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EXAMPLE OF A NON-SEPARABLE B_0-SPACE IN WHICH EVERY BOUNDED SET IS SEPARABLE

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J. Dieudonné has given in [1] an example of a non-complete locally convex linear space which is non-separable but every bounded subset of it is separable, and has asked if this phenomenon may occur in B_0-spaces (i. e. complete locally convex, linear metric spaces)? An example of a complete linear metric space (but not locally convex) which has the above-mentioned property is given by C. Bessaga and S. Rolewicz in [2].

The aim of our note is to give a positive answer to this question. Our proof, as well as the proofs by Dieudonné and Bessaga-Rolewicz, is based on the continuum hypothesis. Our construction makes use of some ideas from paper [2] of Bessaga and Rolewicz.

We introduce, in the class S of all sequences (of the type ω) of positive numbers, the relation $\prec$ as follows:

$(f_n) \prec (g_n)$ iff $f_n < g_n$ for all sufficiently large indices n. We will say that a subclass $S_1 \subset S$ is *limited* when there exists a sequence greater (in the sense of the relation $\prec$) than each element of S_1. Obviously all denumerable subclasses are limited.

LEMMA. *There exists a non-denumerable class S_0 such that each of its non-denumerable subclasses is non-limited.*

Proof. Let us well order all elements of the class S in the transfinite sequence of the type ω_1 (here we make use of the continuum hypothesis):

$$(1) \qquad (f_n^\alpha) \quad \text{where} \quad \alpha < \omega_1.$$

Then we define the other transfinite sequence of the type ω_1 as follows: (g_n^β) is the first term in sequence (1) which is greater (in the sense of $\prec$) than all (f_n^α) for $\alpha < \beta$. It is easy to see that the class S_0 consisting of all terms of the transfinite sequence (g_n^β), $\beta < \omega_1$, has the required property.

Now we will consider the space X of all transfinite sequences (x_β) ($\beta < \omega_1$, x_β are real) for which the pseudonorms

(2) $$\|(x_\beta)\|_n \overset{\mathrm{df}}{=} \sum_{\beta<\omega_1} g_n^\beta \cdot |x_\beta|$$

are finite for all $n = 0, 1, 2, \ldots$; the g_n^β have the same meaning as in the proof of the Lemma.

We shall prove the following

THEOREM. *The space X is a non-separable B_0-space, and every bounded subset of X is separable.*

Proof. The non-separability of X is clear. Let Z be an arbitrary bounded subset of X. Then there exists a sequence of positive numbers (m_n) $(n < \omega)$ such that:

(3) $$Z \subset \{(x_a): \|(x_a)\|_n \leqslant m_n \quad \text{for} \quad n = 0, 1, \ldots\}.$$

Now we will denote by O_k (k is a fixed natural) the set of all ordinals β_0 for which there exists a transfinite sequence $(x_a) \epsilon Z$, such that its β_0-coordinate is greater than $1/k$, i. e., $x_{\beta_0} > 1/k$. From formulas (2) and (3) we get

$$\frac{1}{k} g_n^{\beta_0} < \sum_{\beta<\omega_1} g_n^0 |x_\beta| \leqslant m_n \quad \text{for} \quad n = 0, 1, \ldots,$$

whence for $\beta_0 \epsilon O_k$ we have $(g_n^{\beta_0}) \prec (k \cdot m_n)$, and by the Lemma the set O_k is denumerable. Consequently the union $\bigcup_{k=1}^{\infty} O_k$ is also denumerable and we will denote by γ the smallest ordinal greater than all the terms of $\bigcup_{k=1}^{\infty} O_k$.

It immediately follows from the definitions of the set O_k and the ordinal γ that

(4) if $(x_a) \epsilon Z$, then $x_a = 0$ for any $a \geqslant \gamma$.

Hence the set Z is containded in the separable subspace X_γ of all sequences (x_a) such that $x_a = 0$, for any $a \geqslant \gamma$. Thus Z is separable.

REFERENCES

[1] J. Dieudonné, *Bounded sets in (F)-spaces*, Proceedings of the American Mathematical Society 6 (1955), p. 729-731.

[2] C. Bessaga and S. Rolewicz, *On bounded sets in F-spaces*, Colloquium Mathematicum 9 (1961), p. 89-91.

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SUR UN PROBLÈME, POSÉ PAR C. RYLL-NARDZEWSKI, CONCERNANT LES SÉLECTEURS À MESURE MAXIMUM

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C. Ryll-Nardzewski a posé le problème suivant [3]:

There is a decomposition of the interval $\langle 0, 1\rangle$ into subsets E_a ($a \epsilon A$ and A is an arbitrary set of indices). Let us consider all Lebesgue measurable sets of the form $S = \{p_a\}_{a\epsilon A}$ where $p_a \epsilon E_a$. Let it be that $\mu = \sup_S |S|$ ($|S|$ denotes the measure of S). Is this upper limit attained on a set? How is it in the case $E_a = \{t: 0 \leqslant t \leqslant 1,\ f(t) = a\}$ ($f(t)$ is Lebesgue measurable)?

Ce problème reçoit ici une réponse affirmative dans le cas particulier $E_a = \{t: 0 \leqslant t \leqslant 1,\ f(t) = a\}$, où $f(t)$ est mesurable au sens de Lebesgue. Dans le cas considéré, il existe une désintégration ([2], § 3, th. 1) de la mesure de Lebesgue ν relative à l'application f, soit $\nu = \int \lambda_a d\mu(a)$; et on a $\nu(S) = \int \lambda_a(S \cap f^{-1}(a)) d\mu(a)$ pour tout ensemble S considéré ([1], § 3, théorème 1); on démontre qu'on peut partager $[0, 1]$ en deux parties mesurables disjointes U et V telles que les λ_a soient atomiques sur U et diffuses sur V ([1], § 5, nº 10); et on n'aura à considérer ici que l'ensemble U, puisque, pour tout ensemble S, $S \cap V$ est de mesure nulle. On est donc ramené à démontrer, en changeant quelque peu les notations, le

THÉORÈME 1. *Soient Y et Z deux espaces topologiques localement compacts à bases dénombrables d'ensembles ouverts, ν une mesure positive bornée sur Z, φ une application ν-mesurable de Z dans Y, μ l'image de ν par φ, $\nu = \int_Y \lambda_\eta d\mu(\eta)$ une désintégration de ν relative à φ. On suppose λ_η atomique pour tout $\eta \epsilon Y$. Alors parmi les sections de Z pour φ qui sont ν-mesurables, il en existe une au moins dont la mesure est maximale.*

Nous allons démontrer le

THÉORÈME 1'. *Avec les notations du théorème 1, soit A le sous-ensemble de Z qui rencontre chaque classe $\varphi^{-1}(\eta)$ suivant l'ensemble de points de $\varphi^{-1}(\eta)$ dont la mesure pour λ_η est maximale; alors A es ν-mesurable.*