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ABSTRACT ALGEBRAS IN WHICH ALL ELEMENTS ARE INDEPENDENT

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1. Introduction. Let $\mathfrak{A} = (A; \boldsymbol{F})$ be an *abstract algebra*, i. e. a set A of elements and a class $\boldsymbol{F}$ of fundamental operations. Every f from $\boldsymbol{F}$ is a function of several variables which associates with each system $x_1, x_2, \ldots, x_n$ of elements of A an element $f(x_1, x_2, \ldots, x_n)$ belonging to A. If $\boldsymbol{F} = \{f_1, f_2, \ldots, f_k\}$ we often write $(A; f_1, f_2, \ldots, f_k)$ instead of $(A; \boldsymbol{F})$. We denote by $\boldsymbol{A}^{(n)}(\mathfrak{A})$ the class of all *algebraic operations* of n variables, i. e. the smallest class of operations containing all identity operations

$$(1) \qquad e_k^{(n)}(x_1, x_2, \ldots, x_n) = x_k \qquad (k = 1, 2, \ldots, n)$$

and closed under the composition with the fundamental operations. The class of all algebraic operations will be denoted by $\boldsymbol{A}(\mathfrak{A})$. Two algebras $\mathfrak{A} = (A; \boldsymbol{F})$ and $\overline{\mathfrak{A}} = (A; \overline{\boldsymbol{F}})$ having the same class of algebraic operations will be treated here as identical. Further, if $\boldsymbol{A}(\mathfrak{A}) \subset \boldsymbol{A}(\overline{\mathfrak{A}})$, we say that $\mathfrak{A}$ is a *subsystem* of $\overline{\mathfrak{A}}$.

We shall call the identity operations (1) also *trivial operations*. If all algebraic operations are trivial, then the algebra will be called *trivial*.

Let us consider an n-element subset $I = \{a_1, a_2, \ldots, a_n\}$ of A. We say that I is a set of *independent* elements, or else that all elements of I are *independent* if for any $f, g \in \boldsymbol{A}^{(n)}(\mathfrak{A})$ the equality $f(a_1, a_2, \ldots, a_n) = g(a_1, a_2, \ldots, a_n)$ implies the identity of f and g in A. The properties of the notion of independence are given in [1] and [2].

All elements of a trivial algebra are obviously independent. S. Świerczkowski has proved that for algebras having at least three elements the converse implication is true: if all elements are independent, then the algebra is trivial ([5], p. 501, and [6], Theorem 1). In a recent note [3], using a complete characterization of two-element algebras given by Post [4], we described non-trivial two-element algebras in which all elements are independent. Namely, denoting by $\mathscr{I}$ the class of all such algebras, with fixed elements 0 and 1, we proved that $\mathscr{I}$ consists of three algebras, $\mathfrak{P}$, $\mathfrak{P}_*$ and $\mathfrak{P}^*$ (defined below). In this paper we present a complete proof of the result quoted. The proof involves some of Post's ideas as presented in [4]; however, no particular result of [4] is explicitly used. We hope that this will be appreciated by the reader, the more so as the reasonings in [4] are rather complicated.

In what follows we shall consider algebras possessing only the elements 0 and 1. The set $T = \{0, 1\}$ can be regarded as a Boolean algebra with 0 as a neutral element. By $\cup$, $\cap$, $'$ and $\dot{-}$ we denote the elementary Boolean operations: joint, meet, complementation and symmetric subtraction. A T-valued operation f of n variables running over T is said to be *homogeneous* if for every mapping τ of T into itself the equality

$$f(\tau(x_1), \tau(x_2), \ldots, \tau(x_n)) = \tau(f(x_1, x_2, \ldots, x_n))$$

holds. Since each mapping of T into itself is a composition of two mappings $x \to x'$ and $x \to 0$ $(x \in T)$, *an operation f is homogeneous if and only if*

$$f(x_1, x_2, \ldots, x_n) = f'(x_1', x_2', \ldots, x_n') \tag{2}$$

and

$$f(0, 0, \ldots, 0) = 0. \tag{3}$$

For an arbitrary abstract algebra it was proved in [1] (p. 733) that I is a set of independent elements if and only if each mapping of I into A can be extended to a homomorphism of the subalgebra generated by I into A. Applying this result to the algebras in question we get the following statement:

The elements 0 and 1 of an algebra $\mathfrak{A}$ are independent if and only if all algebraic operations from $\boldsymbol{A}(\mathfrak{A})$ are homogeneous.

It is very easy to verify that the only operations of two variables satisfying conditions (2) and (3) are trivial ones. Furthermore, it is obvious that the triviality of all operations of two variables implies the independence of the elements 0 and 1. Consequently, *the elements 0 and 1 are independent if and only if each operation from $\boldsymbol{A}^{(2)}(\mathfrak{A})$ is trivial.*

2. The algebras $\mathfrak{P}_*$, $\mathfrak{P}^*$ and $\mathfrak{P}$. We define three T-valued nontrivial operations p_*, p^* and p of three variables running over T by the following conditions:

$$p_*(x, x, y) = p_*(x, y, x) = p_*(y, x, x) = y, \tag{4}$$

$$p^*(x, x, y) = p^*(x, y, x) = p^*(y, x, x) = x, \tag{5}$$

$$p(x, x, y) = x, \quad p(x, y, x) = p(y, x, x) = y. \tag{6}$$

The three algebras

$$\mathfrak{P}_* = (T; p_*), \quad \mathfrak{P}^* = (T; p^*), \quad \mathfrak{P} = (T; p)$$

were considered by Post [4]. It follows directly from (4), (5), (6) and from the definition of algebraic operations of two variables that all these operations in the three algebras considered are trivial. Thus $\mathfrak{P}_*, \mathfrak{P}^*, \mathfrak{P} \in \mathscr{I}$. Further, from the formulas

$$\begin{cases} p_*(x, y, z) = p(p(x, y, z), p(x, z, y), x), \\ p^*(x, y, z) = p(x, y, p(x, y, z)), \\ p(x, y, z) \;\;= p_*(x, y, p^*(x, y, z)) \end{cases} \tag{7}$$

we get the equality $\mathfrak{P} = (T; p_*, p^*)$. Consequently, the algebras $\mathfrak{P}_*$ and $\mathfrak{P}^*$ are subsystems of the algebra $\mathfrak{P}$.

We shall now describe the structure of the algebras $\mathfrak{P}_*$, $\mathfrak{P}^*$ and $\mathfrak{P}$.

Algebra $\mathfrak{P}_*$. The fundamental operation p_* can be expressed in terms of the symmetric difference as follows:

$$p_*(x, y, z) = x \dot{-} y \dot{-} z. \tag{8}$$

Hence we infer that *the class $\boldsymbol{A}(\mathfrak{P}_*)$ consists of all operations f of the form*

$$f(x_1, x_2, \ldots, x_n) = x_{j_1} \dot{-} x_{j_2} \dot{-} \ldots \dot{-} x_{j_k},$$

where $1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant n$ and k is an odd integer.

Algebra $\mathfrak{P}^*$. The fundamental operation p^* can be defined by means of the joint and the meet operations as follows:

$$p^*(x, y, z) = (x \cap y) \cup (x \cap z) \cup (y \cap z). \tag{9}$$

Hence we infer that all algebraic operations from $\boldsymbol{A}(\mathfrak{P}^*)$ can be expressed in terms of the operations $\cup$ and $\cap$. Now we shall prove that *the class $\boldsymbol{A}(\mathfrak{P})^*$ consists of all homogeneous operations which can be expressed in terms of the operations $\cup$ and $\cap$.*

We present the proof which is an adaptation of a part of paper [4] by Post. In order to prove our statement it is sufficient to show that every homogeneous operation which can be expressed in terms of the operations $\cup$ and $\cap$ belongs to $\boldsymbol{A}(\mathfrak{P}^*)$. Let f be such an operation of n variables. First we shall prove that for every pair $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ of systems of elements of T satisfying the condition

$$f(a_1, a_2, \ldots, a_n) = 1 = f(b_1, b_2, \ldots, b_n) \tag{10}$$

there exists an index k $(1 \leqslant k \leqslant n)$ such that $a_k = b_k = 1$. Contrary to this, let us suppose that $a_j \cap b_j = 0$ for all indices j $(1 \leqslant j \leqslant n)$. In other words, we have $a_j' = 1$ whenever $b_j = 1$. Since the operation f can be expressed in terms of $\cup$ and $\cap$, we have, according to (10), the equality $f(a_1', a_2', \ldots, a_n') = 1$, which, by virtue of (2), implies $f(a_1, a_2, \ldots, a_n) = 0$. But this contradicts (10).

Formula (9) can be rewritten in the form $p^*(x,y,z) = x\cap(y\cup z)\cup(y\cap z)$. Moreover, the compositions g_m $(m = 2, 3, \ldots)$ defined by formulas

$$g_2(x_0, x_1, x_2) = p^*(x_0, x_1, x_2),$$

$$g_m(x_0, x_1, \ldots, x_m) = p^*\big(x_0, x_m, g_{m-1}(x_0, x_1, \ldots, x_{m-1})\big) \qquad (m \geqslant 3)$$

can be written in the form

$$g_m(x_0, x_1, \ldots, x_m) = \Big(x_0 \cap \bigcup_{j=1}^{m} x_j\Big) \cup \bigcap_{j=1}^{m} x_j .$$

Let $a_{1,s}, a_{2,s}, \ldots, a_{n,s}$ $(s = 1, 2, \ldots, r)$ be the family of all systems of elements from T satisfying the condition $f(a_{1,s}, a_{2,s}, \ldots, a_{n,s}) = 1$. By J_s we denote the set of indices j for which $a_{j,s} = 1$. We have proved that

$$J_{s_1} \cap J_{s_2} \neq 0 \qquad (1 \leqslant s_1, s_2 \leqslant r). \tag{11}$$

Let us introduce the following sequence $f_1, f_2, \ldots, f_r$ of compositions of operations g_m and trivial operations:

$$f_1(x_1, x_2, \ldots, x_n) = x_{j_1}, \quad \text{where} \quad j_1 \epsilon J_1,$$

$$f_k(x_1, x_2, \ldots, x_n) = \Big(f_{k-1}(x_1, x_2, \ldots, x_n) \cap \bigcup_{j\epsilon J_k} x_j\Big) \cup \bigcap_{j\epsilon J_k} x_j \qquad (2 \leqslant k \leqslant r).$$

Obviously, all the operations $f_1, f_2, \ldots, f_r$ belong to $\boldsymbol{A}(\mathfrak{P}^*)$. Moreover, from (11) we get the equalities

$$1 = f_s(a_{1,s}, a_{2,s}, \ldots, a_{n,s}) = f_{s+1}(a_{1,s}, a_{2,s}, \ldots, a_{n,s}) = \ldots$$
$$= f_r(a_{1,s}, a_{2,s}, \ldots, a_{n,s}) \qquad (s = 1, 2, \ldots, r).$$

In other words, $f_r(a_1, a_2, \ldots, a_n) = 1$ whenever $f(a_1, a_2, \ldots, a_n) = 1$. Since both operations are homogeneous, the last implication implies the following one: $f_r(a_1, a_2, \ldots, a_n) = 0$ whenever $f(a_1, a_2, \ldots, a_n) = 0$. Thus $f = f_r$ and, consequently, $f \epsilon \boldsymbol{A}(\mathfrak{P}^*)$, which completes the proof.

Algebra $\mathfrak{P}$. We know that all algebraic operations from $\boldsymbol{A}(\mathfrak{P})$ are homogeneous. The converse implication is also true. Namely, we shall prove that *the class $\boldsymbol{A}(\mathfrak{P})$ consists of all homogeneous operations*.

Now let f be a homogeneous operation of n variables. Since every T-valued operation defined on T is a Boolean polynomial, we can write

$$f(1, x_2, \ldots, x_n) = \bigcup_{(i_2, i_3, \ldots, i_n)\epsilon J} \ \bigcap_{k=2}^{n} x_k^{i_k}, \tag{12}$$

where J is a set of $(n-1)$-tuples of 0's and 1's and $x^0 = x'$, $x^1 = x$ $(x \epsilon T)$. Since, by (2) and (3), $f(1, 1, \ldots, 1) = 1$, the set J contains the $(n-1)$-tuple $(1, 1, \ldots, 1)$. Writing the operation p in terms of Boolean operations

$$p(x,y,z) = (x\cap y\cap z)\cup(x\cap y\cap z')\cup(x'\cap y\cap z')\cup(x\cap y'\cap z'),$$

we have $p(1, y, z_k) = y \cup z'$. Hence we infer that for every k-tuple $(j_1, j_2, \ldots, j_k) \neq (0, 0, \ldots, 0)$ the class $\boldsymbol{A}(\mathfrak{P})$ contains an operation $g_{j_1, j_2, \ldots, j_k}$ of $k+1$ variables such that

$$g_{j_1, j_2, \ldots, j_k}(1, x_1, \ldots, x_k) = \bigcup_{s=1}^{k} x_s^{j_s}.$$

Thus for every $(n-1)$-tuple $(i_2, i_3, \ldots, i_n) \neq (1, 1, \ldots, 1)$ there exists an operation $h_{i_2, i_3, \ldots, i_n}$ in $\boldsymbol{A}^{(n)}(\mathfrak{P})$ such that

$$h_{i_2, i_3, \ldots, i_n}(1, x_2, x_3, \ldots, x_n) = \Big(\bigcap_{k=2}^{n} x_k^{i_k}\Big)'.$$

Further, from the equality $p(y, z, 1) = y \cap z$ it follows that there exists an operation $h \epsilon \boldsymbol{A}^{(n)}(\mathfrak{P})$ such that

$$h(1, x_2, \ldots, x_n) = \bigcap_{k=2}^{n} x_k .$$

Let J be the set appearing in (12). By a suitable composition of the operations $g_{j_1, j_2, \ldots, j_k}$, $h_{i_2, i_3, \ldots, i_n}$ and h we get an operation $f_J \epsilon \boldsymbol{A}^{(n)}(\mathfrak{P})$ such that

$$f_J(1, x_2, \ldots, x_n) = \bigcup_{(i_2, i_3, \ldots, i_n)\epsilon J} \ \bigcap_{k=2}^{n} x_k^{i_k}.$$

Hence and from (12) we obtain the equality $f(1, x_2, \ldots, x_n) = f_J(1, x_2, \ldots, x_n)$, whence, by virtue of the homogeneity of f and f_J, the equality $f = f_J$ follows. Thus $f \epsilon \boldsymbol{A}(\mathfrak{P})$, which completes the proof.

3. Characterization of the class $\mathscr{J}$. We have seen that all the algebras $\mathfrak{P}_*$, $\mathfrak{P}^*$ and $\mathfrak{P}$ belong to $\mathscr{J}$. The complete description of this class is given by the following

THEOREM. $\mathscr{J} = \{\mathfrak{P}_*, \mathfrak{P}^*, \mathfrak{P}\}$.

Before proving the theorem we shall prove some lemmas. In what follows the algebra $\mathfrak{A}$ is supposed to belong to $\mathscr{J}$.

LEMMA 1. *One of the relations*

$$\boldsymbol{A}^{(3)}(\mathfrak{A}) = \boldsymbol{A}^{(3)}(\mathfrak{P}), \qquad \boldsymbol{A}^{(3)}(\mathfrak{A}) = \boldsymbol{A}^{(3)}(\mathfrak{P}^*), \qquad \boldsymbol{A}^{(3)}(\mathfrak{A}) \subseteq \boldsymbol{A}^{(3)}(\mathfrak{P}_*),$$

holds.

Proof. Since $\boldsymbol{A}(\mathfrak{P})$ consists of all homogeneous operations, we have the inclusion $\boldsymbol{A}(\mathfrak{A}) \subset \boldsymbol{A}(\mathfrak{P})$ and, consequently,

$$\boldsymbol{A}^{(3)}(\mathfrak{A}) \supset \boldsymbol{A}^{(3)}(\mathfrak{P}). \tag{13}$$

First let us suppose that $\boldsymbol{A}^{(3)}(\mathfrak{A})$ contains a non-symmetric operation f depending on every variable. Since operations of two variables are trivial, each of the operations $f(x,x,y)$, $f(x,y,x)$ and $f(y,x,x)$ is equal to either x or y. Further, by the non-symmetry of f, among these operations there exist two different ones. Without loss of generality we may suppose that

(14) $$f(x, x, y) = x, \quad f(x, y, x) = y.$$

Since the operation f depends on the last variable, there exists a system a_1, a_2, a_3 such that $f(a_1, a_2, a_3) \neq f(a_1, a_2, a_3')$. Consequently, $f(y, x, y) \neq f(y, x, x)$, which together with (14) yields $f(y, x, x) = y$. Taking into account formula (6), we have the equality $f = p$. Thus $A^{(3)}(\mathfrak{P}) \supset A^{(3)}(\mathfrak{A})$. Comparing this inclusion with (13), we get the equality $A^{(3)}(\mathfrak{A}) = A^{(3)}(\mathfrak{P})$.

Now let us suppose that each operation g from $A^{(3)}(\mathfrak{A})$ depending on every variable is symmetric. Then all operations $g(x, x, y)$, $g(x, y, x)$ and $g(y, x, x)$ are equal to either x or y. By (4) and (5) in the first case we have the equality $g = p_*$ and in the second $g = p^*$. Because of formula (7) the class $A^{(3)}(\mathfrak{A})$ contains exactly one of the operations p_* and p^*. Thus we have either $A^{(3)}(\mathfrak{A}) = A^{(3)}(\mathfrak{P}_*)$ or $A^{(3)}(\mathfrak{A}) = A^{(3)}(\mathfrak{P}^*)$.

Finally, if $A^{(3)}(\mathfrak{A})$ does not contain any operation depending on every variable, then all operations of three variables are trivial and, of course, $A^{(3)}(\mathfrak{A}) \subset A^{(3)}(\mathfrak{P}_*)$, which completes the proof.

LEMMA 2. *If $A^{(3)}(\mathfrak{A}) \subset A^{(3)}(\mathfrak{P}_*)$, then there exists no operation in $A^{(4)}(\mathfrak{A})$ depending on every variable.*

Proof. Contrary to this, let us suppose that there exists an operation f in $A^{(4)}(\mathfrak{A})$ depending on every variable. First we shall prove the equalities

(15) $$f(y, y, y, x) = f(y, y, x, y) = f(y, x, y, y) = f(x, y, y, y) = x.$$

By the symmetry of our assumption it suffices to prove the last equality. Since f depends on the first variable we can find a system a_1, a_2, a_3, a_4 of elements of T such that $f(a_1, a_2, a_3, a_4) \neq f(a_1', a_2, a_3, a_4)$. Among the elements a_2, a_3, a_4 at least two are identical. Without loss of generality we may suppose that $a_2 = a_3$. Hence it follows that the operation $f(x, y, y, z)$ depends on x. We know that the only operations in $A^{(3)}(\mathfrak{P}_*)$ depending on the first variable are x and $x \dot{-} y \dot{-} z$. Thus, by the inclusion $A^{(3)}(\mathfrak{A}) \subset A^{(3)}(\mathfrak{P}_*)$, $f(x, y, y, z) = x$ or $f(x, y, y, z) = x \dot{-} y \dot{-} z$. Obviously, in both cases we have the equality $f(x, y, y, y) = x$, which completes the proof of (15).

From (15) it follows directly that the operations $f(x_1, x_2, x_3, x_3)$ and $f(x_1, x_1, x_3, x_4)$ depend on x_1, x_2 and x_3, x_4 respectively. Since they belong to $A^{(3)}(\mathfrak{P}_*)$, we have the equalities

$$f(x_1, x_2, x_3, x_3) = x_1 \dot{-} x_2 \dot{-} x_3,$$

$$f(x_1, x_1, x_3, x_4) = x_1 \dot{-} x_3 \dot{-} x_4.$$

From the first equality we get $f(0, 0, 1, 1) = 1$ and from the second one $f(0, 0, 1, 1) = 0$, which is impossible. The Lemma is thus proved.

LEMMA 3. *If $A^{(3)}(\mathfrak{A}) \subset A^{(3)}(\mathfrak{P}_*)$, then every operation f from $A^{(n)}(\mathfrak{A})$ $(n \geqslant 3)$ satisfies the equality*

(16) $$f(x_1, x_2, x_3, x_4, \ldots, x_n) = f(x_1, x_1, x_1, x_4, \ldots, x_n) \dot{-} \\ \dot{-} f(x_1, x_2, x_1, x_4, \ldots, x_n) \dot{-} f(x_1, x_1, x_3, x_4, \ldots, x_n).$$

Proof. We shall prove this equality by induction on n. It is very easy to verify (16) for trivial operations and for operations of the form $f(x_1, x_2, \ldots, x_n) = x_{j_1} \dot{-} x_{j_2} \dot{-} x_{j_3}$ $(1 \leqslant j_1 < j_2 < j_3 \leqslant n)$. Thus (16) holds for all operations from $A(\mathfrak{P}_*)$ depending on at most three variables. Hence, in particular, we get (16) for $n = 3$.

Now let us suppose that $n \geqslant 4$. Identifying in $f(x_1, x_2, \ldots, x_n)$ two variables, x_1 and x_k $(4 \leqslant k \leqslant n)$, we obtain an operation f_k of $n-1$ variables satisfying, by the induction assumption, equality (16). Since

$$f_k(x_1, x_1, x_1, \ldots) = f(x_1, x_1, x_1, \ldots, x_{k-1}, x_1, x_{k+1}, \ldots, x_n),$$

$$f_k(x_1, x_2, x_1, \ldots) = f(x_1, x_2, x_1, \ldots, x_{k-1}, x_1, x_{k+1}, \ldots, x_n),$$

$$f_k(x_1, x_1, x_3, \ldots) = f(x_1, x_1, x_3, \ldots, x_{k-1}, x_1, x_{k+1}, \ldots, x_n),$$

equality (16) holds for any system $a_1, a_2, \ldots, a_n$ of elements of T for which at least one element a_k $(4 \leqslant k \leqslant n)$ is equal to a_1. When $a_1 = a_2$ or $a_1 = a_3$, equality (16) is obvious. It remains to verify it in the case $a_1 \neq a_k$ $(k = 2, 3, \ldots, n)$. Of course, in this case all elements $a_2, a_3, \ldots, a_n$ must be equal to one another. Identifying in $f(x_1, x_2, \ldots, x_n)$ the variables $x_4, x_5, \ldots, x_n$, we get the operation $g(x_1, x_2, x_3, x_4)$ depending, by Lemma 2, on at most three variables. But we know that every such operation satisfies (16). Hence it follows that (16) holds also in the case $a_1 \neq a_k$ $(k = 2, 3, \ldots, n)$, which completes the proof.

Proof of the theorem. Let $\mathfrak{A}$ be an algebra from $\mathscr{I}$. First let us consider the case $A^{(3)}(\mathfrak{A}) = A^{(3)}(\mathfrak{P})$. Then $p \in A^{(3)}(\mathfrak{A})$ and, consequently, $A(\mathfrak{A}) \supset A(\mathfrak{P})$. We have proved that $A(\mathfrak{P})$ consists of all homogeneous operations, which gives the converse inclusion. Thus $\mathfrak{A} = \mathfrak{P}$.

Now let us suppose that $A^{(3)}(\mathfrak{A}) = A^{(3)}(\mathfrak{P}^*)$. Hence we obtain the relation $p^* \in A^{(3)}(\mathfrak{A})$. Consequently, the inclusion

$$A(\mathfrak{A}) \supset A(\mathfrak{P}^*)$$

holds. In order to prove the converse inclusion it suffices to show that every operation from $A(\mathfrak{A})$ can be expressed in terms of the operations $\cup$ and $\cap$. Every operation f from $A^{(n)}(\mathfrak{A})$ can be written in the form

(17) $$f(x_1, x_2, \ldots, x_n) = \big(x_1 \cap g(x_2, x_3, \ldots, x_n)\big) \cap \big(x_1' \cap h(x_2, x_3, \ldots, x_n)\big),$$

where g and h are T-valued operations of the variables $x_2, x_3, \ldots, x_n$. To prove our statement it suffices to show that $g(a_2, a_3, \ldots, a_n) = 1$ whenever $h(a_2, a_3, \ldots, a_n) = 1$. Contrary to this let us suppose that for a system $a_2, a_3, \ldots, a_n$

$$g(a_2, a_3, \ldots, a_n) = 0 \quad \text{and} \quad h(a_2, a_3, \ldots, a_n) = 1. \tag{18}$$

Replacing in $f(x_1, x_2, \ldots, x_n)$ the variable x_1 by x, the variables x_j by y if $a_j = 1$ and the remaining variables by z, we obtain an operation $f_0(x, y, z)$ belonging to $\boldsymbol{A}^{(3)}(\mathfrak{A})$ and, consequently, to $\boldsymbol{A}^{(3)}(\mathfrak{P}^*)$. Thus f_0 can be expressed in terms of $\cup$ and $\cap$. Hence it follows that also the operation $f_0(x, 1, 0)$ can be defined by means of $\cup$ and $\cap$. But, according to (17) and (18), $f_0(x, 1, 0) = x'$, which gives a contradiction. The equality $\mathfrak{A} = \mathfrak{P}^*$ is thus proved.

Finally, by Lemma 1, the following case remains:

$$\boldsymbol{A}^{(3)}(\mathfrak{A}) \subset \boldsymbol{A}^{(3)}(\mathfrak{P}_*). \tag{19}$$

First we shall prove by induction on n that every operation from $\boldsymbol{A}^{(n)}(\mathfrak{A})$ can be expressed in terms of symmetric difference. For $n = 3$ this follows from (19). The passage from n to $n+1$ results from formula (16), which shows that every operation from $\boldsymbol{A}^{n+1}(\mathfrak{A})$ is a symmetric difference of three algebraic operations each of which depends on n variables. Thus every operation $f \in \boldsymbol{A}^{(n)}(\mathfrak{A})$ can be written in the form

$$f(x_1, x_2, \ldots, x_n) = x_{j_1} \dot{-} x_{j_2} \dot{-} \ldots \dot{-} x_{j_k},$$

where $1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant n$. The integer k is odd because of the equality $f(x, x, \ldots, x) = x$. Hence we get the inclusion

$$\boldsymbol{A}(\mathfrak{A}) \subset \boldsymbol{A}(\mathfrak{P}_*). \tag{20}$$

Further, since $\mathfrak{A} \in \mathscr{I}$, there exists a non-trivial operation in $\boldsymbol{A}(\mathfrak{A})$ of the form $x_1 \dot{-} x_2 \dot{-} \ldots \dot{-} x_n$ where $n \geqslant 3$. Identifying $x_3, x_4, \ldots, x_n$ we get, according to (8), the fundamental operation p_*. Thus $p_* \in \boldsymbol{A}(\mathfrak{A})$ and, consequently, $\boldsymbol{A}(\mathfrak{A}) \supset \boldsymbol{A}(\mathfrak{P}_*)$. Comparing this inclusion with (20) we get the equality $\mathfrak{A} = \mathfrak{P}_*$, which completes the proof of the Theorem.

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