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REMARK ON NON-COMPLEMENTED SUBSPACES OF THE SPACE $m(S)$

BY

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In the sequel $m(S)$ denotes the Banach space of all real-valued bounded functions $f(\cdot)$ defined on an infinite set S, with the norm $\|f\| = \sup_{s \in S} |f(s)|$. Let $m_0(S)$ denote the subspace of $m(S)$ consisting of all such functions $f(\cdot)$ that $\overline{\overline{\{s \in S : |f(s)| > \varepsilon\}}} < \overline{\overline{S}}$ for every $\varepsilon > 0$ ($\overline{\overline{A}}$ denotes the cardinal of a set A).

In this note we shall prove the following result:

THEOREM 1. *There is no projection (= continuous linear idempotent operator) of $m(S)$ onto $m_0(S)$.*

This theorem is a generalization of a well-known result of Philips [6, p. 540] namely that there is no projection of $m = m(N)$ onto its subspace $c_0 = m_0(N)$, where N denotes the set of all integers. The proof of Theorem 1 is based on the following set-theoretical result, due to Sierpiński [7, p. 448]:

(*) Any set S of cardinal $\overline{\overline{S}} \geqslant \aleph_0$ has a family of more than $\overline{\overline{S}}$ subsets, each of power S, such that the intersection of any two of them is of power $< \overline{\overline{S}}$.

1. Proof of Theorem 1. Suppose a contrario that there is a projection P of $m(S)$ onto $m_0(S)$. Then the subspace $p^{-1}(0)$ of $m(S)$ is isomorphic to the quotient space $m(S)/m_0(S)$. We shall prove that it is impossible by showing that

(**) every total set of linear functionals on $m(S)/m_0(S)$ has the power $> \overline{\overline{S}}$.

On the other hand, the family $(\xi_s^*)_{s \in S}$ of all point functionals on $m(S)$ is total over $m(S)$, i. e. if $\xi_s^*(f) = f(s) = 0$, for every s in S, then $f = 0$.

To prove (**) we shall use some properties of the Stone-Čech compactification βS of the set S with discrete topology. We recall (see e. g.

[2, chap. 6] that βS is a topological space whose points are ultrafiltres of subsets of S. The topology in βS is defined by taking the family of all sets

$$F_A = \{p \in \beta S\colon A \in p\} \qquad (A \subset S),$$

as a base for the closed sets. βS is compact and extremally disconected, i. e. the closure of each open set in βS is open. Let $p_s = \{A \subset S\colon s \in A\}$, for $s \in S$. Then p_s is an ultrafilter and the mapping $h\colon s \leftrightarrow p_s$ is a homeomorphism of S into a dense set of βS. This homeomorphism induces the "natural" isometrical isomorphism between $m(S)$ and the space $C(\beta S)$ — of all continuous real-valued functions on βS.

Lemma 1. *Let us write*

$$\gamma S = \{p \in \beta S\colon \text{ if } A \in p, \text{ then } \bar{\bar{A}} = \bar{\bar{S}}\}.$$

Then there is a family $(U_a)_{a \in \mathfrak{a}}$ of subsets of γS such that

(i) *U_a is non-empty and open in the topology induced by βS in γS for $a \in \mathfrak{a}$,*

(ii) *if $a_1 \neq a_2$, then $U_{a_1} \cap U_{a_2} = \emptyset$,*

(iii) *$\bar{\bar{\mathfrak{a}}} > \bar{\bar{S}}$.*

Proof. Let $(A_a)_{a \in \mathfrak{a}}$ with $\bar{\bar{\mathfrak{a}}} > \bar{\bar{S}}$ be a family of subsets of S satisfying the assertion of (*). Let us define

$$U_a = \overline{hA}_a \cap \gamma S \qquad (a \in \mathfrak{a}),$$

where $\overline{hA}_a$ denotes the closure of the image of A_a under the homeomorphism h.

Since βS is extremally disconnected, $\overline{hA}_a$, as a closure of the open set hA_a, is open. Hence U_a is open in the topology induced by βS in γS, for $a \in \mathfrak{a}$, U_a is non-void. Indeed, there is an ultrafilter p_a containing A_a and such that, if $B \in p_a$, then $\bar{\bar{B}} = \bar{\bar{A}}_a = \bar{\bar{S}}$.

To prove (ii) we observe that $\overline{hA} = \{p \in \beta S\colon A \in p\}$, for any subset A of S. Hence if $a_1 \neq a_2$ and $p \in \overline{hA}_{a_1} \cap \overline{hA}_{a_2}$, then $A_{a_1} \in p$, $A_{a_2} \in p$ and $A_{a_1} \cap A_{a_2} \in p$. Since by definition $\bar{\bar{A}}_{a_1} \cap \bar{\bar{A}}_{a_2} < \bar{\bar{S}}$, we have $p \notin \gamma S$. Thus $U_{a_1} \cap U_{a_2} = \Phi$.

Lemma 2. *Let $C(\gamma S)$ denote the space of all continuous real-valued functions on γS. Then*

1. *$C(\gamma S)$ is (isometrically) isomorphic to the space $m(S)/m_0(S)$,*

2. *each total set of linear functionals on $C(\gamma S)$ has the power $> \bar{\bar{S}}$.*

Proof. Let $\hat{f}$ denote the function in $C(\beta S)$ corresponding under the natural isomorphism to the function f in $m(S)$, i. e. $f(s) = \hat{f}(p_s)$, for any s in S. We observe that f is in $m_0(S)$ if and only if $\hat{f}(p) = 0$, for any p in γS. We omit the simple checking of this fact. Let $r\hat{f}$ denote the restriction of $\hat{f}$ to γS, for any f in $m(S)$. Since γS is closed in βS, r maps $C(\beta S)$ onto $C(\gamma S)$ (we apply Tietze's extension theorem [2, p. 18]). On the other hand, the kernel of r consists of all functions vanishing on γS. Thus, by the preceding remark and a theorem of Banach [1, p. 37], $C(\gamma S)$ is isomorphic to $m(S)/m_0(S)$.

Let $(U_a)_{a \in \mathfrak{a}}$ be the family of subsets satisfying the conditions (i)-(iii) of Lemma 1. Since βS and a fortiori γS are zero-dimensional topological spaces, we may assume without loss of generality that

(iv) U_a is a closed open subset in γS, for $a \in \mathfrak{a}$.

Let χ_a denote the characteristic function of U_a $(a \in \mathfrak{a})$. By (iv) χ_a are continuous. Furthermore every countable sequence of different χ_{a_i} $(i = 1, 2, \ldots)$ weakly converges to zero (because, by (ii), $\lim\limits_i \chi_{a_i}(p) = 0$ for any p in γS and $\|\chi_a\| = 1$ for $a \in \mathfrak{a}$). Hence, if ξ^* is a linear functional on $C(\gamma S)$, then the set $\{a \in \mathfrak{a}\colon \xi^*(\chi_a) \neq 0\}$ is countable. Thus, if $(\xi^*_\lambda)_{\lambda \in \Lambda}$ is a family of functionals of power $\bar{\bar{\Lambda}}$ equal to or less than $\bar{\bar{S}}$, then $\overline{\overline{\{a \in \mathfrak{a}\colon \xi^*_\lambda(\chi_a) \neq 0 \text{ for some } \lambda \in \Lambda\}}} \leqslant \bar{\bar{S}}$. Hence, by (iii), there is an a_0 such that $\xi^*_\lambda(\chi_{a_0}) = 0$ for any λ in Λ.

Theorem 1 is an immediate consequence of Lemmas 1 and 2.

2. Corollary. *Let $\aleph_0 \leqslant \aleph_\tau \leqslant \bar{\bar{S}}$ and let $m(S|\aleph_\tau)$ denote the subspace of $m(S)$ of all functions $x(\cdot)$ such that $\overline{\overline{\{s \in S\colon |x(s)| > \varepsilon\}}} < \aleph_\tau$ for any $\varepsilon > 0$. Then there is no projection of $m(S)$ onto $m(S|\aleph_\tau)$.*

Proof. Suppose a contrario that there is a projection P of $m(S)$ onto $m(S|\aleph_\tau)$. Let S_1 be a subset of S with $\bar{\bar{S}}_1 = \aleph_\tau$. Let $P_1 f = f \cdot \chi_{S_1}$ for any f in $m(S|\aleph_\varkappa)$, where χ_{S_1} denotes the characteristic function of S_1. It is easily seen that P_1 is a projection of $m(S|\aleph_\tau)$ onto $m(S_1|\aleph_\tau)$. Hence $P_1 \cdot P$ projects $m(S)$ onto $m(S_1|\aleph_\tau)$. Thus, $m(S)$ having the Hahn-Banach extension property [6, Corollary 7.1.], $m(S_1|\aleph_\tau)$ also has this property. Thus, by [1, p. 94], if $m(S_1|\aleph_\tau)$ is imbedded in any B-space X, then there is a projection of X onto $m(S_1|\aleph_\tau)$. But it contradicts Theorem 1, as $m(S_1|\aleph_\tau) = m_0(S_1)$.

In the particular case of putting in Corollary $S = \langle 0, 1\rangle$ and $\aleph_\tau = \aleph_0$ we obtain the solution of P 309 (see [5]).

3. Remarks. 1° The original proof of Philips cannot be generalized to the case $\bar{\bar{S}} > \aleph_0$. Grothendieck [3, p. 168] has shown that Philips original proof is, in fact, based on the following property (G) which is a feature of the space $m(N)$ but not of the space $c_0 = m_0(N)$:

(G) *If (x^*_n) is a sequence of linear functionals on a B-space X and $\lim\limits_n x^*_n(x) = 0$, for any x in X, then (x^*_n) weakly (with respect to second conjugate space) converges to 0.*

It is easily checked that if $\bar{\bar{S}} \geqslant \aleph_\tau > \aleph_0$, then $m(S|\aleph_2)$ satisfies (G).

2° It follows from Grothendieck's result quoted above that there is no linear continuous mapping of m onto c_0. Moreover, every linear mapping of m into c_0 is compact.

P 351. Let $\aleph_0 \leqslant \aleph_\tau \leqslant \overline{\overline{S}}$. Does there exist a linear mapping of $m(S)$ onto its subspace $m(S|\aleph_\tau)$? (We recall that $m(S|\aleph_\tau)$ denotes the space of all functions $f(\cdot)$ in $m(S)$ such that $\overline{\overline{\{s \in S: |f(s)| > \varepsilon\}}} < \aleph_\tau$, for any $\varepsilon > 0$.)

We note that there is a mapping T of the space $m(S)$ into $m(S|\aleph_1)$ such that

$$\overline{\overline{\bigcup_{f \in m(S)} \{s \in S: (Tf)(s) > \tfrac{1}{2}\}}} > \aleph_0.$$

This may easily be deduced from the following result comunicated to us by prof. C. Ryll-Nardzewski:

Let $\overline{\overline{S}} = \aleph_1$. Then, under the assumption of the continuum hypothesis there exists a family $(\nu_a)_{a \in \mathfrak{a}}$ $(\overline{\overline{\mathfrak{a}}} = \aleph_1)$ of finite-additive set functions with, bounded variation defined on the field of all subsets of the set S such that $\overline{\overline{\{a \in \mathfrak{a}: \nu_a(A) \neq 0\}}} \leqslant \aleph_0$ for any $A \subset S$.

3° It is interesting to compare our result with the following generalization — due to Grünbaum [4] — of the theorem of Sobczyk [8] concerning projections onto c_0.

Let $\overline{\overline{S}} \geqslant \aleph_\tau \geqslant \aleph_0$, let the space $m(S|\aleph_\tau)$ be isomorphically embedded into a B-space X and let the quotient space $X/m(S|\aleph_\tau)$ contain a dense set of cardinality $\leqslant \aleph_\tau$. Then there exists a projection of X onto $m(S|\aleph_\tau)$.

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ON BOUNDED SETS IN F-SPACES

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A subset Z of a metric linear space X is called *bounded* if $\limsup_{t \to 0} \sup_{x \in Z} \varrho(0, tx) = 0$. If every bounded subset of X is compact (i. e. its closure is compact), then X is called a *metrisable Montel space*.

J. Dieudonné [2] proved that every locally convex metrisable Montel space is separable. Using the continuum hypothesis Dieudonné showed in his paper [3] that there is a non-complete locally convex linear metric space which is not separable but every bounded set in this space is separable.

In this paper we prove that Dieudonné's theorem on the separability of Montel spaces is valid in the case of arbitrary metrisable Montel spaces; we also give an example of a non-separable complete linear metric space in which every bounded set is separable. The construction of this example is also based on continuum hypothesis. The problem whether there exist B_0-spaces (i. e., according to [5], locally convex complete metric linear spaces) having this property is still open.

1. THEOREM. *Every metrisable Montel space is separable.*

Proof. Let X be non-separable linear metric space and Z an arbitrary uncountable set in X, such that

(1) $$\varrho(z, z') \geqslant \delta > 0 \quad \text{for} \quad z, z' \in Z, \quad z \neq z'.$$

Let us define the sequence of quasi-norms (Hyers [4], see also Bourgin [1] and Rolewicz [6]) by

$$[x]_n = \inf\{t > 0: \varrho(0, tx) \geqslant 1/n\}, \quad x \in X.$$

It is obvious that a set A, $A \subset X$, is bounded if and only if

(2) $$\sup_{x \in A} [x]_n < \infty \quad (n = 1, 2, \ldots).$$

Since $\overline{\overline{Z}} > \aleph_0$, we can find such $M_1 > 0$ that $Z_1 = Z \cap \{x \in X: [x]_1 < M_1\}$ is uncountable. Further we can define by induction a sequence