où k est un des indices dans (4′); en outre, il faut choisir convenablement la valeur de $\sqrt[p_k]{r^2}$; en même temps, le second membre de (11) est défini par la formule

$$\chi_k(\zeta) = f\Big(\big(\Omega_k^{-1}(\zeta)\big)^{m_k}, \Psi_k\big(\Omega(\zeta)\big)\Big). \tag{12}$$

En particulier, si nous nous limitons aux r positifs et aux x, y réels remplissant les conditions (2′) et (2″), nous voyons que dans les formules (11) n'interviennent que les indices k pour lesquels les fonctions (12) ne s'annulent pas identiquement, car le premier membre de (11) est positif à l'exception du point (0, 0). Chacune de ces fonctions, comme série entière définie dans le voisinage du point zéro et s'annulant en ce point, est bornée en valeur absolue inférieurement par le produit d'une puissance naturelle quelconque de $|\zeta|$ et d'une constante. Étant donné que le nombre de ces fonctions est fini, il existe donc une évaluation analogue par un produit $C|\zeta|^s$, le même pour toutes ces fonctions. En conséquence, pour tout couple de valeurs x, y remplissant les conditions (2′) et (2″), en particulier pour celui pour lequel le minimum de (0) est atteint, nous avons en vertu de (11)

$$f(x, y) \geqslant Cr^{2s/p}, \quad \text{où} \quad p = \min_k p_k.$$

Il en résulte immédiatement l'inégalité annoncée (0).

Remarque 1. L'hypothèse d'analyticité de la fonction $f(x, y)$ est essentielle en ce sens que, même pour une fonction de classe C^∞, le théorème n'est plus vrai, comme l'indique l'exemple $f(x, y) = e^{-1/(x^2+y^2)}$ pour $x^2+y^2 > 0$ et $f(0, 0) = 0$.

Remarque 2. La généralisation au cas des fonctions de plusieurs variables est possible.

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ON DETERMINING BOUNDED SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF ORDER n

BY

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In technical problems we very often deal with differential equations of the form

$$y^{(n)} + \psi_{n-1}(t)y^{(n-1)} + \ldots + \psi_0(t)y = g(t), \tag{1}$$

where the functions $\psi_i(t)$, $i = 1, 2, \ldots, n-1$, and $g(t)$ are given and bounded for $t \in (-\infty, \infty)$.

It is very important to prove the existence of a bounded solution of this equation and to give a method of determining the solutions with an arbitrarily prescribed accuracy. In the case of differential equations of 2nd order the problem has been solved by A. E. Gelman [1] with the help of the method of small parameter.

Here we will give similar results for certain cases of equation (1).

For this purpose the solution of equation (1) will be expressed in the form of a series the terms of which are integrals of a certain sequence of differential equations with constant coefficients.

The estimations of the remainder of this series will also be given, which will enable us to estimate the accuracy of the approximate solution of equation (1).

LEMMA 1. *For an arbitrary sequence of numbers $r_i \neq 0$, $i = 1, 2, \ldots, n$, such that $r_i \neq r_k$ for $i \neq k$ the following relation holds:*

$$\sum_{i=1}^{n} (-1)^{n+i} \frac{r_i^p}{d_i} = \begin{cases} 0 & \text{for} \quad 0 \leqslant p \leqslant n-2, \\ 1 & \text{for} \quad p = n-1, \end{cases}$$

[1] А. Е. Гельман, *О периодических, квазипериодических и ограниченных решениях одного класса линейных дифференциальных уравнений*, Известия Ленинградского Электротехнического Института, 1958, выпуск 35, p. 231-238.

where

$$d_i = \begin{cases} \prod_{k=2}^{n}(r_k - r_1) & \text{for} \quad i = 1, \\ \prod_{k=i+1}^{n}(r_k - r_i)\prod_{k=1}^{i-1}(r_i - r_k) & \text{for} \quad 2 \leqslant i \leqslant n-1, \\ \prod_{k=1}^{n-1}(r_n - r_k) & \text{for} \quad i = n. \end{cases}$$

Proof. Let us consider Vandermond's determinant of order n,

$$(2) \qquad W = \begin{vmatrix} 1, \ldots, 1, \ldots, 1 \\ r_1, \ldots, r_i, \ldots, r_n \\ r_1^2, \ldots, r_i^2, \ldots, r_n^2 \\ \vdots \\ r_1^{n-1}, \ldots, r_i^{n-1}, \ldots, r_n^{n-1} \end{vmatrix} = \prod_{S=2}^{n}\prod_{k=1}^{S-1}(r_S - r_k),$$

and the expression

$$(3) \qquad Q = \sum_{i=1}^{n}(-1)^{n+i} r_i^p W_i,$$

where W_i are minors of determinant (2) corresponding to the elements of the n-th row.

Since

$$(4) \qquad Q = \begin{cases} 0 & \text{for} \quad 0 \leqslant p \leqslant n-2, \\ W & \text{for} \quad p = n-1 \end{cases}$$

and

$$(5) \qquad \frac{W_i}{W} = \frac{1}{d_i},$$

we have

$$\frac{Q}{W} = \sum_{i=1}^{n}(-1)^{n+i} r_i^p \frac{W_i}{W} = \sum_{i=1}^{n}(-1)^{n+i}\frac{r_i^p}{d_i} = \begin{cases} 0 \text{ for } 0 \leqslant p \leqslant n-2, \\ 1 \text{ for } p = n-1. \end{cases}$$

LEMMA 2 ([2]). *If $g(t)$ is a function defined and bounded for $t \in (-\infty, \infty)$, i. e. $|g(t)| \leqslant \bar{g}$, then*

$$f(t) = e^{rt}\int_{t} g(x) e^{-rx} dx,$$

where $\int_t$ denotes $\int_t^{\infty}$ when $\alpha = \mathrm{Re}(r) > 0$ or $\int_{-\infty}^{t}$ when $\alpha = \mathrm{Re}(r) < 0$, is bounded, and the following inequality holds:

$$|f(t)| \leqslant \frac{\bar{g}}{|\alpha|}.$$

([2]) The proof can be found in ([1]).

Let us consider the differential equation

$$(6) \qquad y^{(n)} + A_{n-1}y^{(n-1)} + \ldots + A_1 y' + A_0 y = f(t).$$

If the characteristic equation

$$(7) \qquad r^n + A_{n-1}r^{n-1} + \ldots + A_1 r + A_0 = 0$$

of differential equation (6) has single roots r_i, then the general solution of the homogeneous equation can be written in the form

$$(8) \qquad y(t) = \sum_{i=1}^{n} C_i e^{r_i t}.$$

Then the general solution of equation (6) can be found by the method of the variation of constants. Therefore the following conditions must be satisfied:

$$(9) \qquad \begin{cases} \sum_{i=1}^{n} C_i' r_i^k e^{r_i t} = 0 \quad \text{for} \quad k = 0, 1, \ldots, n-2, \\ \sum_{i=1}^{n} C_i' r_i^{n-1} e^{r_i t} = f(t). \end{cases}$$

Solving system (9) we obtain

$$(10) \qquad C_i = (-1)^{n+i}\frac{1}{d_i}\int f(t) e^{-r_i t} dt.$$

Thus the general solution of equation (6) can be written as follows:

$$(11) \qquad y(t) = \sum_{i=1}^{n}(-1)^{n+i}\frac{e^{r_i t}}{d_i}\int f(t) e^{-r_i t} dt.$$

If we put into formula (11) definite integrals of the form $\int_{a}^{t}$ instead of indefinite ones, then (11) will represent a particular solution of equation (6).

After these remarks we can state

THEOREM 1. *If the characteristic equation (7) of the differential equation (6) has single roots r_i such that $\alpha_i = \mathrm{Re}(r_i) \neq 0$ and $f(t)$ is bounded for $t \in (-\infty, \infty)$, then there exists a solution $y = y(t)$ of equation (6)* *satisfying*

$$(12)\qquad |y^{(k)}(t)| \leqslant \bar{f}\sum_{i=1}^{n}\left|\frac{r_i^k}{d_i a_i}\right| \quad \text{for} \quad k = 0, 1, \ldots, n-1,$$

where $\bar{f} = \sup_t |f(t)|$.

Proof. Let us consider the solution

$$(13)\qquad y(t) = \sum_{i=1}^{n}(-1)^{n+i}\frac{e^{r_i t}}{d_i}\operatorname{sgn}(-a_i)\int_t f(x)^{-r_i x}dx,$$

of the differential equation (6).

By Lemma 2 we have

$$(14)\qquad |y(t)| \leqslant \sum_{i=1}^{n}\frac{1}{|d_i|}\left|e^{r_i t}\int_t f(x)e^{-r_i x}dx\right| \leqslant \sum_{i=1}^{n}\frac{\bar{f}}{|d_i a_i|} = \bar{f}\sum_{i=1}^{n}\left|\frac{r_i^0}{d_i a_i}\right|.$$

Differentiating (13) k-times and using Lemma 1 we obtain

$$(15)\qquad y^{(k)}(t) = \sum_{i=1}^{n}(-1)^{n+i}\frac{r_i^k}{d_i}e^{r_i t}\operatorname{sgn}(-a_i)\int_t f(x)e^{-r_i x}dx$$

for $k \leqslant n-1$.

By Lemma 2 we have

$$(16)\qquad |y^{(k)}(t)| \leqslant \sum_{i=1}^{n}\left|\frac{r_i^k}{d_i}\right|\left|e^{r_i t}\int_t f(x)e^{-r_i x}dx\right| \leqslant \bar{f}\sum_{i=1}^{n}\left|\frac{r_i^k}{d_i a_i}\right|.$$

Let us now consider the differential equation

$$(17)\qquad y^{(n)} + [A_{n-1} + \lambda\varphi_{n-1}(t)]y^{(n-1)} + \ldots + [A_0 + \lambda\varphi_0(t)]y = f(t).$$

Let

$$(18)\qquad y(t) = \sum_{k=0}^{\infty}\lambda^k y_k(t)$$

be a certain solution of the differential equation (17). If we substitute series (18) in equation (17) we shall see that the functions $y_k(t)$ must fulfil the following equations:

$$(19)\qquad \begin{aligned} &y_0^{(n)} + A_{n-1}y_0^{(n-1)} + \ldots + A_0 y_0 = f(t),\\ &y_k^{(n)} + A_{n-1}y_k^{(n-1)} + \ldots + A_0 y_k = -\varphi_{n-1}(t)y_{k-1}^{(n-1)} - \ldots - \varphi_0(t)y_{k-1}.\end{aligned}$$

LEMMA 3. *If $f(t)$ and $\varphi_k(t)$ for $k = 0, 1, \ldots, n-1$ are bounded for $t \in (-\infty, \infty)$, and $y_k(t)$ are bounded solutions of equation (19), then*

$$(20)\qquad \bar{f}_k \leqslant S\bar{f}_{k-1},$$

where

$$(21)\qquad S = \sum_{j=0}^{n-1}\sum_{i=1}^{n}\bar{\varphi}_j\left|\frac{r_i^j}{d_i a_i}\right|,$$

whereas

$$\bar{\varphi}_j = \sup_t |\varphi_j(t)| \quad \text{for} \quad j = 0, 1, \ldots, n-1,$$

and

$$\bar{f}_k = \sup_t |y_k^{(n)} + A_{n-1}y_k^{(n-1)} + \ldots + A_0 y_k| = \sup_t |-\varphi_{n-1}(t)y_{k-1}^{(n-1)} - \ldots - \varphi_0(t)y_{k-1}|,$$

for $k = 1, 2, \ldots$ *and* $\bar{f}_0 = \bar{f}$.

Clearly by theorem 1 we have

$$\begin{aligned}\bar{f}_k &= \sup_t |-\varphi_{n-1}(t)y_{k-1}^{(n-1)} - \ldots - \varphi_0(t)y_{k-1}|\\ &\leqslant \bar{\varphi}_{n-1}\bar{f}_{k-1}\sum_{i=1}^{n}\left|\frac{r_i^{n-1}}{d_i a_i}\right| + \ldots + \bar{\varphi}_0\bar{f}_{k-1}\sum_{i=1}^{n}\left|\frac{r_i^0}{d_i a_i}\right|\\ &= \bar{f}_{k-1}\sum_{j=0}^{n-1}\sum_{i=1}^{n}\bar{\varphi}_j\left|\frac{r_i^j}{d_i a_i}\right| = \bar{f}_{k-1}\cdot S.\end{aligned}$$

THEOREM 2. *Suppose we are given the differential equation*

$$(22)\qquad y^{(n)} + [A_{n-1} + \varphi_{n-1}(t)]y^{(n-1)} + \ldots + [A_0 + \varphi_0(t)]y = f(t),$$

where $\varphi_i(t)$ for $i = 0, 1, \ldots, n-1$, and $f(t)$ are bounded for $t \in (-\infty, \infty)$.

Let the characteristic equation (7) have single roots r_i such that $a_i = \operatorname{Re}(r_i) \neq 0$.

If

$$S = \sum_{j=0}^{n-1}\sum_{i=1}^{n}\bar{\varphi}_j\left|\frac{r_i^j}{d_i a_i}\right| < 1,$$

then equation (22) has a bounded solution for $t \in (-\infty, \infty)$ given by the series

$$(23)\qquad y(t) = \sum_{k=0}^{\infty}y_k(t),$$

where $y_k(t)$ are bounded solutions of differential equations (19) and series (23) converges uniformly in the interval $-\infty < t < \infty$.

Moreover, the remainder

(24) $$R_k = y_{k+1}+y_{k+2}+\dots$$

of series (23) *can be estimated either by*

(25) $$|R_k| \leqslant \frac{\sum_{i=1}^{n}\left|\frac{1}{d_i a_i}\right|}{1-S}\sup_t|-\varphi_{n-1}(t)y_k^{(n-1)}-\dots-\varphi_0(t)y_k|$$

or by

(26) $$|R_k| \leqslant \bar{f}\frac{S^{k+1}}{1-S}\sum_{i=1}^{n}\left|\frac{1}{d_i a_i}\right|.$$

Proof. If $S<1$, then

(27) $$\sum_{k=0}^{\infty}\bar{f}_k \leqslant \bar{f}_0\cdot\frac{1}{1-S}.$$

This follows from Lemma 3, because

(28) $$\bar{f}_k \leqslant S\bar{f}_{k-1} \leqslant \dots \leqslant S^k\bar{f}_0.$$

From the convergence of series (27) and from theorem 1 follows the uniform convergence of the series

(29) $$\sum_{k=0}^{\infty}y_k^{(p)}(t) \quad \text{for} \quad p = 0, 1, \dots, n-1.$$

For we have

$$\sum_{k=0}^{\infty}|y_k^{(p)}(t)| \leqslant \sum_{k=0}^{\infty}\left(\bar{f}_k\sum_{i=1}^{n}\left|\frac{r_i^p}{d_i a_i}\right|\right) = \left(\sum_{i=1}^{n}\left|\frac{r_i^p}{d_i a_i}\right|\right)\sum_{k=0}^{\infty}\bar{f}_k \leqslant \frac{\bar{f}_0}{1-S}\sum_{i=1}^{n}\left|\frac{r_i^p}{d_i a_i}\right|.$$

Equations (19) and the uniform convergence of series (29) imply the uniform convergence of series

(30) $$\sum_{k=0}^{\infty}y_k^{(n)}(t).$$

We shall now show that (23) is a bounded integral of the differential equation (22). From equations (19) it follows that

$$\sum_{k=0}^{\infty}[y_k^{(n)}+A_{n-1}y_k^{(n-1)}+\dots+A_0y_k]+\sum_{k=0}^{\infty}[\varphi_{n-1}(t)y_k^{(n-1)}+\dots+\varphi_0(t)y_k] = f(t),$$

whence, because of the uniform convergence of series (29) and (30), we have

$$\Big(\sum_{k=0}^{\infty}y_k\Big)^{(n)}+[A_{n-1}+\varphi_{n-1}(t)]\Big(\sum_{k=0}^{\infty}y_k\Big)^{(n-1)}+\dots+[A_0+\varphi_0(t)]\Big(\sum_{k=0}^{\infty}y_k\Big) = f(t),$$

which shows that the function given by formula (23) is the solution of the differential equation (22).

Since $y_k(t)$ are bounded and, moreover, series (23) is uniformly convergent, the function given by this series is bounded. Let us now estimate the remainder R_k:

$$|R_k| \leqslant |y_{k+1}|+|y_{k+2}|+\dots \leqslant \left(\sum_{i=1}^{n}\left|\frac{1}{d_i a_i}\right|\right)(\bar{f}_{k+1}+\bar{f}_{k+2}+\dots)$$

$$= \bar{f}_{k+1}\sum_{i=1}^{n}\left|\frac{1}{d_i a_i}\right|\left(1+\frac{\bar{f}_{k+2}}{\bar{f}_{k+1}}+\frac{\bar{f}_{k+3}}{\bar{f}_{k+1}}+\dots\right)$$

$$= \bar{f}_{k+1}\sum_{i=1}^{n}\left|\frac{1}{d_i a_i}\right|(1+S^1+S^2+\dots) = \frac{\bar{f}_{k+1}}{1-S}\sum_{i=1}^{n}\left|\frac{1}{d_i a_i}\right|.$$

Since

$$\bar{f}_{k+1} = \sup_t|\varphi_{n+1}(t)y_k^{(n-1)}+\dots+\varphi_0(t)y_k|,$$

we have

$$|R_k| \leqslant \frac{1}{1-S}\sum_{i=1}^{n}\left|\frac{1}{d_i a_i}\right|\sup_t|\varphi_{n-1}(t)y_k^{(n-1)}+\dots+\varphi_0(t)y_k|.$$

This proves inequality (25). Inequality (26) results from (25) and (28).

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