6)\), we have the equality \(f = p\). Thus \(A^{(0)}(\mathcal{P}) \supset A^{(0)}(\mathcal{P})\). Comparing this inclusion with \((13)\), we get the equality \(A^{(0)}(\mathcal{P}) = A^{(0)}(\mathcal{P})\).

Now let us suppose that each operation \(g\) from \(A^{(0)}(\mathcal{P})\) depending on every variable is symmetric. Then all operations \(g(x, x, y)\), \(g(x, y, x)\) and \(g(y, x, x)\) are equal to either \(x\) or \(y\). By \((4)\) and \((5)\) in the first case we have the equality \(g = p\) and in the second \(g = p^*\). Because of formula \((7)\), the class \(A^{(0)}(\mathcal{B})\) contains exactly one of the operations \(p\) and \(p^*\). Thus we have either \(A^{(0)}(\mathcal{B}) = A^{(0)}(\mathcal{P})\) or \(A^{(0)}(\mathcal{B}) = A^{(0)}(\mathcal{P})\).

Finally, if \(A^{(0)}(\mathcal{B})\) does not contain any operation depending on every variable, then all operations of three variables are trivial and, of course, \(A^{(0)}(\mathcal{B}) \subseteq A^{(0)}(\mathcal{P})\), which completes the proof.

**Lemma 2.** If \(A^{(0)}(\mathcal{B}) \subseteq A^{(0)}(\mathcal{P})\), then there exists no operation in \(A^{(0)}(\mathcal{B})\) depending on every variable.

Proof. Contrary to this, let us suppose that there exists an operation \(f\) in \(A^{(0)}(\mathcal{B})\) depending on every variable. First we shall prove the equalities

\[(15) \quad f(y, y, y, x) = f(y, y, x, y) = f(y, y, y, y) = f(x, y, y, y) = x.\]

By the symmetry of our assumption it suffices to prove the last equality. Since \(f\) depends on the first variable we can find a system \(a_1, a_2, a_3\) of elements of \(T\) such that \(f(a_1, a_2, a_3, a_4) \neq f(a_1, a_2, a_3, a_5)\). Among the elements \(a_1, a_2, a_3\) at least two are identical. Without loss of generality we may suppose that \(a_2 = a_3\). Hence it follows that the operation \(f(x, y, x, y)\) depends on \(x\). We know that the only operations in \(A^{(0)}(\mathcal{P})\) depending on the first variable are \(x\) and \(x \land \neg x\). Thus, by the inclusion \(A^{(0)}(\mathcal{B}) \subseteq A^{(0)}(\mathcal{P})\), \(f(x, y, x, y) = x\) or \(f(x, y, y, x) = x \land \neg x\). Obviously, in both cases we have the equality \(f(x, y, y, y) = x\), which completes the proof of \((15)\).

From \((15)\) it follows directly that the operations \(f(x_1, x_2, x_3, x_4)\) and \(f(x_1, x_2, x_3, x_4)\) depend on \(x_1, x_2\) and \(x_3, x_4\) respectively. Since they belong to \(A^{(0)}(\mathcal{P})\), we have the equalities

\[f(x_1, x_2, x_3, x_4) = x_1 \land \neg x_2 \land x_3,\]

\[f(x_1, x_2, x_3, x_4) = x_1 \land \neg x_2 \land x_4.\]

From the first equality we get \(f(0, 0, 1, 1) = 1\) and from the second one \(f(0, 0, 1, 1) = 0\), which is impossible. The Lemma is thus proved.

**Lemma 3.** If \(A^{(0)}(\mathcal{B}) \subseteq A^{(0)}(\mathcal{P})\), then every operation \(f\) from \(A^{(0)}(\mathcal{B})\) on \(3\) satisfies the equality

\[(16) \quad f(x_1, x_2, x_3, \ldots, x_n) = f(x_1, x_2, x_3, \ldots, x_n) \land \neg f(x_1, x_2, x_3, \ldots, x_n).\]

Proof. We shall prove this equality by induction on \(n\). It is very easy to verify \((16)\) for trivial operations and for operations of the form \(f(x_1, x_2, \ldots, x_n) = x_1 \land \neg x_2 \land \ldots \land \neg x_n\) (\(1 \leq j_1 < j_2 < \ldots < j_n \leq n\)). Thus \((16)\) holds for all operations from \(A^{(0)}(\mathcal{P})\) depending on at most three variables. Hence, in particular, we get \((16)\) for \(n = 3\).

Now let us suppose that \(n \geq 4\). Identifying in \(f(x_1, x_2, \ldots, x_n)\) two variables, \(x_1\) and \(x_2\) (\(4 \leq k \leq n\)), we obtain an operation \(f_k\) of \(n - 1\) variables satisfying, by the induction assumption, equality \((16)\). Since

\[f_k(x_1, x_2, x_3, \ldots, x_n) = f(x_1, x_2, x_3, \ldots, x_{j_1 - 1}, x_{j_2}, x_{j_1 + 1}, \ldots, x_n),\]

\[f_k(x_1, x_2, x_3, \ldots, x_n) = f(x_1, x_2, x_3, \ldots, x_{j_1 - 1}, x_{j_2}, x_{j_1 + 1}, \ldots, x_n),\]

equality \((16)\) holds for any system \(a_1, a_2, \ldots, a_n\) of elements of \(T\) for which at least one element \(a_k\) (\(4 \leq k \leq n\)) is equal to \(a_1\) or \(a_2\). When \(a_1 = a_2\) or \(a_1 = a_2\), equality \((16)\) is obvious. It remains to verify it in the case \(a_1 = a_2 = a_k\). (\(k \geq 3\), \(3 \leq n\)). Of course, in this case all elements \(a_1, a_2, \ldots, a_n\) must be equal to one another. Identifying in \(f(x_1, x_2, x_3, \ldots, x_n)\) the variables \(x_1, x_2, \ldots, x_k\), we get the operation \(g(x_1, x_2, x_3, \ldots, x_n)\) depending, by Lemma 2, on at most three variables. But we know that every such operation satisfies \((16)\). Hence it follows that \((16)\) holds also in the case \(a_1 = a_2 = a_k\). (\(k \geq 2\), \(3 \leq n\)), which completes the proof.

**Proof of the theorem.** Let \(\mathcal{B}\) be an algebra from \(\mathcal{F}\). First let us consider the case \(A^{(0)}(\mathcal{B}) = A^{(0)}(\mathcal{P})\). Then \(\mathcal{B} \subseteq A^{(0)}(\mathcal{B})\) and, consequently, \(\mathcal{B} \subseteq A^{(0)}(\mathcal{B})\). We have proved that \(A^{(0)}(\mathcal{B})\) consists of all homogeneous operations, which gives the converse inclusion. Thus \(\mathcal{B} = \mathcal{P}\).

Now let us suppose that \(A^{(0)}(\mathcal{B}) = A^{(0)}(\mathcal{P})\). Hence we obtain the relation \(\mathcal{B} \supseteq A^{(0)}(\mathcal{P})\). Consequently, the inclusion

\[A^{(0)}(\mathcal{B}) \supseteq A^{(0)}(\mathcal{P})\]

holds. In order to prove the converse inclusion it suffices to show that every operation from \(A^{(0)}(\mathcal{B})\) can be expressed in terms of the operations \(\cup\) and \(\cap\). Every operation \(f\) from \(A^{(0)}(\mathcal{B})\) can be written in the form

\[(17) \quad f(x_1, x_2, \ldots, x_n) = [x_1 \cup g(x_2, x_3, \ldots, x_n)] \cap [x_1 \cap h(x_2, x_3, \ldots, x_n)],\]
where $g$ and $h$ are $T$-valued operations of the variables $x_1, x_2, \ldots, x_n$. To prove our statement it suffices to show that $g(a_1, a_2, \ldots, a_n) = 1$ whenever $h(a_1, a_2, \ldots, a_n) = 1$. Contrary to this let us suppose that for a system $a_1, a_2, \ldots, a_n$

\begin{equation}
(18) \quad g(a_1, a_2, \ldots, a_n) = 0 \quad \text{and} \quad h(a_1, a_2, \ldots, a_n) = 1.
\end{equation}

Replacing in $f(x_1, x_2, \ldots, x_n)$ the variable $x_1$ by $x$, the variables $x_2$ by $y$ if $a_2 = 1$ and the remaining variables by $z$, we obtain an operation $f_1(x, y, z)$ belonging to $A^{0}(\mathcal{H})$ and, consequently, to $A^{0}(\mathcal{P}_4)$. Thus $f_1$ can be expressed by means of $\cup$ and $\cap$. Hence it follows that the operation $f_1(x, 1, 0) = x'$, which gives a contradiction. The equality $\mathcal{A} = \mathcal{P}_4$ is thus proved.

Finally, by Lemma 1, the following case remains:

\begin{equation}
(19) \quad A^{0}(\mathcal{H}) \subset A^{0}(\mathcal{P}_4).
\end{equation}

First we shall prove by induction on $n$ that every operation from $A^{0}(\mathcal{H})$ can be expressed in terms of symmetric difference. For $n = 3$ this follows from (19). The passage from $n$ to $n + 1$ results from formula (16), which shows that every operation from $A^{n+2}(\mathcal{H})$ is a symmetric difference of three algebraic operations each of which depends on $n$ variables. Thus every operation $f$ from $A^{n}(\mathcal{H})$ can be written in the form

\begin{equation}
(20) \quad f(a_1, a_2, \ldots, a_n) = a_{j_1} \cup a_{j_2} \cap \ldots \cap a_{j_k},
\end{equation}

where $1 \leq j_1 < j_2 < \ldots < j_k \leq n$. The integer $k$ is odd because of the equality $f(x, x, \ldots, x) = x$. Hence we get the inclusion

\begin{equation}
A^{0}(\mathcal{H}) \subset A^{0}(\mathcal{P}_4).
\end{equation}

Further, since $A \times A$, there exists a non-trivial operation in $A(\mathcal{H})$ of the form $a_1 \cup a_2 \cup \ldots \cup a_n$, where $n \geq 4$. Identifying $a_1, a_2, \ldots, a_n$ we get, according to (8), the fundamental operation $p_s$. Thus $p_s \in A(\mathcal{H})$ and, consequently, $A(\mathcal{H}) \subset A(\mathcal{P}_4)$. Comparing this inclusion with (20) we get the equality $\mathcal{A} = \mathcal{P}_4$, which completes the proof of the Theorem.

\section*{References}


