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TO KARL MENGER ON HIS 60-TH BIRTHDAY

A MAPPING-ALGEBRA WITH INFINITELY MANY OPERATIONS

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In this note we propose to develop an algebra of mappings relative to any given non-empty set S. This algebra has many analogies to (and was in part suggested by) combinatory logic. At the same time, however, there are certain basic differences: the most striking of these are a hierarchical structuring of the elements of the algebra and the presence in it of a denumerable infinity of interrelated associative operations.

1. Functions over the (non-empty) set S (briefly, functions) of degree d and rank r are defined recursively as follows:

Definition 1. (a) A *function over S of degree* 0, *rank* r ($\geqslant 1$) is a string of length r (i. e., an ordered r-tuple) of elements of S.

(b) A *function over S of degree d* (>0), *rank* r ($\geqslant 1$) is mapping whose domain is S and whose values are functions of degree $d-1$, rank r.

The set of all functions of degree d and rank r will be denoted by $\boldsymbol{F}_{dr}(S)$. The union of all the sets $\boldsymbol{F}_{dr}(S)$, for $d = 0, 1, 2, \ldots, r = 1, 2, \ldots$, will be denoted by $\boldsymbol{F}(S)$. If F belongs to $\boldsymbol{F}_{dr}(S)$, then we write $\deg F = d$, $\operatorname{rank} F = r$. In addition, we set $\deg F - \operatorname{rank} F = \operatorname{ind} F$ (the *index* of F). Note that $\operatorname{ind} F$ may be any integer, whereas $\deg F$ is a non-negative, and $\operatorname{rank} F$ is a positive, integer.

If a belongs to S and F belongs to $\boldsymbol{F}_{dr}(S)$ ($d > 0$), then the value of F at a, which we denote by Fa, is well-defined and belongs to $\boldsymbol{F}_{d-1,r}(S)$. If $\deg F = 0$ and $\operatorname{rank} F = r$, i. e., if F is the string $a_1 \ldots a_r$, then Fa is as yet undefined; we remedy this by setting $Fa = a_1 \ldots a_r a$, whence $\deg(Fa) = 0$ and $\operatorname{rank}(Fa) = r+1$.

THEOREM 1. *For $d > 0$, there is a one-one correspondence between $\boldsymbol{F}_{dr}(S)$ and the set of all mappings of S^d into S^r, where S^n denotes the n-fold Cartesian product of S with itself. In particular, $\boldsymbol{F}_{d1}(S)$ corresponds* (1) *to the set of all d-place functions on S.*

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(1) This correspondence was first exploited by Schönfinkel [10].

We now introduce an equivalence relation, denoted by $=$, into $\boldsymbol{F}(S)$. This equivalence relation, which cuts across the partition of $\boldsymbol{F}(S)$ into the sets $\boldsymbol{F}_{dr}(S)$, is defined as follows:

Definition 2. (a) If $\deg F_1 = \deg F_2 = 0$, then $F_1 = F_2$ if and only if F_1 and F_2 are identical.

(b) If either $\deg F_1 > 0$ or $\deg F_2 > 0$, then $F_1 = F_2$ if and only if $F_1 a = F_2 a$ for all a in S.

The structure of this equivalence relation is brought out by the following

THEOREM 2. *If F_1 and F_2 belong to $\boldsymbol{F}(S)$ and if $F_1 = F_2$, then* $\operatorname{ind} F_1 = \operatorname{ind} F_2$. *If $F_1 = F_2$ and either* $\deg F_1 = \deg F_2$ *or* $\operatorname{rank} F_1 = \operatorname{rank} F_2$, *then F_1 and F_2 are identical. If F belongs to $\boldsymbol{F}_{dr}(S)$, then there is a unique member of $\boldsymbol{F}_{d+1,r+1}(S)$ equivalent to F.*

It follows that in every equivalence class in $\boldsymbol{F}(S)$ there is a unique element which is, simultaneously, of minimal degree and of minimal rank. This element will be referred to as the *basic function* of the class. The set of all basic functions will be denoted by $\boldsymbol{B}(S)$, while the set of all basic functions in $\boldsymbol{F}_{dr}(S)$ will be denoted by $\boldsymbol{B}_{dr}(S)$. (N. B. — if S has more than one element, then each of the sets $\boldsymbol{B}_{dr}(S)$ is non-empty.) Among the basic functions are j_S, the member of $\boldsymbol{B}_{11}(S)$ defined by: $j_S a = a$, for all a in S; and I_S, the member of $\boldsymbol{B}_{21}(S)$ defined by: $I_S a = j_S$, for all a in S. In the sequel, j_S and I_S will usually be abbreviated to j and I, respectively.

Definition 3. If F and G belong to $\boldsymbol{B}(S)$, then their *composite*, denoted by FG, is defined as follows:

(a) If $\deg G = 0$, so that $G = a_1 a_2 \ldots a_{r-1} a_r$, for some $a_1, a_2, \ldots, a_{r-1}, a_r$ in S, then FG is the unique member of $\boldsymbol{B}(S)$ given by

$$FG = \big(\ldots\big((Fa_1)a_2\big)\ldots a_{r-1}\big)a_r.$$

(b) If $\deg G > 0$, then FG is the unique member of $\boldsymbol{B}(S)$ such that

$$(FG)a = F(Ga)$$

for all a in S.

THEOREM 3. *Under composition, $\boldsymbol{B}(S)$ is a semigroup with unit. The unit is j. Furthermore, for any functions F, G in $\boldsymbol{B}(S)$, we have*

$$\operatorname{ind}(FG) = \operatorname{ind} F + \operatorname{ind} G.$$

Since composition is associative, and therefore power-associative, powers of functions in $\boldsymbol{B}(S)$ may be defined in the standard way:

$$F^0 = j, \quad F^{n+1} = FF^n$$

for any F in $\boldsymbol{B}(S)$ and any non-negative integer n.

We are now in a position to introduce an infinite number of binary operations on $\boldsymbol{B}(S)$ (2).

Definition 4. For any non-negative integer n, we define the operation $\boldsymbol{n}$, of order n, as follows:

$$F\mathbf{0}G = FG, \quad (F\boldsymbol{n{+}1}G)a = Fa\boldsymbol{n}Ga,$$

for any F, G in $\boldsymbol{B}(S)$ and any a in S (3).

THEOREM 4. *Under each of the operations $\boldsymbol{n}$, defined above, the set $\boldsymbol{B}(S)$ is a semigroup with unit. The unit for $\boldsymbol{n}$ is I^n. Furthermore, for any F, G in $\boldsymbol{B}(S)$, we have*

$$\operatorname{ind}(F\boldsymbol{n}G) = \operatorname{ind} F + \operatorname{ind} G - n.$$

THEOREM 5 (Interassociativity). *If F, G, H are in $\boldsymbol{B}(S)$ and $m \leqslant n$, then*

$$(F\boldsymbol{m}G)\boldsymbol{n}H = F\boldsymbol{m}(G\boldsymbol{n}H).$$

THEOREM 6 (Quasi-distributivity). *If F, G, H are in $\boldsymbol{B}(S)$ and $m \geqslant n + \operatorname{rank}(Ha_1 \ldots a_n)$, then*

$$(F\boldsymbol{m}G)\boldsymbol{n}H = (F\boldsymbol{n}H)\boldsymbol{p}(G\boldsymbol{n}H),$$

where $p = m - n + \operatorname{ind} H$. In particular, if $n = \operatorname{ind} H$, then $p = m$ and the quasi-distributivity reduces to actual distributivity.

The pair $A(S) = \{\boldsymbol{B}(S), \{\boldsymbol{n}\}\}$, where $\{\boldsymbol{n}\}$ is the set of operations of Definition 4, is the *basic function algebra* (over S).

2. As previously mentioned, many features of the algebra $A(S)$ have their counterparts in combinatory logic (4). For example, if $K = j\mathbf{1}I^2$, then for any a, b in S, we have

$$Ka = (j\mathbf{1}I^2)a = ja\mathbf{0}I^2a = jaI^2a = aI^2a = aIIa = aIj = aI,$$

(2) Strictly speaking, here, as well as in Definition 3, what we combine are the *equivalence classes* in $\boldsymbol{F}(S)$ rather than the basic functions of $\boldsymbol{B}(S)$. It has seemed desirable, however, to work throughout with individual representatives of these equivalence classes, and generally with the unique representative of each which is in $\boldsymbol{B}(S)$.

(3) The definition of the operations $\boldsymbol{n}$ was suggested by the definition of a similar operation introduced by McKiernan in his theory of operators. For an outline of some aspects of this theory, see [5] and [6]. The usage of boldface numerals to denote the operations follows that of Aczél, Belousov and Hosszú in [1].

(4) In view of the analogies between $A(S)$ and combinatory logic, as well as the relation between $A(S)$ and some of the work of Menger (see footnotes 7 and 8), it is possible that the algebras $A(S)$ will serve as a bridge between the views of the combinatory logicians and those of Menger. Cf. some of the statements in Curry and Feys [4] on pp. 11, 80f., 105.

so that

$$Kab = aIb = aj = a.$$

More generally, if $K_n = j\mathbf{n}I^{2n}$, (so that K_1 coincides with K), then we find that for any F, G in $\mathbf{B}_{dr}(S)$,

$$K_r F\mathbf{d}G = F.$$

Indeed, if the functions F_k, $k = 1, \ldots, m$, are in $\mathbf{B}_{dr}(S)$ and if $1 \leqslant n \leqslant m$, then

$$K_r^{m-n} I^{r(n-1)} F_1 \mathbf{d} F_2 \mathbf{d} \ldots \mathbf{d} F_n \mathbf{d} \ldots \mathbf{d} F_m = F_n.$$

It is thus possible to construct a function which selects the n-th member of a string of functions in $\mathbf{B}_{dr}(S)$. More complicated combinations of K's and I's will select a particular F, or a combination of F's, from an arbitrary string. However, unlike, say, the universal constancy operator $\mathbf{K}$ of a combinatory logic (5), the "selector" or "constancy" functions of $A(S)$ must be tailored to the structure of the strings from which they are to select. This lack of universality is offset, on the other hand, by the ease of manipulation and construction in $A(S)$ which arises from the associativity and interassociativity of all the operations — features not present in combinatory logics.

Variables (in this case, variables ranging over S) can be "eliminated" in $A(S)$, just as in combinatory logics (6). Consider, for example, the composite 2-place function $F(G_1, G_2, G_3)$, which is formed by substituting the three 2-place functions G_1, G_2, G_3 into the 3-place function F, and ordinarily defined by writing:

$$F(G_1, G_2, G_3)(x, y) = F\big(G_1(x, y), G_2(x, y), G_3(x, y)\big),$$

for all x, y in S. In the symbolism of $A(S)$, this composite function can be defined without "variables" by writing:

$$F(G_1, G_2, G_3) = FG_1\mathbf{2}G_2\mathbf{2}G_3.$$

More generally, an ordered m-tuple of n-place functions $G_1, G_2, \ldots, G_m$ may be substituted into an m-place function F to yield an n-place function $F(G_1, G_2, \ldots, G_m)$ by a procedure which is precisely and succinctly characterized by the equation (7)

$$F(G_1, G_2, \ldots, G_m) = FG_1\mathbf{n}G_2\mathbf{n}\ldots\mathbf{n}G_m.$$

(5) Cf. [2], pp. 4, 58; [3], p. 371; [4], pp. 153, 188f.; [9], p. 7; [10] (where the corresponding object is denoted by 'C').

(6) This "elimination of variables" is called the *raison d'être* of combinatory logic by Curry in [3], p. 371.

(7) On p. 302 of his book [7], K. Menger calls this type of substitution (m, n)-substitution. A second type of substitution of m functions into an m-place function

If the original set S has some internal structure, then this structure is reflected in the algebra $A(S)$. For example, if S is the set of real numbers R, then the set $\mathbf{B}_{11}(R)$ contains the *linear* functions, L_{ab}, which are defined, for a, b in R, by: $L_{ab}c = a + b\cdot c$, for any c in R. Accordingly, the set $\mathbf{B}_{21}(R)$ contains the function Σ (addition), defined by: $\Sigma a = L_{a1}$, for any a in R, and the function Π (multiplication), defined by: $\Pi a = L_{0a}$, for any a in R. It follows that,

$$\Sigma ab = a + b, \qquad \Pi ab = a\cdot b,$$

for any a, b in R. Furthermore,

$$\Sigma f\mathbf{1}g = f + g, \qquad \Pi f\mathbf{1}g = f\cdot g,$$

for any f, g in $\mathbf{B}_{11}(R)$. Thus the pointwise sum and product of any two ordinary real-valued functions on the reals are both definable within the algebra $A(R)$. The same is true for more complicated sums and products, e. g., of n-place functions.

Addition and multiplication of real numbers are commutative and associative operations, and these properties of R, as well as others, are reflected in the fact that Σ and Π satisfy certain functional equations. This suggests a possible use of the algebras $A(S)$ in establishing a natural classification of functional equations which, in turn, would be a first step towards a systematic theory of such equations. For example, the classical functional equation of Cauchy, viz.,

$$f(x + y) = f(x) + f(y),$$

can, in the symbolism of $A(R)$, be written in the form

$$f\Sigma = \Sigma f\mathbf{1}fI.$$

By analogy with differential equations, this could be called a functional equation of order 1, since the order of the highest operation that appears is 1. Examples with 2-place functions are also immediate. The equations of commutativity and associativity, expressed classically in the respective forms (8),

$$F(x, y) = F(y, x), \qquad F\big(F(x, y), z\big) = F\big(x, F(y, z)\big),$$

is briefly discussed on the same page. These two types of substitution are also considered by Menger on p. 461ff. of his paper [8], and distinguished there, as "intersection substitution" and "product substitution", respectively. Product substitution, and even more general types of substitution as well, can also be defined in $A(S)$ without the use of "variables".

(8) Cf. also Menger ([7], p. 302; [8], p. 462), where the associative laws of addition and multiplication are expressed, in terms of product substitution (cf. footnote 7), essentially in the form $F[F, j] = F[j, F]$.

take on the respective forms,

$$F = FI2K, \quad FF = F1FI.$$

The functional equation of commutativity thus has order 2, while the superficially more complicated functional equation of associativity has the lower order 1.

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A SUFFICIENT CONDITION FOR INDEPENDENCE

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The present paper originated in an attempt to decide whether the well known notion of independence of continuous functions comes under the general scheme of such notions given by E. Marczewski (1). Our theorem gives a positive answer to this and to a class of similar questions.

By an *algebra* we mean what is also called a general algebraic system, i. e. a set A together with some A-valued functions of finitely many variables defined over A (also called *operations*). The notion of independence in an algebra is taken from the paper of E. Marczewski and it is assumed that the reader is acquainted with it.

THEOREM. *Let A be an arbitrary set with at least two elements, $\bar{\bar{A}} \geqslant 2$. Let $\boldsymbol{J} \subset 2^A$ be a hereditary family of finite subsets of A and let $S = \{x : \{x\} \notin \boldsymbol{J}\}$. Then the condition $\bar{\bar{S}} \geqslant \bar{\bar{\boldsymbol{J}}}$ implies the existence of an algebra with the set of elements A in which $\boldsymbol{J}$ is the class of exactly all finite independent subsets.*

If we wish $\boldsymbol{J}$ to be the family of all finite independent subsets of an algebra, then clearly every element of S has to form a dependent set, i. e. it has to be self-dependent. Thus our theorem says that if $\boldsymbol{J}$ allows for sufficiently many self-dependent elements, then it can be the family of all finite independent subsets in a suitable algebra over A.

If $\bar{\bar{S}} \geqslant \bar{\bar{\boldsymbol{J}}}$ does not hold, then there may be no algebra over A in which $\boldsymbol{J}$ is the set of exactly all finite independent subsets. There is no such algebra if $A = \{a, b, c\}$ and $\boldsymbol{J} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ ($\emptyset$ denotes the empty set). Assuming the contrary, we have from the self-dependence of c that there is an operation $f(x)$ such that $f(x) \neq x$ holds for some $x \in A$. Then it follows from the independence of a that $f(a) \neq a$. Since

(1) E. Marczewski, *A general scheme of the notions of independence in mathematics*, Bulletin de l'Académie Polonaise des Sciences, Série des Sciences Mathématiques, Astronomiques et Physiques, 6 (1958), p. 731-736.