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ON CHANGE OF VARIABLE IN THE DENJOY-PERRON INTEGRAL (I)

BY

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This paper contains some theorems concerning change of variable in the Denjoy-Perron integral. These theorems are a generalization of Karták's results in this direction [1]. In the sequel we shall use the notation and terminology of [2]. We begin by proving the following

LEMMA 1. *Let F be a function defined on an interval $[a, b]$ and derivable at each point of a set E such that $|F[E]| = 0$. Then $F'(x) = 0$ almost everywhere on E.*

Proof. Let A be the set of derivability of F. On account of Theorem 4.2, p. 112, [2], A is measurable and therefore $A = \sum_{n=1}^{\infty} A_n + H$ where each A_n is closed and H is of measure zero. In view of [2], Theorem 10.5, p. 235, we may assume that F is AC_* on each A_n. Let F_n be, for each positive integer n, the function which coincides with F at the points A_n and is linear in the intervals continuous to A_n. Since $|F_n[E \cdot A_n]| = 0$ and since F_n is AC on the smallest interval containing A_n, we have $F_n'(x) = 0$ almost everywhere on $E \cdot A_n$. Hence, we obtain $F'(x) = 0$ almost everywhere on $E \cdot A_n$, for $n = 1, 2, \ldots$ It is easy to see that this completes the proof.

THEOREM 1. *Let f be D_*-integrable on $[a, b]$ and φ be derivable almost everywhere on $[c, d]$ such that $\varphi\big[[c, d]\big] \subset [a, b]$. If the function $G = F(\varphi)$, where F is an indefinite D_*-integral of f on $[a, b]$, is ACG_* on $[c, d]$, then for every t, $c < t \leqslant d$, the function $f(\varphi)\varphi'$ is D_*-integrable on $[c, t]$ and*

$$(1) \qquad (D_*) \int_{\varphi(c)}^{\varphi(t)} f(x)\,dx = (D_*) \int_c^t f\big(\varphi(t)\big)\varphi'(t)\,dt \text{ [1]}.$$

[1] Theorem 1 remains true for the Denjoy-Khintchine integral when we replace in it the ordinary derivability of φ by approximate derivability and require G to be ACG on $[c, d]$.

Proof. It is easy to see that it is enough to prove that

$$G'(t) = f(\varphi(t))\varphi'(t) \tag{2}$$

almost everywhere on $[c, d]$. Let T be the set of points at which φ and G are derivable; moreover, let X be the set of x at which F is derivable and $F'(x) = f(x)$. We clearly have $|T_1| = |X_1| = 0$, where $T_1 = [c, d]-T$ and $X_1 = [a, b]-X$. It is easy to see that (2) holds at each t belonging to $T\cdot T_2$, where $T_2 = \varphi^{-1}[X]$. Therefore it is enough to show that (2) is also satisfied almost everywhere on $T_3 = T-T_2$. For this purpose, we shall prove that

$$G'(t) = \varphi'(t) = 0 \tag{3}$$

almost everywhere on T_3. Since $\varphi[T_3]\subset X_1$ and F fulfils condition (N), we have $|G[T_3]| = |\varphi[T_3]| = 0$. Now it is enough to use Lemma 1 to obtain the required result.

LEMMA 2. *Let a function G satisfy the following conditions:*
(a) *G is continuous and fulfils condition* (N) *on* $[c, d]$,
(b) *there exists a function g which is D_*-integrable on $[c, d]$ and such that $G'(t) = g(t)$ at each point t at which the derivative $G'(t)$ exists, except perhaps those of a set of measure zero.*

Then the function G is ACG_* on $[c, d]$.

Proof. This lemma follows at once from [3], Theorem 8, p. 145.

THEOREM 2. *Let f be a function which is D_*-integrable on $[a, b]$ and let F be an indefinite D_*-integral of f on $[a, b]$. Further, let φ be a function which is continuous, derivable almost everywhere, fulfils condition* (N) *on $[c, d]$ and is such that $\varphi[[c, d]]\subset[a, b]$. Then the following conditions are equivalent:*
(i) *$G = F(\varphi)$ is ACG_* on $[c, d]$,*
(ii) *$f(\varphi)\varphi'$ is D_*-integrable on $[c, d]$ and* (1) *holds,*
(iii) *$f(\varphi)\varphi'$ is D_*-integrable on $[c, d]$* [2].

Proof. Since, on account of Theorem 1, (i) implies (ii) and (ii) clearly implies (iii), it is enough to show that (iii) implies (i). For this purpose, in view of Lemma 2, it is enough to prove that (2) holds almost everywhere on the set of derivability of G. But this can be shown by means of the same argument as in the proof of Theorem 1.

A function F defined on an interval $[a, b]$ will be termed L_* with a positive constant M on a set $E\subset[a, b]$ if $|F(x_2)-F(x_1)|\leqslant M|x_2-x_1|$ whenever at least one of the points x_1, x_2 belongs to E.

A function F defined on an interval $[a, b]$ will be termed LG_* on this interval if $[a, b]$ is expressible as the sum of a finite or denumerable sequence of sets on each of which F is L_*.

[2] Theorem 2 remains true for the Denjoy-Khintchine integral when we change its formulation in a suitable way, see [1] on p. 99.

Now we shall prove

THEOREM 3. *In order that a function F be LG_* on $[a, b]$, it is necessary and sufficient that the Dini derivates of F be finite at each point of the interval $[a, b]$.*

Proof. The necessity of this condition is trivial; therefore we have only to prove it sufficient. Let the Dini derivates of F be finite at each point of $[a, b]$. Since then F is obviously continuous on $[a, b]$, it follows that for each x belonging to $[a, b]$ there exists a positive number $M(x)$ such that $|F(t)-F(x)|\leqslant M(x)|t-x|$ holds for each $t\in[a, b]$. Let us define sets E_n in the following way:

$$E_n = \{x\colon M(x)\leqslant n, a\leqslant x\leqslant b\}.$$

We see that $[a, b] = \sum_{n=1}^{\infty} E_n$ and that F is L_* on each E_n. This completes the proof.

Now we shall prove two theorems concerning superposition of functions ACG_*.

THEOREM 4. *Let F be LG_* on an interval $[a, b]$. Then, for every function φ which is ACG_* on an interval $[c, d]$ and such that $\varphi[[c, d]]\subset[a, b]$, the function $G = F(\varphi)$ is ACG_* on $[c, d]$.*

Proof. We can express the interval $[a, b]$ as the sum of a sequence of sets E_n on each of which F is L_* with a constant M_n. Let us put $T_n = \varphi^{-1}[E_n]$. Since φ is ACG_* we have $T_n = \sum_{k=1}^{\infty} T_{n,k}$ and φ is AC_* on each $T_{n,k}$. Further, since G is clearly continuous, it is enough to prove that G is AC_* on each $T_{n,k}$. For this purpose, let $\{I_p\}$ be any finite sequence of non-overlapping intervals whose end-points a_p, b_p belong to fixed $T_{n,k}$. Now, for every interval $[a'_p, b'_p]\subset I_p$, we have

$$\begin{aligned}|G(b'_p)-G(a_p)| &\leqslant M_n|\varphi(b'_p)-\varphi(a_p)|,\\ |G(a_p)-G(a'_p)| &\leqslant M_n|\varphi(a_p)-\varphi(a'_p)|.\end{aligned} \tag{4}$$

By (4) we obtain $O(G; I_p)\leqslant 2M_n O(\varphi; I_p)$. It is evident that this completes the proof.

Let us remark that, on account of the preceding theorem, every function LG_* is also ACG_*. In order to establish the converse of Theorem 4 (or even a slightly stronger assertion), we shall prove two lemmas.

LEMMA 3. *Let F be a function defined on an interval $[a, b]$ and let at least one of its Dini derivates be infinite at $x_0\in[a, b]$. Then there exists a sequence $\{x_n\}$, $x_n\in[a, b]$ such that $\sum_{n=1}^{\infty}|x_n-x_0| < +\infty$ but $\sum_{n=1}^{\infty}|F(x_n)-F(x_0)| = +\infty$.*

Proof. It is easy to see that there exists a sequence $\{\bar{x}_n\}$ such that $\bar{x}_n \in [a, b]$, $\lim_n \bar{x}_n = x_0$, $\bar{x}_n \neq x_0$ and

$$(5) \qquad |F(\bar{x}_n) - F(x_0)| \geqslant n^2 |\bar{x}_n - x_0| \quad \text{for} \quad n = 1, 2, \ldots$$

Let $\{k_i\}$ be an increasing sequence of natural numbers such that $|\bar{x}_{k_i} - x_0| \leqslant 1/i^2$ for $i = 1, 2, \ldots$ There exists a sequence $\{l_i\}$ of natural numbers such that

$$(6) \qquad \frac{1}{i^2} \leqslant l_i |\bar{x}_{k_i} - x_0| < \frac{2}{i^2} \quad \text{for} \quad i = 1, 2, \ldots$$

Let us define the sequence $\{x_n\}$ as follows:

$$x_n = \bar{x}_{k_i} \quad \text{for} \quad 1 + s_i \leqslant n \leqslant s_{i+1},$$

where $s_i = \sum_{k=0}^{i-1} l_k$ (provided that $l_0 = 0$). We shall show that $\{x_n\}$ is the required sequence. In fact, by (6) we have

$$\sum_{n=1}^{\infty} |x_n - x_0| \leqslant \sum_{i=1}^{\infty} \frac{2}{i^2} < +\infty$$

and by (5) and (6) we have

$$\sum_{n=1+s_i}^{s_{i+1}} |F(x_n) - F(x_0)| \geqslant l_i k_i^2 |\bar{x}_{k_i} - x_0| \geqslant 1.$$

LEMMA 4. *If a series* $\sum_{n=1}^{\infty} c_n$ *with non-negative terms is divergent, then there exists a sequence* $\{N_k\}$ *of disjoint, denumerable subsets of natural numbers* N *such that* $\sum_{k=1}^{\infty} N_k = N$ *and* $\sum_{n \in N_k} c_n = +\infty$ *for* $k = 1, 2, \ldots$

Proof. It is evident that there exists an increasing sequence $\{p_i\}$ of natural numbers such that $\sum_{k=1+p_{i-1}}^{p_i} c_k \geqslant 1$ for $1, 2, \ldots$ (provided that $p_0 = 0$). Let $N = \sum_{k=1}^{\infty} B_k$ where B_k are disjoint and denumerable. Then the sequence $\{N_k\}$ where $N_k = \sum_{i \in B_k} \{n: 1 + p_{i-1} \leqslant n \leqslant p_i;\ n \in N\}$ is the required one.

THEOREM 5. *Suppose that a function* F *defined on an interval* $[a, b]$ *has the following property: for every function* φ *which is* AC *on an interval* $[c, d]$ *and such that* $\varphi[[c, d]] \subset [a, b]$, *the superposition* $G = F(\varphi)$ *is* ACG_* *on* $[c, d]$; *then the function* F *is* LG_* *on* $[a, b]$.

Proof. Suppose, to the contrary, that F is not LG_* on $[a, b]$. Then, on account of Theorem 3 and Lemma 3, there exists a sequence $\{x_n\}$ and a point x_0 such that $x_n \in [a, b]$, $x_0 \in [a, b]$ and $\sum_{n=1}^{\infty} |x_n - x_0| < +\infty$ but $\sum_{n=1}^{\infty} |F(x_n) - F(x_0)| = +\infty$. Let $\{I_k\}$ be the sequence of all different rational subintervals of $(0, 1)$. It is easy to see that for each natural k there exists a sequence $\{w_i^{(k)}\}$ of rational numbers from $(0, 1)$ such that $w_i^{(k)} \in I_k$ for $i = 1, 2, \ldots$ and $w_i^{(k)} \neq w_s^{(n)}$ for $(i, k) \neq (s, n)$. Further, let $\{N_k\}$ be the sequence given by Lemma 4 applied to the series $\sum_{n=1}^{\infty} |F(x_n) - F(x_0)|$. Let us put $a_n = w_{f_k(n)}^{(k)}$ for $n \in N_k$ where, for each natural k, the function $f_k(n)$ establishes a one-to-one correspondence between N_k and the set of all natural numbers. By the definition of $\{a_n\}$ the following proposition is true:

(7) *For every interval* $I \subset (0, 1)$ $\sum_{n=1}^{\infty} |F(x_{k_n}) - F(x_0)| = +\infty$, *where the sequence* $\{k_n\}$ *is such that* $\{a_{k_n}\}$ *is a subsequence of* $\{a_n\}$ *whose all terms belong to* I.

Let C be an arbitrary, perfect, non-dense set whose bounds are c and d. Now, since there exists a one-to-one order preserving correspondence between the set of all rational numbers of $(0, 1)$ and the set of all intervals contiguous to C, by (7) we infer that there exists a sequence $\{P_n\}$ of different intervals contiguous to C such that

(8) *For every portion* K *of* C *we have* $\sum_{n=1}^{\infty} |F(x_{k_n}) - F(x_0)| = +\infty$, *where* $\{P_{k_n}\}$ *is a subsequence of* $\{P_n\}$ *consisting of all intervals contiguous to* $\bar{K}$ *which are terms of the sequence* $\{P_n\}$.

We may clearly suppose that F is continuous. Let us define a function φ as follows:

$$\varphi(t) = \begin{cases} x_n \text{ at } t \text{ equal to the centre of } P_n, \\ x_0 \text{ at } t \text{ belonging to } [c, d] - \sum_{n=1}^{\infty} \operatorname{int}(P_n), \\ \text{linear in } \operatorname{int}(P_n). \end{cases}$$

Since the series $\sum_{n=1}^{\infty} |x_n - x_0|$ is convergent, φ is AC on $[c, d]$. Further, since $\varphi[[c, d]] \subset [a, b]$, the function $G = F(\varphi)$ should be ACG_* on $[c, d]$ and in particular, on C. By (8) and Theorem 9.1 of [2], p. 233, since $O(G; P_n) \geqslant |F(x_n) - F(x_0)|$, for $n = 1, 2, \ldots$, we find that G is not ACG_* on C. This completes the proof.

By Theorems 2, 4, and 5 we obtain the following

COROLLARY. *Let f be D_*-integrable on $[a, b]$. In order that for every function φ which is ACG_* on an interval $[c, d]$ and such that $\varphi[[c, d]] \subset \subset [a, b]$, the function $f(\varphi)\varphi'$ be D_*-integrable on $[c, d]$ and (1) hold, it is necessary and sufficient that an indefinite D_*-integral of f on $[a, b]$ be the function LG_*.*

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ON A RECURRENCE RELATION

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In the present paper we shall consider the recurrence relation

(1) $$x_{n+1}+x_n = b_n,$$

in which the sequence b_n is given and x_n is to be determined. Of course, the sequence x_n can be found in infinitely many ways. We may choose arbitrarily the term x_0 and then the whole sequence x_n will be uniquely determined by relation (1). However, we shall prove that under suitable assumptions there exists only one sequence x_n fulfilling relation (1) and an additional condition. In what follows all occurring sequences are supposed to be real.

For an arbitrary sequence a_n we denote (as usual) by Δa_n the difference

$$\Delta a_n \overset{\text{df}}{=} a_{n+1}-a_n.$$

Further, we define the successive iterates of the operator Δ by the relations

$$\Delta^0 a_n \overset{\text{df}}{=} a_n, \quad \Delta^{\nu+1} a_n \overset{\text{df}}{=} \Delta\Delta^\nu a_n, \quad \nu = 0, 1, 2, \ldots$$

Of course, the operator Δ^1 coincides with the operator Δ.

The purpose of the present note is to prove the following

THEOREM. *If (for a certain $r \geqslant 1$) the terms $\Delta^{r+1} b_n$ have a constant sign, and for a certain positive integer $p \leqslant r$*

(2) $$\lim_{n\to\infty} \Delta^p b_n = 0,$$

then there exists exactly one sequence x_n such that the terms $\Delta^r x_n$ have a constant sign, and relation (1) holds. This sequence is given by the formula

(3) $$x_n = \sum_{\nu=0}^{p-1} \frac{(-1)^\nu}{2^{\nu+1}} \Delta^\nu b_n + \frac{(-1)^p}{2^p} \sum_{\nu=0}^{\infty} (-1)^\nu \Delta^p b_{n+\nu}.$$