PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

COLLOQUIUM MATHEMATICUM

VOL. IX 1962 FASC. 1

ALGEBRA OF FORMALIZED LANGUAGES

BY

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This paper is of a methodological character. It contains a part of the theory of propositional and predicate calculi, developed from a purely algebraic point of view. The languages of propositional and predicate calculi are considered as certain abstract algebras with operations determined by logical connectives and quantifiers. The set of terms is treated in a similar way. From the methodological point of view, the method just mentioned causes an elimination of proofs by induction with respect to the length of formulas. Inductive arguments are used only to state that some operations on sets of formulas are homomorphisms (theorems 2.2, 7.2, 7.3, 7.4, 9.1, 9.2), and that some algebras of formulas or terms are free (theorems 2.1, 6.1, 7.1). Other theorems (3.1, 3.2, 3.3, 8.1, 8.2, 8.3, 8.4, 9.3, 9.4), traditionally proved by induction with respect to the length of formulas, are now proved without the use of any inductive argument. Each of those theorems has the form of an equality between two expressions. The proof is based on the fact that both sides of the equality can be interpreted as homomorphisms in suitable algebras. These homomorphisms coincide on a set of generators. Consequently they are equal, i. e. the equality holds. Thus many inductive reasonings are now replaced by the purely algebraic argument that every mapping from generators of an algebra into another similar algebra has at most one homomorphic extension over the whole algebra (1.2 and 4.2). Another algebraic argument frequently used in this paper is that the superposition of two homomorphisms is a homomorphism (1.1 and 4.1).

The developed part of the theory of propositional and predicate calculi which is presented here treats of the interpretation of formulas and terms as mappings in suitable algebras or sets. We recall that, given a Boolean algebra A, every formula α from a propositional calculus can be interpreted as a mapping $\alpha_A: A\times\ldots\times A \to A$. Similarly, in the case of a predicate calculus, if X is a given set, A is a Boolean algebra and R is a realization of functors as mappings from $X\times\ldots\times X$ into X, and of predicates as mappings from $X\times\ldots\times X$ into A, then every formula α can be interpreted as a mapping $\alpha_R\colon X\times\ldots\times X\to A$, and every term τ can be interpreted as a mapping $\tau_R\colon X\times\ldots\times X\to X$. The mappings α_A, α_R, τ_R play an important part, e. g. in algebraic proofs of the completeness theorems for the classical propositional and predicate calculi. These mappings also play an important part in the investigation of non-classical propositional and predicate calculi, Boolean algebras being replaced by algebras of another kind (e. g. by relatively pseudo-complementative lattices for the intuitionistic calculi, etc.) (1).

The case of predicate calculi is much more complicated than that of propositional calculi. In the case of predicate calculi there are various possibilities of algebraic interpretations of quantifiers. One of the possible interpretations is that in the theory of polyadic algebras or of cylindrical algebras (2). The interpretation assumed in this paper is of another type. It requires to assign, with the formalized language of a predicate calculus, two kinds of algebras called Q-algebras and X-algebras respectively (§ 6 and § 7). They are algebras with infinite operations, i. e. operations performed on some infinite sets of formulas. The general theory of algebras with operations performed on some infinite subsets is the subject of § 4. The algebraic investigation of formalized languages of predicate calculi is the main part of this paper (§§ 5-9). The algebraic investigation of formalized languages of propositional calculi and languages of terms is only of an auxiliary importance (§§ 2, 3).

The paper is written in an abstract way, the degree of generality being greater than is necessary for applications in Mathematical Logic. The great degree of generality is assumed because it brings out the essential algebraic aspects of the paper without causing any complications. The close connection with concrete problems in Mathematical Logic is shown in Examples 1-6.

Many ideas appearing in this paper are well known among specialists

(1) The truth-table method applied to two-valued and many-valued propositional calculi can be considered as the first form of interpretation of formulas α as mappings α_A in suitable algebras A. For the case of the Heyting and Lewis propositional calculus, see especially McKinsey [1], McKinsey and Tarski [1-3], Rieger [1]. The interpretation of formulas α (in the Heyting predicate calculus) as mappings α_R was first used by Mostowski [1] to prove the non-deducibility of some formulas. For systematic applications of this interpretation to the proof of the completeness of the classical and non-classical predicate calculi and related questions, see Henkin [1], Rasiowa [1-3], Rasiowa and Sikorski [1-4], Rieger [2], Sikorski [1], Stone [1].

(2) Halmos [1-3]; Henkin [2], Tarski [1], Tarski and Thompson [1]. Algebraic aspects of Logic have been the subject of many papers. See e. g. Łoś [1], Robinson [1].

interested in algebraic methods in Mathematical Logic (3). It has seemed useful to collect and systematize these ideas in a paper. The notion of X-algebras which plays a fundamental role in the investigation of predicate calculi seems to be the only essentially new notion in this paper.

TERMINOLOGY AND NOTATION

The words *mapping* and *function* have always the same meaning in this paper. We write $f\colon X\to Y$ to indicate that f is a mapping defined on a set X with values in a set Y. Usually, if f denotes a mapping, then $f(x)$ is used to denote the value of f at a point x. Sometimes we also write fx or f_x instead of $f(x)$. If, for every $x\in X$, y_x is an element in a set Y, then the function y, which assigns to every $x\in X$ the element y_x, will also be denoted by $\{y_x\}_{x\in X}$. The set of all such functions $y=\{y_x\}_{x\in X}$, i. e. the set of all functions $y\colon X\to Y$, will be denoted by Y^X. Clearly Y^X is the Cartesian product of $\overline{\overline{X}}$ replicas of the set Y. The element y_x will sometimes be called the *x-th coordinate* of the point $y=\{y_x\}_{x\in X}\in Y^X$. If X is the set of the integers $1,\ldots,m$, then we write Y^m instead of Y^X.

If $\sim$ is an equivalence relation in a set X, then, for every $x\in X$, $|x|$ denotes the set of all x' such that $x\sim x'$, i. e. the equivalence class containing x, and $A/\!\sim$ denotes the set of equivalence classes $|x|$, $x\in X$. If $\approx$ is another equivalence relation in X, the corresponding equivalence classes are denoted by $\|x\|$.

In the sequel we shall examine a fixed set $\mathcal{S}$ whose elements will be called *signs*. A finite sequence formed from signs $s_i\in\mathcal{S}$ $(i=1,\ldots,n)$ will be written

$$s_1s_2\ldots s_n.$$

Finite sequences of signs will usually be denoted by Greek letters α,β,γ,τ (with indices if necessary) and called *expressions*.

If α denotes an expression $s_1\ldots s_n$ and β denotes an expression $s_1'\ldots s_m'$, then $\alpha\beta$ will denote the expression

$$s_1\ldots s_n s_1'\ldots s_m'.$$

(3) For instance, during the preparation of this paper I listened to a talk of A. Mostowski at the Polish Mathematical Society, who suggested also a similar interpretation of quantifiers as infinite operations (without introducing the X-algebras).

Some ideas and notation in this paper are the result of my discussions with H. Rasiowa during the preparation of our book *Mathematics of Metamathematics* (to appear in Monografie Matematyczne).

The meaning of notation like

$$\alpha\beta\gamma, \quad \alpha s\beta, \quad s\alpha s'\beta, \quad s\alpha_1\alpha_2\ldots\alpha_m \quad \text{etc.}$$

(where α, β, γ $\alpha_1, \ldots, \alpha_m$ are expressions and s, s' are signs) is similar.

Let α be an expression

$$\alpha_1 s_1 \alpha_2 s_2 \ldots \alpha_n s_n \alpha_{n+1}$$

where $s_1, \ldots, s_n$ are signs and $\alpha_1, \ldots, \alpha_{n+1}$ are finite sequences of signs (some of which may be empty), and let $\beta_1, \ldots, \beta_n$ be certain expressions. The expression

$$\alpha_1 \beta_1 \alpha_2 \beta_2 \ldots \alpha_n \beta_n \alpha_{n+1}$$

is said to be obtained from α by the *simultaneous replacement* of the occurrences of signs s_i $(i = 1, \ldots, n)$ in α by expressions $\beta_1, \ldots, \beta_n$.

Let α be an expression, let $s_1, \ldots, s_m$ be signs, and let $\beta_1, \ldots, \beta_m$ be expressions. The expression obtained from α by the simultaneous replacement of every occurrence of s_1 by β_1, of every occurrence of s_2 by $\beta_2, \ldots$, and of every occurrence of s_m by β_m, will be denoted by

(1) $$\alpha(s_1/\beta_1, \ldots, s_m/\beta_m).$$

Expression (1) will be called the *result of the substitution of* $\beta_1, \ldots, \beta_m$ *for* $s_1, \ldots, s_m$ *in* α.

Sometimes, in order to emphasize that an expression α contains, perhaps, some signs $s_1, \ldots, s_m$, we shall denote it by

$$\alpha(s_1, \ldots, s_m).$$

Then substitution (1) will also be denoted, more suggestively, by

$$\alpha(\beta_1, \ldots, \beta_m).$$

The letter M will always denote the set of all non-negative integers.

PART I

ALGEBRA OF FORMALIZED LANGUAGES OF THE ZERO ORDER

§ 1. Abstract algebras. We assume that the reader is familiar with the notion of abstract algebra. We recall only some fundamental definitions to fix the terminology.

Any mapping $o: A^m \to A$ is called an *m-argument operation* in a set A $(m = 0, 1, 2, \ldots)$. The case $m = 0$ is admissible: a 0-argument operation is a constant $o \in A$.

A subset $A' \subset A$ is said to be *closed under an m-argument operation* o in A provided

$$o(a_1, \ldots, a_m) \in A' \quad \text{for all} \quad a_1, \ldots, a_m \in A'.$$

By an *abstract algebra* or, simply, *algebra* we understand any pair

(1) $$\{A, \{o_\varphi\}_{\varphi\in\Phi}\}$$

where A is a non-empty set and, for every $\varphi \in \Phi$, o_φ is an operation in A. The cardinal of Φ may be arbitrary; in particular, the set Φ may be empty.

To simplify notation, we shall not strictly distinguish between algebra (1) and the set A of its elements.

Any subset $A' \subset A$ closed with respect to all the operations o_φ $(\varphi \in \Phi)$ is called a *subalgebra* of algebra (1). A set $G \subset A$ is said to *generate the algebra* A (see (1)), or: to be *a set of generators for* A provided A is the only subalgebra containing G.

Let (1) and

(2) $$\{B, \{o'_\varphi\}_{\varphi\in\Phi'}\}$$

be abstract algebras, o_φ and o'_φ being respectively an m_φ-argument operation and an m'_φ-argument operation. If $\Phi' = \Phi$ and $m'_\varphi = m_\varphi$ for every φ, then algebras (1) and (2) are said to be *similar*. Usually, if (2) is similar to (1), then the corresponding operations o'_φ will be denoted by the same symbol o_φ, i. e. we shall then write

(3) $$\{B, \{o_\varphi\}_{\varphi\in\Phi}\}$$

instead of (2).

A mapping $h: A \to B$ is said to be a *homomorphism* from an algebra (1) into a similar algebra (3) provided

$$h(o_\varphi(a_1, \ldots, a_{m_\varphi})) = o_\varphi(h(a_1), \ldots, h(a_{m_\varphi}))$$

for all $\varphi \in \Phi$ and all $a_1, \ldots, a_{m_\varphi} \in A$. A one-to-one homomorphism from A onto B is said to be an *isomorphism*. If there exists an isomorphism h from A onto B, then A and B are said to be isomorphic, and h^{-1} is an isomorphism from B onto A.

1.1. *If A, B, C are similar algebras, and*

$$h: A \to B, \quad g: B \to C$$

are homomorphisms, then the superposition

$$gh: A \to C$$

is also a homomorphism.

1.2. *Let G be a set of generators for an algebra A, and let B be an algebra similar to A. If a mapping $f: G \to B$ can be extended to a homomorphism $h: A \to B$, then this homomorphic extension h is unique.*

Let $\mathfrak{K}$ be a class of similar algebras. An algebra $A \in \mathfrak{K}$ is said to be $\mathfrak{K}$*-free* if it contains a set G of generators such that every mapping

$f\colon G \to B$, where B is any algebra in $\mathfrak{K}$, can be extended to a homomorphism $h\colon A \to B$. Then G is said to be a set of *$\mathfrak{K}$-free generators for A*.

1.3. *If A and A' are $\mathfrak{K}$-free algebras with sets G and G' of $\mathfrak{K}$-free generators respectively, and the sets G, G' have the same cardinal, then A and A' are isomorphic. More precisely: every one-to-one mapping from G onto G' can be extended to an isomorphism from A onto A'.*

§ 2. Formalized languages of the zero order. By an *alphabet of the zero order* we shall understand any ordered pair

$$\mathscr{A}_0 = \{V, \{\Phi_m\}_{m \in M}\},$$

where

1° $V, \Phi_0, \Phi_1, \Phi_2, \ldots$ are disjoint sets,

2° the set V is not empty.

Elements of the set

$$\mathscr{S}_0 = V \cup \Phi_0 \cup \Phi_1 \cup \Phi_2 \cup \ldots$$

are called *signs* of the alphabet $\mathscr{A}_0$. Elements in V will be called *variables* and denoted by v. Elements in

$$\Phi = \Phi_0 \cup \Phi_1 \cup \Phi_2 \cup \ldots$$

are called *functors* or *connectives* and denoted by $\boldsymbol{o}$. More precisely, any $\boldsymbol{o} \in \Phi_m$ is called an *m-argument functor* or *connective*.

As was stated on p. 3, we can form expressions from signs in $\mathscr{S}_0$, i. e. finite sequences of signs of $\mathscr{A}_0$. Let $\mathscr{T}$ be the smallest set of finite sequences τ of signs of $\mathscr{A}_0$ such that

a) all one-element sequences v, where v is in V, are in $\mathscr{T}$;

b) if $\boldsymbol{o} \in \Phi_m$ and $\tau_1, \ldots, \tau_m$ are in $\mathscr{T}$, then the expression $\boldsymbol{o}\tau_1 \ldots \tau_m$ is also in $\mathscr{T}$.

By a), V is a subset of $\mathscr{T}$.

If elements $\boldsymbol{o} \in \Phi$ are called connectives, then expressions in $\mathscr{T}$ are usually called *formulas* (more precisely: *formulas of the zero order*). If elements $\boldsymbol{o} \in \Phi$ are called functors, then the expressions in $\mathscr{T}$ are usually called *terms*.

The ordered pair

$$\mathscr{L}_0 = \{\mathscr{A}_0, \mathscr{T}\}$$

is said to be a *formalized language of the zero order*. More precisely, $\mathscr{L}_0$ is said to be the *formalized language based on the alphabet $\mathscr{A}_0$*.

Example 1. Suppose that V is infinite, Φ_1 contains only one sign **N** called the *negation sign*, Φ_2 contains only three signs **D**, **C**, **I** called respectively the *disjunction sign*, the *conjunction sign* and the *implication sign*, the remaining sets $\Phi_0, \Phi_3, \Phi_4, \ldots$ being empty. Then $\mathscr{T}$ is the set of all formulas of a propositional calculus (in the sense usually adopted in Mathematical Logic), V being the set of all propositional variables. More exactly, $\mathscr{T}$ is the set of formulas of a propositional calculus where Łukasiewicz's parenthesis-less notation is applied.

Another important interpretation of $\mathscr{T}$ will be given in § 5, Example 4.

The set $\mathscr{T}$ can be conceived as an abstract algebra

$$\{\mathscr{T}, \{\boldsymbol{o}\}_{\boldsymbol{o} \in \Phi}\} \tag{1}$$

with the following definition of operations: if $\boldsymbol{o} \in \Phi_m$, then the expression $\boldsymbol{o}\tau_1 \ldots \tau_m$ is said to be the result of the m-argument operation $\boldsymbol{o}$ performed on elements (expressions) $\tau_1, \ldots, \tau_m$ in $\mathscr{T}$. In symbols,

$$\boldsymbol{o}(\tau_1, \ldots, \tau_m) = \boldsymbol{o}\tau_1 \ldots \tau_m \quad \text{for} \quad \tau_1, \ldots, \tau_m \in \mathscr{T}. \tag{2}$$

In other words, every m-argument functor $\boldsymbol{o}$ determines, in a natural way, an m-argument operation (2) in $\mathscr{T}$. This operation is denoted by the same symbol $\boldsymbol{o}$. Algebra (1) is the set $\mathscr{T}$ with all the operations determined by functors.

Algebra (1) is called the *algebra of formulas in $\mathscr{L}_0$*, or *the algebra of terms*, according to the name adopted for expressions in $\mathscr{T}$.

Let $\mathfrak{K}_0$ be the class of all algebras similar to the algebra $\mathscr{T}$. According to the convention on p. 5, if $A_0 \in \mathfrak{K}_0$, then the operation in A which corresponds to operation (2) in $\mathscr{T}$ will be denoted by the same symbol $\boldsymbol{o}$, i. e. any algebra $A_0 \in \mathfrak{K}_0$ will be written in the form

$$\{A_0, \{\boldsymbol{o}\}_{\boldsymbol{o} \in \Phi}\}.$$

2.1. *The algebra $\{\mathscr{T}, \{\boldsymbol{o}\}_{\boldsymbol{o} \in \Phi}\}$ is $\mathfrak{K}_0$-free, the set V being the set of $\mathfrak{K}_0$-free generators.*

In fact, every mapping $f\colon V \to A_0$ $(A_0 \in \mathfrak{K}_0)$ can be extended to a mapping $h\colon \mathscr{T} \to A_0$ defined by induction on the length of $\tau \in \mathscr{T}$, as follows:

(i) for every v in V, $h(v) = f(v)$;

(ii) if $h(\tau_i)$ is defined for some $\tau_i \in \mathscr{T}$, $i = 1, \ldots, m$, and if τ is the expression $\boldsymbol{o}\tau_1 \ldots \tau_m$, where $\boldsymbol{o} \in \Phi_m$, then

$$h(\tau) = \boldsymbol{o}\big(h(\tau_1), \ldots, h(\tau_m)\big).$$

The uniqueness of $h(\tau)$ follows from the fact that τ can be represented in the form $\boldsymbol{o}\tau_1 \ldots \tau_m$ in exactly one way.

It follows from (i) that h is an extension of f. It follows from (ii) that h is a homomorphism from $\mathscr{T}$ into A_0. It follows from a) and b) that V is a set of generators for $\mathscr{T}$. This proves 2.1.

Consider now the case where A_0 is the algebra $\mathscr{T}$ itself. Any mapping

$$\mathfrak{s}\colon V \to \mathscr{T} \tag{3}$$

is called a *substitution* in the language $\mathscr{L}_0$. By 2.1, the mapping (3) can be uniquely extended to a homomorphism denoted by the same letter $\mathfrak{s}$:

(4) $$\mathfrak{s}: \mathscr{T} \to \mathscr{T},$$

The value of the homomorphism $\mathfrak{s}$ at an element $\tau \in \mathscr{T}$ will be denoted by $\mathfrak{s}\tau$.

Suppose that $v_1, \ldots, v_m$ are all variables appearing in a $\tau \in \mathscr{T}$, and denote τ by $\tau(v_1, \ldots, v_m)$ (see p. 4). Then

(5) $$\mathfrak{s}\tau = \tau(\mathfrak{s}v_1, \ldots, \mathfrak{s}v_m)$$

(see p. 4). In fact, both expressions, considered as functions of $\tau \in \mathscr{T}$, are homomorphisms from $\mathscr{T}$ into itself. They coincide on the set V of generators for $\mathscr{T}$. By 1.2 they are equal.

Identity (5) means that homomorphism (4) coincides with the operation of substitution for variables in the sense defined on p. 4. This remark can be formulated in the form of the following theorem:

2.2. *Every substitution* (4) *in* $\mathscr{L}_0$ *is a homomorphism from the algebra* $\mathscr{T}$ *into itself.*

§ 3. Interpretation of formulas as mappings. Let $\mathscr{L}_0$ be the formalized language of the zero order from § 2, and let

$$\{A_0, \{o\}_{o\in\Phi}\}$$

be an algebra similar to the algebra

$$\{\mathscr{T}, \{o\}_{o\in\Phi}\}$$

discussed in § 2.

Every expression τ in $\mathscr{T}$ determines, in a natural way, a function (of several variables) in A_0 with values in A_0. This function will be denoted by τ_{A_0}. To obtain τ_{A_0} it suffices

A) to interpret variables appearing in τ as variables running through A_0;

B) to interpret signs $o \in \Phi$ in τ as signs of the corresponding operations o in A_0.

By this definition, τ_{A_0} is a mapping

$$\tau_{A_0}: A_0^{V_\tau} \to A_0,$$

where V_τ is the set of all variables appearing in τ. However, every mapping $f: A_0^{V'} \to A_0$, where $V' = (v_1, \ldots, v_n)$ is a subset of V, can be interpreted as a mapping $f: A_0^V \to A_0$ on assuming

$$f(\mathfrak{v}) = f(\mathfrak{v}_{v_1}, \ldots, \mathfrak{v}_{v_n})$$

for every element $\mathfrak{v} = \{\mathfrak{v}_v\}_{v\in V} \in A_0^V$. The element $f(\mathfrak{v})$ depends, of course, only on coordinates $\mathfrak{v}_{v_1}, \ldots, \mathfrak{v}_{v_n}$ of $\mathfrak{v}$.

In particular, τ_{A_0} can be conceived as a mapping

(1) $$\tau_{A_0}: A_0^V \to A_0$$

which to every $\mathfrak{v} \in A_0^V$ assigns an element $\tau_{A_0}(\mathfrak{v}) \in A_0$.

Elements $\mathfrak{v} = \{\mathfrak{v}_v\}_{v\in V}$, i. e. the mappings

(2) $$\mathfrak{v}: V \to A_0$$

will be called *valuations* in A_0.

In this paper we shall replace the above intuitive definition of τ_{A_0} by another definition, equivalent but more precise and more algebraic (see (4) below), which will be used as the starting point for further investigations.

Let $\mathfrak{v}$ be a fixed valuation (2). By 2.1, $\mathfrak{v}$ can be uniquely extended to a homomorphism from the algebra $\mathscr{T}$ into the algebra A_0. Denote this homomorphism by $\mathfrak{v}_{A_0}$. Thus, for every $\tau \in \mathscr{T}$,

(3) $$\mathfrak{v}_{A_0}(\tau)$$

is a well defined element in A_0. If τ is fixed and $\mathfrak{v}$ is variable, then the element (3) is a function of $\mathfrak{v} \in A_0^V$. Denote this function by τ_{A_0}. By definition,

(4) $$\tau_{A_0}(\mathfrak{v}) = \mathfrak{v}_{A_0}(\tau).$$

That is the mapping τ_{A_0} in (1). The easy proof of the equivalence of definitions (1) and (4) is omitted because we shall use only definition (4).

Example 2. Let $\mathscr{L}_0$ be the formalized language of a propositional calculus described in § 2, Example 1. Let A_0 be a Boolean algebra. For any $a, b \in A_0$, let $\mathbf{N}a$ denote the complement of a, let $\mathbf{D}ab$ and $\mathbf{C}ab$ denote the meet and join of a and b respectively, and let $\mathbf{I}ab = \mathbf{DN}ab$. Let $\vee$ denote the unit element of A. Then, for every formula τ in $\mathscr{T}$, τ_{A_0} is identically equal to $\vee$ in every Boolean algebra A_0 (or: in the two-element Boolean algebra A_0) if and only if τ is a propositional tautology. The verification whether τ_{A_0} is identically equal to the unit element in a two-element Boolean algebra A_0 is another formulation of the well-known truth-table method.

The following theorems express fundamental properties of τ_{A_0}.

To formulate the first theorem, let us suppose that

$$\{B_0, \{o\}_{o\in\Phi}\}$$

is another algebra similar to $\{\mathscr{T}, \{o\}_{o\in\Phi}\}$ and that $h: A_0 \to B_0$ is a homomorphism. For every valuation $\mathfrak{v} = \{\mathfrak{v}_v\}_{v\in V}$ in A_0 let $h\mathfrak{v}$ be the valuation $\{h(\mathfrak{v}_v)\}_{v\in V}$ in B_0. Thus $h\mathfrak{v}$ is the superposition of $\mathfrak{v}: V \to A_0$ and $h: A_0 \to B_0$. Under the above hypotheses the following theorem holds:

3.1. *For every* $\tau \in \mathscr{T}$,

(5) $$\tau_{B_0}(h\mathfrak{v}) = h(\tau_{A_0}(\mathfrak{v})).$$

In fact, for every fixed valuation $\mathfrak{v}$, both sides of (5), considered as functions of $\tau \in \mathscr{T}$, are homomorphisms from the algebra $\mathscr{T}$ into the algebra B_0 by (4) and 1.1. These homomorphisms coincide on the set V of generators for $\mathcal{T}$. Thus they are equal by 1.2.

Now let $\mathfrak{s}$ be a substitution in $\mathcal{L}_0$ (see § 2, p. 8). The substitution $\mathfrak{s}$ induces a mapping

(6) $$\mathfrak{s}_{A_0}: A_0^V \to A_0^V$$

defined as follows: for every valuation $\mathfrak{v} = \{\mathfrak{v}_v\}_{v\in V}$ in A_0, $\mathfrak{s}_{A_0}\mathfrak{v}$ is the valuation $\{\mathfrak{s}v_{A_0}(\mathfrak{v})\}_{v\in V}$. We recall that $\mathfrak{s}v$ is the value of $\mathfrak{s}$ at $v \in V_0$, i.e. an expression in $\mathcal{T}$, and $\mathfrak{s}v_{A_0}$ is the mapping determined by the expression $\mathfrak{s}v$. Thus $\mathfrak{s}_{A_0}\mathfrak{v}$ is a point in A_0^V whose v-th coordinate is equal to the value of the mapping $\mathfrak{s}v_{A_0}$ at $\mathfrak{v}$. Under the above notation the following theorem holds:

3.2. *For every* $\tau \in \mathcal{T}$,

(7) $$\mathfrak{s}\tau_{A_0}(\mathfrak{v}) = \tau_{A_0}(\mathfrak{s}_{A_0}\mathfrak{v}).$$

The left side of (7) is the value of the mapping $\mathfrak{s}\tau_{A_0}$ determined by the expression $\mathfrak{s}\tau$ (the result of the substitution $\mathfrak{s}$ in τ — see p. 8) at the point $\mathfrak{v}$.

The proof of 3.2 is similar to that of 3.1. By (4), 2.2 and 1.1 the left side of (7), considered as a function of τ, is a homomorphism from $\mathcal{T}$ into A_0. By (4) the right side of (7) is also a homomorphism from $\mathcal{T}$ into A_0. By the definition of (6), both homomorphisms coincide on the set V of generators for $\mathcal{T}$. By 1.2 they are equal.

Suppose now that $\sim$ is an equivalence relation in $\mathcal{T}$ and that with every $\boldsymbol{o} \in \Phi_m$ $(m = 0, 1, 2, \ldots)$ there is associated an m-argument operation in $\mathcal{T}/\sim$, denoted by the same letter $\boldsymbol{o}$, in such a way that the natural mapping

(8) $$h(\tau) = |\tau| \in \mathcal{T}/\sim \qquad (\tau \in \mathcal{T})$$

is a homomorphism from the algebra $\{\mathcal{T}, \{\boldsymbol{o}\}_{\boldsymbol{o}\in\Phi}\}$ onto the algebra $\{\mathcal{T}/\sim, \{\boldsymbol{o}\}_{\boldsymbol{o}\in\Phi}\}$. Every substitution $\mathfrak{s}: V \to \mathcal{T}$ induces a corresponding valuation $\mathfrak{s}'$ in the algebra $T/\sim$, viz.

(9) $$\mathfrak{s}' = \{|\mathfrak{s}v|\}_{v\in V}.$$

The letter $\mathfrak{i}$ will denote the identity substitution:

(10) $$\mathfrak{i}: V \to \mathcal{T}.$$

Under the above hypotheses the following theorem holds:

3.3. *For every* τ *in* $\mathcal{T}$,

(11) $$\tau_{T/\sim}(\mathfrak{s}') = |\mathfrak{s}\tau|.$$

In particular,

(12) $$\tau_{T/\sim}(\mathfrak{i}') = |\tau|.$$

In fact, both sides of (11), when considered as functions of τ, are homomorphisms from the algebra $\mathcal{T}$ into the algebra $\mathcal{T}/\sim$ on account of (4), (8), 2.2 and 1.1. Since they coincide on the set V of generators for $\mathcal{T}$, they are equal by 1.2.

Example 3. Let $\mathcal{L}_0$ be the language of a propositional calculus described in § 2, Example 1. Write $\tau_1 \sim \tau_2$ for $\tau_1, \tau_2 \in \mathcal{T}$ if and only if both formulas $\mathbf{I}\tau_1\tau_2$ and $\mathbf{I}\tau_2\tau_1$ are propositional tautologies. Then the equalities

$$\mathbf{N}(|\tau|) = |\mathbf{N}\tau|, \qquad \mathbf{D}(|\tau_1|, |\tau_2|) = |\mathbf{D}\tau_1\tau_2|,$$
$$\mathbf{C}(|\tau_1|, |\tau_2|) = |\mathbf{C}\tau_1\tau_2|, \qquad \mathbf{I}(|\tau_1|, |\tau_2|) = |\mathbf{I}\tau_1\tau_2|$$

defines some operations **N**, **D**, **C**, **I** in $\mathcal{T}/\sim$, corresponding to the operation **N**, **D**, **C**, **I** in $\mathcal{T}$. Mapping (8) is a homomorphism. Note that $\mathcal{T}/\sim$ is a Boolean algebra with complementation **N**, join **D** and meet **C**. Identities (11) and (12) play a fundamental role in the Boolean proof of the completeness theorem for the propositional calculus.

PART II

ALGEBRA OF FORMALIZED LANGUAGES OF THE FIRST ORDER

§ 4. Generalized abstract algebras. Let A be a non-empty set. By a *generalized operation* in A we shall understand any mapping

$$O: \mathfrak{D} \to A$$

where $\mathfrak{D}$ is a class of non-void subsets of A. Thus O assigns to every set $S \in \mathfrak{D}$ an element $OS \in A$. The class $\mathfrak{D}$ is called the *domain of the generalized operation* O in question, according to the terminology assumed generally for mappings. Sets $S \in \mathfrak{D}$ are called *sets admissible for the operation* O. It is important that sets S in $\mathfrak{D}$ can be infinite. To underline this fact we also call O an *infinite operation* in A (operations defined on p. 4 are called *finite operations*). If a set $S \in \mathfrak{D}$ is given in the form of an indexed set $\{a_t\}_{t\in T}$, then we write

(1) $$O_{t\in T}\, a_t$$

instead of OS.

A subset $A' \subset A$ is said to be *closed with respect to a generalized operation* O in A provided that, for every set $S \subset A'$,

$$S \in \mathfrak{D} \text{ implies } OS \in A'.$$

By a *generalized abstract algebra*, or simply: *generalized algebra*, we shall understand any triple

(2) $$\{A, \{o_\varphi\}_{\varphi\in\Phi}, \{O_\psi\}_{\psi\in\Psi}\}$$

where A is a non-empty set, o_φ is a finite operation in A for every $\varphi \in \Phi$, and O_ψ is an infinite operation in A for every $\psi \in \Psi$. The cardinals of the sets Φ and Ψ can be arbitrary; in particular, the set Φ or Ψ can be empty.

To simplify the notation, we shall not strictly distinguish between algebra (2) and the set A of its elements.

Of course, if (2) is a generalized algebra, then

$$\{A, \{o_\varphi\}_{\varphi\in\Phi}\} \tag{3}$$

is an algebra in the sense defined in § 1, p. 5.

Any subset $A' \subset A$ closed with respect to all the operations o_φ ($\varphi\in\Phi$) and all the generalized operations O_ψ ($\psi\in\Psi$) is called a *subalgebra* of the generalized algebra (2). A set $G \subset A$ is said to *generate the generalized algebra* (2), or: to be a *set of generators for* (2), provided A is the only subalgebra containing G.

Let (2) and

$$\{B, \{o'\}_{\varphi\in\Phi}, \{O'_\psi\}_{\psi\in\Psi'}\} \tag{4}$$

be generalized algebras. Algebras (2) and (4) are said to be *similar* if $\Psi' = \Psi$ and the algebras

$$\{A, \{o_\varphi\}_{\varphi\in\Phi}\}, \quad \{B, \{o'_\varphi\}_{\varphi\in\Phi'}\}$$

are similar in the sense defined in § 1, p. 5 (the last condition implies that $\Phi' = \Phi$). Usually, if (4) is similar to (2), then the corresponding operations o'_φ will be denoted by the same symbol o_φ, and the corresponding generalized operations O'_ψ will be denoted by the same symbols O_ψ as in A, i. e. we shall write

$$\{B, \{o_\varphi\}_{\varphi\in\Phi}, \{O_\psi\}_{\psi\in\Psi}\} \tag{5}$$

instead of (4).

A mapping $h: A \to B$ is said to be a *homomorphism* from algebra (1) into a similar algebra (5) provided it is a homomorphism of the algebra $\{A, \{o_\varphi\}_{\varphi\in\Phi}\}$ into the algebra $\{B, \{o_\varphi\}_{\varphi\in\Phi}\}$ in the sense defined in § 1, p. 5, and, moreover, for every $\psi\in\Psi$,

$$h(O_\psi S) = O_\psi h(S) \tag{6}$$

for every set S admissible for O_ψ. The last condition should be understood as follows: If S is admissible for O_ψ in A, then the set $h(S)$ is admissible for the corresponding operation O_ψ in B, and equality (6) holds.

The following generalizations of 1.1 and 1.2 are true:

4.1. *If A, B, C are generalized algebras, and*

$$h: A \to B, \quad g: B \to C$$

are homomorphisms, then the superposition

$$gh: A \to C$$

is also a homomorphism.

4.2. *Let G be a set of generators for a generalized algebra A, and let B be a generalized algebra similar to A. If a mapping $f: G \to B$ can be extended to a homomorphism $h: A \to B$, this homomorphic extension h is unique.*

Let $\mathfrak{K}$ be a class of similar generalized algebras. A generalized algebra A similar to algebras in $\mathfrak{K}$ is said to be a *generalized $\mathfrak{K}$-free algebra* if it contains a set G of generators such that every mapping $f: G \to B$, where B is any algebra in $\mathfrak{K}$, can be extended to a homomorphism $h: A \to B$. Then G is said to be a set of *generalized $\mathfrak{K}$-free generators for A.*

Note that, in contrast to an analogous definition on p. 6-7, we do not require here that $A\in\mathfrak{K}$. Consequently no analogue of 1.3 holds for generalized $\mathfrak{K}$-free algebras.

A generalized algebra $\{A', \{o'_\varphi\}_{\varphi\in\Phi}, \{O'_\psi\}_{\psi\in\Psi}\}$ is said to be an *extension* of a similar algebra $\{A, \{o_\varphi\}_{\varphi\in\Phi}, \{O_\varphi\}_{\psi\in\Psi}\}$ provided A is a subset of A' and

1° for every $\varphi\in\Phi$ and for all elements $a_1, \ldots, a_m \in A$

$$o_\varphi(a_1, \ldots, a_m) = o'_\varphi(a_1, \ldots, a_m),$$

o_φ and o'_φ being supposed to be m-argument operations ($m = 0, 1, 2, \ldots$);

2° for every $\psi\in\Psi$, if a set $S \subset A$ is admissible for O_ψ, then it is also admissible for O'_ψ and

$$O'_\psi S = O_\psi S.$$

A generalized algebra $\{A, \{o_\varphi\}_{\varphi\in\Phi}, \{O_\psi\}_{\psi\in\Psi}\}$ is said to be a *complete algebra* provided the class of all non-empty subsets of A is the common domain for all the generalized operations O_ψ in A.

Every generalized algebra $\{A, \{o_\varphi\}_{\varphi\in\Phi}, \{O_\varphi\}_{\psi\in\Psi}\}$ can be extended to a similar complete algebra $\{A', \{o'_\varphi\}_{\varphi\in\Phi}, \{O'_\psi\}_{\psi\in\Psi}\}$ in various ways. Note the following special way of extension: Let a_0 be a fixed element, $a_0 \notin A$, and let $A' = A \cup (a_0)$. Operations in A' are defined as follows:

$$o'_\varphi(a_1, \ldots, a_m) = \begin{cases} o_\varphi(a_1, \ldots, a_m) & \text{if} \quad a_1, \ldots, a_m \in A, \\ a_0 & \text{otherwise,} \end{cases}$$
$$O'_\psi S = \begin{cases} O_\psi S & \text{if} \quad S \subset A \text{ and } S \text{ is admissible for } O_\psi, \\ a_0 & \text{otherwise.} \end{cases} \tag{7}$$

In other words, the result of a finite or infinite operation in A' is equal to a_0 unless the operation in question is feasible in A.

According to the convention on p. 12, if A' is an extension of A, then corresponding finite and infinite operations in A' will often be denoted by the same symbols as operations in A.

§ 5. Formalized languages of the first order. By an *alphabet of the first order* we shall understand an ordered system

$$\mathscr{A} = \{V, \{\Phi_m\}_{m\in M}, \{\Pi_m\}_{m\in M}, \{C_m\}_{m\in M}, Q, \overline{V}\},$$

where

1° $V, \Phi_0, \Phi_1, \Phi_2, \ldots, \Pi_0, \Pi_1, \Pi_2, \ldots, C_0, C_1, C_2, \ldots, Q, \overline{V}$ are disjoint sets,

2° V, $\Pi_0 \cup \Pi_1 \cup \Pi_2 \cup \ldots$ and $C_0 \cup C_1 \cup C_2 \cup \ldots$ are not empty sets,

3° if the set Q is not empty, then the set $\overline{V}$ is infinite.

We shall use the notation

$$\Phi = \Phi_0 \cup \Phi_1 \cup \Phi_2 \cup \ldots, \qquad \Pi = \Pi_0 \cup \Pi_1 \cup \Pi_2 \cup \ldots,$$
$$C = C_0 \cup C_1 \cup C_2 \cup \ldots, \qquad \mathcal{S} = V \cup \Phi \cup \Pi \cup C \cup Q \cup \overline{V}.$$

Elements in V will be called *variables* or, more precisely, *free individual variables* and will be denoted by v with indices. Elements in Φ will be called *functors* and denoted by $\mathbf{o}$; more precisely, elements in Φ_m will be called *m-argument functors*. Elements in Φ_0 will also be called *individual constants*. Elements in Π will be called *predicates* and denoted by Π; more precisely, elements in Π_m will be called *m-argument predicates*. Elements in C will be called *connectives* and denoted by o; more precisely, elements in C_m will be called *m-argument connectives*. Elements in Q will be called *quantifiers* and denoted by O. Elements in $\overline{V}$ will be called *bound individual variables* and denoted by ξ, η, ζ.

All elements in $\mathcal{S}$ are called *signs* of the alphabet $\mathcal{A}$.

It follows from 1°, 2° that the ordered pair

$$\mathcal{A}_0 = \{V, \{\Phi_m\}_{m \in M}\}$$

is an alphabet of the zero order, with the set $\mathcal{S}_0 = V \cup \Phi$ of signs. By § 3 a), b), we can form the corresponding set $\mathcal{T}$ of formulas of the order zero and the language $\mathcal{L}_0 = \{\mathcal{S}_0, \mathcal{T}\}$ of the order zero. In Part II of this paper, expressions in $\mathcal{T}$ will always be called *terms* and denoted by τ.

By means of signs in $\mathcal{S}$ we will also form other expressions, called *formulas of the first order*, or simply: *formulas*. Viz. the set $\mathcal{F}$ of all formulas of the first order is the smallest set of expressions formed from signs in $\mathcal{S}$, such that

A) if $\pi \in \Pi_m$ and $\tau_1, \ldots, \tau_m$ are in $\mathcal{T}$, then the expression

$$\pi\tau_1 \ldots \tau_m \tag{1}$$

is in $\mathcal{F}$;

B) if $o \in C_m$ and $a_1, \ldots, a_m$ are in $\mathcal{F}$, then the expression

$$o a_1 \ldots a_m$$

is in $\mathcal{F}$;

C) If $O \in Q$, $v \in V$, $a(v)$ is an expression in $\mathcal{F}$, and $\xi \in \overline{V}$ does not appear in $a(v)$, then the expression

$$O\xi a(\xi)$$

is in $\mathcal{F}$.

Formulas of form (1) are said to be *elementary*.

Formulas in $\mathcal{F}$ will be denoted by a, β, γ.

The triple

$$\mathcal{L} = \{\mathcal{A}, \mathcal{T}, \mathcal{F}\}$$

is said to be a *formalized language of the first order*. More precisely, $\mathcal{L}$ is said to be the *formalized language based on the alphabet* $\mathcal{A}$. The language

$$\mathcal{L}_0 = \{\mathcal{A}_0, \mathcal{T}\}$$

is called the *language of terms* of $\mathcal{L}$.

Example 4. Suppose that V and $\overline{V}$ are infinite sets, C_1 contains only one sign **N**, called the *negation sign*, C_2 contains only three signs: **D**, **C**, **I**, called respectively the *disjunction sign*, the *conjunction sign* and the *implication sign*, all the sets $C_0, C_3, C_4, \ldots$ being empty. Suppose also that Q contains only two signs **E** and **U** called respectively the *existential quantifier* and the *universal quantifier*, and that Φ and Π are any sets satisfying hypotheses mentioned in 1° and 2°. Then $\mathcal{T}$ and $\mathcal{F}$ are respectively the set of all terms and the set of all formulas (in the sense usually adopted in Mathematical Logic) of the predicate calculus based on the alphabet $\mathcal{A}$, Łukasiewicz's parenthesis-less notation being adopted.

By § 2, the set $\mathcal{T}$ of all terms will be conceived as an abstract algebra

$$\{\mathcal{T}, \{\mathbf{o}\}_{\mathbf{o} \in \Phi}\} \tag{2}$$

with operations defined by § 2 (2). This algebra will now be called the *algebra of terms of the language* $\mathcal{L}$.

The set $\mathcal{F}$ of all formulas can also be considered as an abstract algebra, viz. as the algebra

$$\{\mathcal{F}, \{o\}_{o \in C}\} \tag{3}$$

with the following definition of operations: if $o \in C_m$, then the formula $o a_1 \ldots a_m$ is said to be the *result* of the m-argument operation o performed on elements (formulas) $a_1, \ldots, a_m$ in $\mathcal{F}$. In symbols,

$$o(a_1, \ldots, a_m) = o a_1 \ldots a_m. \tag{4}$$

In other words, every m-argument connective o determines an m-argument operation (4) in $\mathcal{F}$. This operation is denoted by the same symbol o. Algebra (3) is the set $\mathcal{F}$ with all the operations determined by connectives.

However, algebra (3), called the *algebra of formulas of* $\mathcal{L}$, does not play any essential part in the examination of the formalized language $\mathcal{L}$ of the first order because it does not take into consideration the quantifiers which should also be interpreted as operations on formulas (see C)). To include quantifiers into algebraic consideration, we shall define in §§ 6 and 7 two kinds of generalized algebras which play an essential part in the algebraic investigation of the language $\mathcal{L}$ of the first order.

§ 6. The Q-algebra of formulas. Formulas a, β in $\mathcal{F}$ are said to be *syntactically equivalent*, in symbols

$$a \sim \beta,$$

provided there exists a formula γ in $\mathscr{F}$ such that

$$\alpha = \gamma(\zeta_1/\xi_1, \ldots, \zeta_n/\xi_n),$$

$$\beta = \gamma(\zeta_1/\eta_1, \ldots, \zeta_n/\eta_n)$$

(see the notation on p. 4) where $\xi_1, \ldots, \xi_n$, $\eta_1, \ldots, \eta_n$, $\zeta_1, \ldots, \zeta_n$ are bound individual variables. In other words, α and β are syntactically equivalent if and only if one of them can be obtained from the other by changing some bound variables.

E. g. if $\alpha(v)$ is a formula ($v \in V$), then formulas

(1) $$O\xi\alpha(\xi), \quad O\eta\alpha(\eta)$$

(where $O \in Q$, $\xi, \eta \in \overline{V}$, ξ, η do not appear in $\alpha(v)$) are syntactically equivalent, but formulas $\alpha(v)$ and $O\xi\alpha(\xi)$ are not. Similarly, if $\alpha(v_1, v_2)$ is a formula, then

$$O\xi O\eta\alpha(\xi, \eta), \quad O\eta O\xi\alpha(\eta, \xi)$$

are syntactically equivalent, but

$$O\xi O\eta\alpha(\xi, \eta), \quad O\eta O\xi\alpha(\xi, \eta)$$

are not.

It is easy to see that $\sim$ is an equivalence relation in $\mathscr{F}$. Let

(2) $$F = \mathscr{F}/\sim.$$

We recall (see p. 3) that F is composed of all equivalence classes $|\alpha|$ of the relation $\sim$ (α in $\mathscr{F}$).

We shall always consider the set F as a generalized algebra

(3) $$\{F, \{o\}_{o\in C}, \{O\}_{O\in Q}\}$$

with the following definition of operations:

(i) Every m-argument connective $o \in C_m$ ($m = 0, 1, 2, \ldots$) determines an m-argument operation in F, denoted by the same symbol o. The result $o(|\alpha_1|, \ldots, |\alpha_m|)$ of the operation o performed on some elements $|\alpha_1|, \ldots, |\alpha_m| \in F$ is the element $|o\alpha_1 \ldots \alpha_m| \in F$ determined by the formula $o\alpha_1 \ldots \alpha_m$. In symbols,

(4) $$o(|\alpha_1|, \ldots, |\alpha_m|) = |o\alpha_1 \ldots \alpha_m| \quad \text{for} \quad \alpha_1, \ldots, \alpha_m \quad \text{in } \mathscr{F}.$$

(ii) Every quantifier $O \in Q$ determines an infinite operation in F, denoted by the same symbol O. All these infinite operations O have the same domain $\mathfrak{D}$. A subset S of F is in the domain $\mathfrak{D}$ if and only if there exists a formula $\alpha(v)$ (with $v \in V$) such that S is composed of all the elements

(5) $$|\alpha(\tau)| \in F, \quad \text{where} \quad \tau \text{ is any term in } \mathscr{T}.$$

Roughly speaking, the only sets (5) admissible for O are sets of substitutions of all terms for a fixed individual variable v in a fixed formula $\alpha(v)$. The elements (5) of an admissible set $S \in \mathfrak{D}$ are indexed, in a natural way, by $\tau \in \mathscr{T}$. Therefore the result of an operation O ($O \in Q$) performed on the set S of all the elements (5) will be denoted by

$$O_{\tau\in\mathscr{T}}|\alpha(\tau)|$$

according to § 4 (1). The result is defined by the equality

(6) $$O_{\tau\in\mathscr{T}}|\alpha(\tau)| = |O\xi\alpha(\xi)|,$$

where ξ is any bound individual variable which does not appear in $\alpha(v)$. Thus the result is the element in F which is determined by the formula $O\xi\alpha(\xi)$ (see § 5 C)). The element $|O\xi\alpha(\xi)|$ does not depend on the choice of ξ (see (1)).

Algebra (3) will be called the *quantifier algebra of the formalized language* $\mathscr{L}$, or briefly the Q-*algebra of* $\mathscr{L}$.

Observe that if α is an elementary formula (i. e. is of the form § 5 (1)) and $\alpha \sim \beta$ for a formula $\beta \in \mathscr{F}$, then β is identical with α. Thus, if α is elementary, the equivalence class $|\alpha|$ contains only the formula α. In the sequel we shall not distinguish between α and $|\alpha|$ if α is elementary. The set of all elementary formulas in $\mathscr{F}$ will be denoted by $\mathscr{E}$. By the above convention,

$$\mathscr{E} \subset F.$$

The following theorem is an analogue of 2.1:

6.1. *The* Q-*algebra* (3) *is a generalized* $\mathfrak{K}$-*free algebra in the class* $\mathfrak{K}$ *of all complete algebras similar to* (3), *the set* $\mathscr{E}$ *of all elementary formulas being the set of* $\mathfrak{K}$-*free generators.*

The proof of 6.1 is similar to that of 2.1.

Observe that no analogue of 6.1 and 2.1 is valid for the algebra $\mathscr{F}$ of formulas of $\mathscr{L}$ (see § 5 (3)).

Observe also that no analogue of 2.2 holds for the Q-algebra F (it holds, however, for the algebra $\mathscr{F}$). This is a serious defect of the Q-algebra F, which causes some difficulties in applications of F to problems in Mathematical Logic. To avoid those difficulties (and also for some other reasons) we shall introduce in § 7 another kind of generalized algebras formed from formulas.

§ 7. The X-algebra of formulas. Let $\mathscr{L} = \{\mathscr{A}, \mathscr{T}, \mathscr{F}\}$ be the formalized language examined in §§ 5 and 6, and let X be a non-empty set. We shall form a new alphabet $\mathscr{A}_X$ and a new language $\mathscr{L}_X$ of the first order in the following way.

Take a fixed set X', disjoint with the sets $\mathscr{S}, \mathscr{T}, \mathscr{F}$ of signs, terms and formulas in $\mathscr{L}$, and such that the cardinals of X' and X are equal (if X is disjoint from $\mathcal{S}, \mathcal{T}, \mathcal{F}$, we can assume $X' = X$; however, in applications the set X can be identical with $\mathcal{T}$ — see § 9). Establish a one-to-one correspondence between elements in X and X'. If x is an element in X, then x' will denote the element in X', corresponding to x. Consequently elements in X' will be denoted by symbols x' where x denotes corresponding elements in X. The element x' is sometimes called the *name* of $x \epsilon X$.

The alphabet $\mathcal{A}_X$ mentioned above is obtained from the alphabet $\mathcal{A}$ by adding the set X' to the set Φ_0 of all individual constants in $\mathcal{A}$. $\mathcal{L}_X$ is the formalized language (of the first order) based on the alphabet $\mathcal{A}_X$. The sets of all terms and formulas in $\mathcal{L}_X$ will be denoted by $\mathcal{T}_X$ and $\mathcal{F}_X$ respectively. By definition, $\mathcal{L}_X = \{\mathcal{A}_X, \mathcal{T}_X, \mathcal{F}_X\}$.

As in § 6, we write

$$\alpha \sim \beta$$

for some formulas α, β in $\mathcal{F}_X$ if these formulas are syntactically equivalent. F_X will denote the set $\mathcal{F}_X/\!\sim$ of all equivalence classes $|\alpha|$ where $\alpha \epsilon \mathcal{F}_X$.

We shall always consider the set F_X as a generalized algebra

(1) $$\{F_X, \{o\}_{o \epsilon C}, \{O\}_{O \epsilon Q}\}$$

with the following definition of the operations:

(i) The definition of finite operations o is the same as in § 6 (i). Every m-argument connective $o \epsilon C_m$ $(m = 0, 1, 2, \ldots)$ determines an m-argument operation in F_X, denoted by the same symbol o and defined by the equality

(2) $$o(|\alpha_1|, \ldots, |\alpha_m|) = |o\alpha_1 \ldots \alpha_m| \quad \text{for} \quad \alpha_1, \ldots, \alpha_m \text{ in } \mathcal{F}_X.$$

(ii) Infinite operations O are defined in a slightly different way from those in § 6 (ii). Every quantifier $O \epsilon Q$ determines an infinite operation in F_X, denoted by the same symbol O. All these infinite operations O have the same domain $\mathfrak{D}$. A subset S of F_X is in the domain $\mathfrak{D}$ if and only if there exists a formula $\alpha(v)$ in F_X (with $v \epsilon V$) such that S is composed of all the elements

(3) $$|\alpha(x')| \epsilon F_X \quad \text{where} \quad x' \text{ is any element in } X'.$$

Roughly speaking, the only sets (3) admissible for O are sets of substitutions of all added individual constants (names) x' for a fixed variable v in a fixed formula $\alpha(v)$ in $\mathcal{F}_X$. The elements (3) of an admissible set $S \epsilon \mathfrak{D}$ are indexed, in a natural way, by $x' \epsilon X'$. Therefore the result of an operation O $(O \epsilon Q)$ performed on the set S of all the elements (3) we denote by

$$O_{x' \epsilon X'} |\alpha(x')|$$

according to § 4 (1). The result is defined by the equality

(4) $$O_{x' \epsilon X} |\alpha(x')| = |O\xi\alpha(\xi)|$$

where ξ is any bound individual variable which does not appear in $\alpha(v)$.

Definition (4) is correct in the case where the set X' (i. e. the set X) has at least two elements, since then set (3) determines uniquely the element on the right side of (4). If X' has only one element, say x_0', i. e. if X has only one element x_0, then some modifications or further explanations are necessary because (4) then depends on the way of interpretation of set (3) as an indexed set. For instance, for any $\pi \epsilon \Pi_2$, $|O\xi\pi\xi x_0'|$ is the result of the operation O performed on the one-element set $(\pi x_0' x_0')$ considered as the indexed set $\{\pi x' x_0'\}_{x' \epsilon X}$, and the element $|O\xi\pi x_0'\xi|$ is the result of the operation O performed on the same one-element set $(\pi x_0' x_0')$ considered now as the indexed set $\{\pi x_0' x'\}_{x' \epsilon X'}$. Since we do not intend to complicate the main idea by the particular case of one-element sets, we have given the definition in the above form. The reader may suppose in the sequel that X always has at least two elements.

Observe that the hypothesis of a fixed one-to-one correspondence between sets X' and X is not essential in this paper. It can be replaced by the hypothesis of a fixed transformation from X' onto X, which maps $x' \epsilon X'$ onto $x \epsilon X$ (some theorems will then need obvious modifications). Then if X has only one element, we may suppose that X' has at least two-elements. Then the above difficulty in definition (4) does not appear at all.

The generalized algebra (1) will be called the *X-algebra of the language $\mathcal{L}$*. Observe that the X-algebra of $\mathcal{L}$ does not coincide with the Q-algebra of $\mathcal{L}_X$ (see the difference between the admissible sets (3) and § 6 (5)). Both generalized algebras are similar to the Q-algebra F of $\mathcal{L}$.

The symbol $\mathcal{E}_X$ will denote the set of all elementary formulas in the language $\mathcal{L}_X$. By a convention similar to that on p. 17 we shall consider $\mathcal{E}_X$ as a subset of F_X.

The following theorem is an analogue of 6.1 and can be proved in the same way:

7.1. *The X-algebra* (1) *is a generalized $\mathfrak{K}$-free algebra in the class $\mathfrak{K}$ of all complete algebras similar to* (1), *the set $\mathcal{E}_X$ of all elementary formulas in $\mathcal{F}_X$ being the set of $\mathfrak{K}$-free generators.*

As in § 2, by a *substitution in $\mathcal{L}$* we shall understand every mapping $\mathfrak{s}: V \to \mathcal{T}$, and by a *substitution in $\mathcal{L}_X$* we shall understand any mapping $\mathfrak{s}: V \to \mathcal{T}_X$. Since every substitution in $\mathcal{L}$ is also a substitution in $\mathcal{L}_X$, it suffices to investigate substitutions in $\mathcal{L}_X$.

Let

(5) $$\mathfrak{s}: V \to \mathcal{T}_X$$

be a substitution. If α is a formula in $\mathcal{F}_X$, and $v_1, \ldots, v_n$ are all free individual variables in α, then $\mathfrak{s}'\alpha$ will denote the formula

$$\alpha(v_1/\mathfrak{s}v_1, \ldots, v_n/\mathfrak{s}v_n)$$

(see the notation on p. 4) where $\mathfrak{s}v$ is the value of the mapping $\mathfrak{s}$ at

$v \in V$. Denoting, more suggestively, α by $\alpha(v_1, \ldots, v_n)$ we can write the definition of $\mathfrak{s}'\alpha$ in the form of the equality

$$(6) \qquad \mathfrak{s}'\alpha(v_1, \ldots, v_n) = \alpha(\mathfrak{s}v_1, \ldots, \mathfrak{s}v_n).$$

By definition,

$$\mathfrak{s}': \mathcal{F}_X \to \mathcal{F}_X.$$

Moreover, if $\mathfrak{s}$ is a substitution in $\mathcal{L}$, then $\mathfrak{s}'$ maps $\mathcal{F}$ into $\mathcal{F}$.

The equality

$$(7) \qquad \mathfrak{s}''(|\alpha|) = |\mathfrak{s}'\alpha| \qquad (\alpha \text{ in } \mathcal{F}_X)$$

defines a mapping

$$(8) \qquad \mathfrak{s}'': \mathcal{F}_X \to \mathcal{F}_X.$$

The following theorem is an analogue of 2.2:

7.2. *For every substitution* (5), *mapping* (8) *is a homomorphism from the X-algebra of $\mathcal{L}$ into itself.*

The proof is by an easy verification.

Let Y be a non-empty set. In the same way we can form the language $\mathcal{L}_Y = \{\mathcal{A}_Y, \mathcal{T}_Y, \mathcal{F}_Y\}$ by adding a set Y', satisfying conditions mentioned above, to the set Φ_0 of individual constants in the language $\mathcal{L}$. If y is an element in Y, then y' denotes the corresponding element in Y' in the fixed one-to-one correspondence between Y and Y'.

Any mapping

$$f: X \to Y$$

induces two mappings:

$$(9) \qquad f': \mathcal{T}_X \to \mathcal{T}_Y,$$

$$(10) \qquad f^*: \mathcal{F}_X \to \mathcal{F}_Y.$$

The mappings (9) and (10) are defined as follows. If τ is any term in $\mathcal{T}_X$, and $x_1', \ldots, x_n'$ are all elements in X' which appears in α, then $f'\tau$ is the term in $\mathcal{T}_Y$:

$$\tau(x_1'/y_1', \ldots, x_n'/y_n'),$$

where $y_i = f(x_i)$ for $i = 1, \ldots, n$ (for notation, see p. 4). Similarly, if α in any formula in $\mathcal{F}_X$ and $x_1', \ldots, x_n'$ are all the elements in X' which appear in α, then $f^*\alpha$ is the formula

$$\alpha(x_1'/y_1', \ldots, x_n'/y_n')$$

where $y_i = f(x_i)$ for $i = 1, \ldots, n$.

The mapping f^* induces a mapping

$$(11) \qquad f^{**}: F_X \to F_Y$$

defined as follows:

$$(12) \qquad f^{**}(|\alpha|) = |f^*\alpha| \quad \text{for} \quad \alpha \in F_X.$$

Observe that

$$(13) \qquad f'\tau = \tau \quad \text{for} \quad \tau \in \mathcal{T},$$

$$(14) \qquad f^*\alpha = \alpha \quad \text{and} \quad f^{**}(|\alpha|) = |\alpha| \quad \text{for} \quad \alpha \in \mathcal{F}.$$

7.3. *For every mapping f from X into* (*onto*) Y, *mapping* (9) *is a homomorphism of the algebra $\mathcal{T}_X$ into* (*onto*) *the algebra $\mathcal{T}_Y$.*

7.4. *For every mapping f from X onto Y, mapping* (12) *is a homomorphism of the generalized algebra F_X onto the generalized algebra F_Y.*

The proof of 7.3 and 7.4 is by an easy verification. The hypothesis that f maps X onto Y in 7.4 ensures that f^{**} maps admissible sets onto admissible sets.

§ 8. Realizations. The interpretation of formulas as mappings. Let $\mathcal{L} = \{\mathcal{A}, \mathcal{T}, \mathcal{F}\}$ be the formalized language of the first order, described in § 5. The letter A will always denote a complete algebra similar to the Q-algebra F of $\mathcal{L}$. According to the convention on p. 18 we shall denote operations in A by the same symbols as the corresponding operations in F. Thus A is a complete algebra

$$(1) \qquad \{A, \{o\}_{o \in C}, \{O\}_{O \in Q}\}.$$

By a *realization of the language $\mathcal{L}$ in a non-void set X and in a complete algebra* (1) we shall understand any mapping R defined on $\Phi \cup \Pi$ such that

1° R assigns, to every m-argument functor $\boldsymbol{o} \in \Phi_m$ $(m = 0, 1, 2, \ldots)$, an m-argument operation $\boldsymbol{o}_R$ in X, i. e. a mapping

$$\boldsymbol{o}_R: X^m \to X.$$

2° R assigns, to every m-argument predicate $\pi \in \Pi_m$ $(m = 0, 1, 2, \ldots)$, an m-argument function π_R defined in X with values in A, i. e. a mapping

$$\pi_R: X^m \to A.$$

Every realization R of $\mathcal{L}$ in X turns out the set X into an abstract algebra, viz. the algebra

$$(2) \qquad \{X, \{\boldsymbol{o}_R\}_{o \in \Phi}\}.$$

Denote this algebra by X_R and consider the language $\mathcal{L}_0 = \{\mathcal{S}_0, \mathcal{T}\}$ of terms of $\mathcal{L}$ (see p. 15 and p. 6). This language being of the zero order, on account of § 3 every term $\tau \in \mathcal{T}$ determines uniquely a mapping

$$(3) \qquad \tau_{X_R}: X^V \to X.$$

We recall that elements $\mathfrak{v} = \{\mathfrak{v}_v\}_{v\in V} \in X^V$ are called *valuations in X*.

For brevity, we shall denote mapping (3) by τ_R. By definition

(4) $$\tau_R(\mathfrak{v}) = \tau_{X_R}(\mathfrak{v}) \in X \quad \text{for} \quad \mathfrak{v} \in X^V.$$

The notation τ_R is correct since R determines uniquely X and, consequently, X_R.

Given any realization R of $\mathscr{L}$ in a set $X \neq 0$ and in a complete algebra A (see (1)), every formula α in $\mathscr{F}$ determines uniquely a mapping

(5) $$\alpha_R \colon X^{V_\alpha} \to A,$$

where V_α is the set of all free individual variables appearing in α (if V_α is empty, then α_R is a constant element in A). To obtain α_R it suffices:

A) to interpret all free and bound variables in α as variables running through the set X;

B) to interpret every m-argument functor $\boldsymbol{o} \in \Phi_m$ $(m = 0, 1, 2, \ldots)$ appearing in α as the mapping $\boldsymbol{o}_R \colon X^m \to X$;

C) to interpret every m-argument predicate $\pi \in \Pi_m$ $(m = 0, 1, 2, \ldots)$ appearing in α as the mapping $\pi_R \colon X^m \to A$;

D) to interpret every m-argument connective $o \in C_m$ $(m = 0, 1, 2, \ldots)$ as the corresponding operation o in A;

E) to interpret every expression $O\xi$ (where $O \in Q$ and $\xi \in \overline{V}$) as the infinite operation $O_{\xi \in X}$ in A.

Every mapping $f \colon X^{V'} \to A$, where $V' = (v_1, \ldots, v_n)$ is a subset of V, can be interpreted as a mapping $f \colon X^V \to A$ on assuming

$$f(\mathfrak{v}) = f(\mathfrak{v}_{v_1}, \ldots, \mathfrak{v}_{v_n})$$

for every valuation $\mathfrak{v} = \{\mathfrak{v}_v\}_{v\in V}$. The element $f(\mathfrak{v})$ depends, of course, only on coordinates $\mathfrak{v}_{v_1}, \ldots, \mathfrak{v}_{v_n}$ of $\mathfrak{v}$.

In particular, mapping (5) can be interpreted as a mapping

(6) $$\alpha_R \colon X^V \to A.$$

In this paper we shall replace the above intuitive definition of (6) by another definition equivalent but more precise, (see (9) below) which will be used as the starting point for further investigations. To formulate this definition, it is more convenient to deal with the extended language $\mathscr{L}_X$ described in § 7, instead of $\mathscr{L}$, and to define α_R for all formulas α in $\mathscr{F}_X$.

We assume all notations from § 7, in particular X' is the set described on p. 17-18. The languages $\mathscr{L}$ and $\mathscr{L}_X$ have the same set Π of predicates. Their sets of functors are different: the set of functors of $\mathscr{L}_X$ was obtained from the set Φ of functors of $\mathscr{L}$ by adding the set X' of individual constants (names of elements $x \in X$). Consequently, every realization R of $\mathscr{L}$ in X and A determines, in a natural way, a corresponding realization R' of $\mathscr{L}_X$ in X and A on assuming

$$x'_{R'} = x \quad \text{for all} \quad x' \in X',$$
$$\boldsymbol{o}_{R'} = \boldsymbol{o}_R \quad \text{for all} \quad \boldsymbol{o} \in \Phi,$$
$$\pi_{R'} = \pi_R \quad \text{for all} \quad \pi \in \Pi.$$

We recall that x and x' are corresponding elements (in X and X' respectively) in the fixed one-to-one correspondence between sets X and X' (see p. 18).

For simplicity, the natural extension R' of R will be denoted, in the sequel, by the same letter R.

Every fixed valuation $\mathfrak{v} \in X^V$ determines uniquely a mapping

(7) $$\mathfrak{v}_R \colon \mathscr{E}_X \to A,$$

where $\mathscr{E}_X$ is the set of all elementary formulas in $\mathscr{L}_X$; viz. for every elementary formula $\pi\tau_1 \ldots \tau_m$ (see § 5 (1)) we define

$$\mathfrak{v}_R(\pi\tau_1 \ldots \tau_m) = \pi_R(\tau_{1R}(\mathfrak{v}), \ldots, \tau_{nR}(\mathfrak{v})),$$

where, for any term τ, $\tau_R(\mathfrak{v})$ is defined by (4), the language $\mathscr{L}$ being replaced in (4) by $\mathscr{L}_X$.

By 7.1, the mapping (7) can be uniquely extended to a homomorphism from F_X into A. Denote this homomorphism by the same symbol $\mathfrak{v}_R$, and its value at an element $|\sigma| \in F_X$ by

(8) $$\mathfrak{v}_R(|\alpha|).$$

For a fixed formula α in $\mathscr{F}_R$, we can consider element (8) as a function of $\mathfrak{v} \in X^V$. Denote this function by α_R. By definition

(9) $$\alpha_R(\mathfrak{v}) = \mathfrak{v}_R(|\alpha|)$$

for $\alpha \in \mathscr{F}_X$ and $\mathfrak{v} \in X^V$. The mapping

(10) $$\alpha_R \colon X^V \to A$$

just defined coincides, in the case where $\alpha \in \mathscr{F} \subset \mathscr{F}_X$, with mapping (6). For arbitrary $\alpha \in \mathscr{F}_X$ definition (10) is equivalent to definition (6) applied to the language $\mathscr{L}_X$ instead of $\mathscr{L}$. We omit the easy proof of the equivalence of (6) and (10) since we shall use, in the sequel, only definition (6).

Example 5. Let $\mathscr{L}$ be the formalized language of the first order described in § 5, Example 4. If A is a complete Boolean algebra, let **N**, **D**, **C**, **I** be operations defined in § 3 Example 2, let $\vee$ be the unit element of A and let **E** and **U** be the infinite join and the infinite meet in A. Then, for every formula α in $\mathscr{F}$, the following statements are equivalent [4]: (i) α is a tautology in the predicate calculus $\mathscr{L}$; (ii) α_R is identically equal to $\vee$ for every interpretation R of $\mathscr{L}$ in every Boolean algebra A

[4] See Rasiowa and Sikorski [1], [3].

and in every set $X \neq 0$; (iii) α_R is identically equal to $\vee$ for every realization R in a fixed complete Boolean algebra A (having at least two elements) and in a fixed infinite set X; (iv) α_R is identically equal to $\vee$ for every realization R in the two-element Boolean algebra and in a fixed infinite set X. The equivalence of (i) and (iv) is the predicate analogue of the truth-table method for propositional calculi mentioned in § 3, Example 2.

Fundamental properties of the mapping α_R are formulated in theorems 8.1, 8.2, 8.4 below. In applications only the case where α is in $\mathscr{F}$ is important for the investigation of the language $\mathscr{L}$. However, for purely technical reasons, it is more convenient to prove them for all formulas α in $\mathscr{F}_X$.

Let

$$\{B, \{o\}_{o\in C}, \{O\}_{O\in Q}\}$$

be another complete algebra, similar to the Q-algebra F, and let $h: A \to B$ be a homomorphism. If R is a realization of $\mathscr{L}$ in X and A, then the following equalities define another realization (denoted by hR) of $\mathscr{L}$ in X and B:

(11) $$\mathbf{o}_{hR} = \mathbf{o}_R \quad \text{for all} \quad \mathbf{o} \in \Phi, \qquad \pi_{hR} = h\pi_R \quad \text{for all} \quad \pi \in \Pi.$$

Thus R and hR coincides on the set of functors. Consequently

(12) $$\tau_R(v) = \tau_{hR}(v) \quad \text{for every} \quad \tau \in T \text{ and } v \in X^V.$$

If π is an m-argument predicate, then π_{hR} is the superposition of h and π_R:

$$\pi_{hR}(x_1, \ldots, x_m) = h(\pi_R(x_1, \ldots, x_m)) \quad \text{for} \quad x_1, \ldots, x_m \in X.$$

8.1. *For every formula $\alpha \in \mathscr{F}_X$ and every valuation $\mathfrak{v} \in X^V$*

(13) $$\alpha_{hR}(\mathfrak{v}) = h(\alpha_R(\mathfrak{v})).$$

For every fixed valuation $\mathfrak{v}$, both sides of (13), when considered as function of $|\alpha| \in F_X$, are homomorphisms from F_X into B on account of (1) and 4.1. By (11) and (12) those homomorphisms have the same value if α is an elementary formula. Since the set $\mathscr{E}_X$ of all elementary formulas in $\mathscr{F}_X$ generates the algebra F_X, both homomorphisms are equal by 4.2.

For every substitution $\mathfrak{s}$ in $\mathscr{L}_X$, let $\mathfrak{s}'$ have the meaning defined in § 7 (6). For every valuation $\mathfrak{v} \in X^V$, let $\mathfrak{s}_R\mathfrak{v}$ be the valuation $\{\mathfrak{s}v_R(\mathfrak{v})\}_{v\in V}$. Thus $\mathfrak{s}_R\mathfrak{v}$ is an abbrevation for $\mathfrak{s}_{X_R}\mathfrak{v}$ in the sense defined in § 3 (6).

8.2. *For every formula α in $\mathscr{F}_X$,*

(14) $$\alpha_R(\mathfrak{s}_R\mathfrak{v}) = \mathfrak{s}'\alpha_R(\mathfrak{v}).$$

Obviously the symbol on the right side of (14) denotes the value of the mapping $\mathfrak{s}'\alpha_R$, determined by the formula $\mathfrak{s}'\alpha$, at the point $\mathfrak{v}$.

To prove 8.2, consider both sides of (14) as functions of $|\alpha| \in F_X$, the valuation $\mathfrak{v}$ being fixed. By (10), 7.2 and 4.1 these functions are homomorphisms from the generalized algebra F_X into the complete algebra A. By 3.2, these homomorphisms have the same value if α is an elementary formula. Since the set $\mathscr{E}_X$ of all elementary formulas generates F_X, both homomorphisms are equal on account of 4.2.

Let R be, as before, a realization of $\mathscr{L}$ in X and A, and let R_0 be another realization of $\mathscr{L}$ in a set $Y \neq 0$ and in the same algebra A. Suppose that f is a mapping from X onto Y such that

(15) $$f(\mathbf{o}_R(x_1, \ldots, x_m)) = \mathbf{o}_{R_0}(f(x_1), \ldots, f(x_m)),$$

(16) $$\pi_R(x_1, \ldots, x_m) = \pi_{R_0}(f(x_1, \ldots, x_m))$$

for every $\mathbf{o} \in \Phi_m$, $\pi \in \Pi_m$, $x_1, \ldots, x_m \in X$ $(m = 0, 1, 2, \ldots)$. For every valuation $\mathfrak{v}$ in X, $f\mathfrak{v}$ denotes the superposition of $\mathfrak{v}: V \to X$ and $f: X \to Y$, i. e. $f\mathfrak{v}$ is the corresponding valuation in Y.

Condition (15) means that f is a homomorphism from the algebra X_R into the algebra Y_{R_0} (see p. 21). Thus, by 3.1 (where $h = f$, and $A_0 = X_R$, $B_0 = Y_{R_0}$),

(17) $$\tau_{R_0}(f\mathfrak{v}) = f(\tau_R(\mathfrak{v}))$$

for every term $\tau \in \mathscr{T}$. More generally, using the notation § 7 (9), (10), (11), (12), we have

8.3. *For every term τ in $\mathscr{T}_X$,*

(18) $$f'\tau_{R_0}(f\mathfrak{v}) = f(\tau_R(\mathfrak{v})).$$

Obviously, the symbol on the left side of (18) denotes the value of the mapping $f'\tau_{R_0}$, determined by the term $f'\tau \in \mathscr{T}_Y$), at the point $f\mathfrak{v} \in Y^V$.

To prove (18) let us observe that both sides of (18), when interpreted as functions of $\tau \in \mathscr{T}_X$, are homomorphisms from $\mathscr{T}_X$ into Y_R on account of (15), 4.1, § 3 (4), and 7.3. They coincide on the set V of all individual variables which generates $\mathscr{T}_X$. By 4.2, they are equal.

Observe that (17) is an immediate consequence of (18) on account of § 7 (13).

8.4. *For every formula $\alpha \in \mathscr{F}_X$,*

(19) $$f^*\alpha_{R_0}(f\mathfrak{v}) = \alpha_R(\mathfrak{v}).$$

In particular, for every $\alpha \in \mathscr{F}$,

(20) $$\alpha_{R_0}(f\mathfrak{v}) = \alpha_R(\mathfrak{v}).$$

The symbol on the left side of (19) denotes, obviously, the value of the mapping $f^*\alpha_{R_0}$, determined by the formula $f^*\alpha$, at the point $f\mathfrak{v} \in Y^V$.

To prove (19), observe that both sides of (19), when interpreted as functions of $|\alpha| \epsilon F_X$, are homomorphisms from F_X into A on account of (9), 4.1 and 7.4. By (18) and (16), they coincide on the set $\mathscr{E}_X$ of elementary formulas which generates F_X. By 4.2, both homomorphisms are equal.

(20) follows immediately from (19) on account of § 7 (14).

In the definition of realizations we have assumed that the generalized algebra A in question is always complete. The purpose of this hypothesis was to ensure that all infinite operations appearing in calculation of $\alpha_R(\mathfrak{v})$ are feasible. Without this hypothesis we cannot state, in the general case, that mapping (7) has a homomorphic extension. However, it can happen, for some special mappings R satisfying 1° and 2°, that (7) has a homomorphic extension in spite of the fact that the generalized algebra A in question is not complete. In this case, the mapping R satisfying 1° and 2° will also be called a *realization of $\mathscr{L}$ in X and A*, and formula (9) yields a definition of α_R.

The above extension of the notion of realization and the mapping α_R is, in a certain sense, not essential. In fact, we have seen on p. 13 that every generalized algebra A can be extended to a complete similar algebra, say A'. In this way, every realization R in the incomplete algebra A can be considered as a realization in the complete algebra A'. In other words, every realization in the sense just defined can be interpreted as a realization in the sense defined on p. 23. However, if only elements in A are values of α_R, and only infinite operations in A are used to calculate $\alpha_R(\mathfrak{v})$, there is no reason to introduce any completion A' of A, and it is more natural to consider R as realization in A.

The fact that any realization in an incomplete algebra A can be conceived as a realization in a complete extension A' of A has important consequences because it enables us to apply to realizations in incomplete algebras all theorems proved for realizations in complete algebras.

Sometimes it is convenient to assume as A' the extension defined in § 4 (7) (p. 13). This extension A' is obtained from A by adding a new element a_0 to A and by assuming a_0 as the value of all finite or infinite operations, except the case where the operation is feasible in A. Hence it follows that if R is any realization of $\mathscr{L}$ in X and A', and $\alpha_R(\mathfrak{v}) \epsilon A$ for every formula α and every valuation $\mathfrak{v}$, then R is a realization of $\mathscr{L}$ in X and A.

Some special realizations in incomplete algebras A will be examined in § 9. The reader can always assume that, in all reasonings, the incomplete algebra A is replaced by the complete algebra A' just mentioned.

§ 9. Realizations in the set of terms. Let $\mathscr{L} = \{\mathscr{A}, \mathscr{T}, \mathscr{F}\}$ be the formalized language (of the first order) described in § 5.

We shall first prove some theorems on the auxiliary language $\mathscr{L}_{\mathscr{T}}$, i. e. the language $\mathscr{L}_X$, where X is the set $\mathscr{T}$ of all terms in $\mathscr{L}$. The language $\mathscr{L}_{\mathscr{T}}$ is based on the alphabet $\mathscr{A}_{\mathscr{T}}$ obtained by adding a set $\mathscr{T}'$ of individual constants (names of terms) to the alphabet $\mathscr{A}$ of the language $\mathscr{L}$. The set $\mathscr{T}'$ is disjoint with $\mathscr{S}$, $\mathscr{T}$ and $\mathscr{F}$, and there is a fixed one-to-one correspondence between all elements in $\mathscr{T}$ and all elements in $\mathscr{T}'$. For any term τ in $\mathscr{T}$, τ' denotes the corresponding element in $\mathscr{T}'$, and conversely.

According to the notation adopted on p. 8, $\mathscr{T}_{\mathscr{T}}$ is the set of all terms in $\mathscr{L}_{\mathscr{T}}$, and $\mathscr{F}_{\mathscr{T}}$ is the set of all formulas in $\mathscr{L}_{\mathscr{T}}$. By definition,

$$\mathscr{L}_{\mathscr{T}} = \{\mathscr{A}_{\mathscr{T}}, \mathscr{T}_{\mathscr{T}}, \mathscr{F}_{\mathscr{T}}\}.$$

$F_{\mathscr{T}}$ will denote the corresponding $\mathscr{T}$-algebra of the language $\mathscr{L}$, i. e. the X-algebra F_X where $X = \mathscr{T}$ (see § 7, p. 18). Of course

$$\mathscr{T} \subset \mathscr{T}_{\mathscr{T}} \quad \text{and} \quad F \subset F_{\mathscr{T}}.$$

The letter σ will denote arbitrary terms in $\mathscr{T}_{\mathscr{T}}$.

If $\sigma \epsilon \mathscr{T}_{\mathscr{T}}$ and the sequence $\tau_1', \ldots, \tau_n'$ is composed of all individual constants from $\mathscr{T}'$ which appear in σ, then the equality

$$f(\sigma) = \sigma(\tau_1'/\tau_1, \ldots, \tau_n'/\tau_n) \tag{1}$$

defines a term $\tau = f(\sigma)$ in $\mathscr{L}$ (for notation, see p. 4).

9.1. *The mapping*

$$f: \mathscr{T}_{\mathscr{T}} \to \mathscr{T}$$

defined by (1) *is a homomorphism of the algebra* $\{\mathscr{T}_{\mathscr{T}}, \{o\}_{o \epsilon \Phi}\}$ *of terms of the language* $\mathscr{L}_{\mathscr{T}}$ *onto the algebra* $\{\mathscr{T}, \{o\}_{o \epsilon \Phi}\}$ *of terms of the language* $\mathscr{L}$. *Moreover,*

$$f(\tau) = \tau \text{ for every term } \tau \text{ in } \mathscr{T}. \tag{2}$$

Similarly, if α is a formula in $\mathscr{L}_{\mathscr{T}}$, and the sequence $\tau_1', \ldots, \tau_n'$ contains all individual constants from $\mathscr{T}'$ which appear in α, then the equality

$$g(\alpha) = \alpha(\tau_1'/\tau_1, \ldots, \tau_n'/\tau_n)$$

defines a formula $\gamma = g(\alpha)$ in $\mathscr{F}$. The mapping $g: \mathscr{F}_{\mathscr{T}} \to \mathscr{F}$ just defined has the following property: if $\alpha \sim \beta$, then $g(\alpha) \sim g(\beta)$. Consequently, the mapping g induces a mapping $h_0: F_{\mathscr{T}} \to F$, viz.

$$h_0(|\alpha|) = |g(\alpha)| \tag{3}$$

for α in $\mathscr{F}_{\mathscr{T}}$.

9.2. *The mapping h_0 defined by* (3) *is a homomorphism of the $\mathscr{T}$-algebra* $\{F_{\mathscr{T}}, \{o\}_{o \epsilon \mathcal{O}}, \{O\}_{O \epsilon Q}\}$ *onto the* Q*-algebra* $\{\mathscr{F}, \{o\}_{o \epsilon \mathcal{O}}, \{O\}_{O \epsilon Q}\}$. *Moreover,*

$$h_0(|\pi\sigma_1 \ldots \sigma_m|) = |\pi f(\sigma_1) \ldots f(\sigma_m)| \tag{4}$$

for every elementary formula in $\mathscr{L}_{\mathscr{T}}$ $(\pi \in \Pi_m, \sigma_1, \ldots, \sigma_m \in \mathscr{T}_{\mathscr{T}})$, *and*

(5) $$h_0(|\alpha|) = |\alpha| \textit{ for every } \alpha \textit{ in } \mathscr{F}.$$

The proof of 9.1 and 9.2 is by an easy verification.

In the next theorem the set $\mathscr{T}_{\mathscr{T}}$ will play the role of the set X in realizations discussed in § 8. Thus by a valuation we shall now mean any mapping $\mathfrak{v}\colon V \to \mathscr{T}_{\mathscr{T}}$. In particular, every mapping $\mathfrak{v}\colon V \to \mathscr{T}$ is a realization.

Every realization is now simultaneously a substitution in $\mathscr{L}_{\mathscr{T}}$. Consequently valuations will be denoted by the letter $\mathfrak{s}$. For every term τ in $\mathscr{T}_{\mathscr{T}}$ and every formula α in $\mathscr{F}_{\mathscr{T}}$, the symbols $\mathfrak{s}\tau$, $\mathfrak{s}'\alpha$ and $\mathfrak{s}''(|\alpha|)$ have the meaning defined § 2 (4), (5), and § 7 (6), (7) (where everything is relativized to the language $\mathscr{L}_{\mathscr{T}}$ and the corresponding language of terms), i. e. they are results of the substitution $\mathfrak{s}$ in τ and α respectively.

The letter $\mathfrak{i}$ will denote the identity mapping from V onto itself, interpreted as a mapping $\mathfrak{i}\colon V \to \mathscr{T}$ or a mapping $\mathfrak{i}\colon V \to \mathscr{T}_{\mathscr{T}}$, i. e. as a substitution or a valuation.

Now let k be a homomorphism of the $\mathscr{T}$-algebra $\{F_{\mathscr{T}}, \{o\}_{o \in C}, \{O\}_{O \in Q}\}$ into a similar generalized algebra $\{A, \{o\}_{o \in C}, \{O\}_{O \in Q}\}$. Let R be a mapping defined on the set $\Phi \cup \mathscr{T}' \cup \Pi$ of all functors and predicate in $\mathscr{L}_{\mathscr{T}}$ as follows:

a) for every m-argument functor $\mathbf{o} \in \Phi_m$ $(m = 0, 1, 2, \ldots)$, $\mathbf{o}_R$ is the corresponding operation in the algebra $\mathscr{T}_{\mathscr{T}}$ of terms of $\mathscr{L}_{\mathscr{T}}$, i. e.

$$\mathbf{o}_R(\sigma_1, \ldots, \sigma_m) = \mathbf{o}\sigma_1 \ldots \sigma_m$$

for $\sigma_1, \ldots, \sigma_m \in \mathscr{T}_{\mathscr{T}}$; moreover,

$$\tau'_R = \tau$$

for every $\tau' \in \mathscr{T}'$.

b) for every m-argument predicate $\pi \in \Pi_m$ $(m = 0, 1, 2, \ldots)$, π_R is the function:

$$\pi_R(\sigma_1, \ldots, \sigma_m) = k(|\pi\sigma_1 \ldots \sigma_m|) \in A.$$

for any terms $\sigma_1, \ldots, \sigma_m$ in $\mathscr{T}_{\mathscr{T}}$.

9.3. *R is a realization of the language $\mathscr{L}_{\mathscr{T}}$ in the set $\mathscr{T}_{\mathscr{T}}$ of all terms of $\mathscr{L}_{\mathscr{T}}$ and in the generalized algebra A. Moreover, for every formula α in $\mathscr{F}_{\mathscr{T}}$ and for every valuation $\mathfrak{s}$,*

(6) $$\alpha_R(\mathfrak{s}) = k(|\mathfrak{s}'\alpha|).$$

In particular,

(7) $$\alpha_R(\mathfrak{i}) = k(|\alpha|).$$

The right side of (6) can be written in the form $k(\mathfrak{s}''(|\alpha|))$. Thus both sides of (6), when interpreted as functions of $|\alpha| \in \mathscr{F}_{\mathscr{T}}$, are homomorphisms from $F_{\mathscr{T}}$ into A, on account of § 8 (9), 4.1 and 7.2. By b), § 8 (7), and 3.2 (applied to the language of terms in $\mathscr{L}_{\mathscr{T}}$), equality (6) holds for every elementary formula α. Since the set of all elementary formulas generates the $\mathscr{T}$-algebra $F_{\mathscr{T}}$, both homomorphisms have the same values for all formulas α.

(7) follows immediately from (6).

The following modification of 9.3 is important in applications.

Let $\approx$ be an equivalence relation in the set $\mathscr{F}$ of all formulas of the language $\mathscr{L}$ such that

(8) $$\text{if } \alpha \sim \beta, \text{ then } \alpha \approx \beta.$$

Thus the equality

(9) $$h_1(|\alpha|) = \|\alpha\| \in \mathscr{F}/\approx \qquad (\alpha \in \mathscr{F})$$

defines a mapping h_1 from $F = \mathscr{F}/\sim$ onto $\mathscr{F}/\approx$. Suppose that with every $o \in C_m$, there is associated an m-argument operation in $\mathscr{F}/\approx$, denoted by the same letter o $(m = 0, 1, 2, \ldots)$, and that with every $O \in Q$ there is associated an infinite operation in $\mathscr{F}/\approx$, denoted by the same letter O. Suppose moreover that

(10) h_1 is a homomorphism from the Q-algebra $\{F, \{o\}_{o \in C}, \{O\}_{O \in Q}\}$ into the generalized algebra $\{\mathscr{F}/\approx, \{o\}_{o \in C}, \{O\}_{O \in Q}\}$.

The set $\mathscr{T}$ of all terms in $\mathscr{L}$ will now play the role of the set X in which realizations are defined. Thus by a valuation we shall now mean any mapping $\mathfrak{s}\colon V \to \mathscr{T}$ which is simultaneously a substitution in $\mathscr{L}$.

Let h be a homomorphism from $\{\mathscr{F}/\approx, \{o\}_{o \in C}, \{O\}_{O \in Q}\}$ into a similar generalized algebra $\{A, \{o\}_{o \in C}, \{O\}_{O \in Q}\}$. Let R_0 be a mapping defined on the set $\Phi \cup \Pi$ of all functors and predicates in $\mathscr{L}$ as follows:

A) for every m-argument functor $o \in C_m$ $(m = 0, 1, 2, \ldots)$, o_{R_0} is the corresponding operation in the algebra $\mathscr{T}$ of terms in $\mathscr{L}$, i. e.

$$o_{R_0}(\tau_1, \ldots, \tau_m) = o\tau_1 \ldots \tau_m \quad \text{for} \quad \tau_1, \ldots, \tau_m \in \mathscr{T};$$

B) For every m-argument predicate $\pi \in \Pi_m$, π_{R_0} is the function

$$\pi_{R_0}(\tau_1, \ldots, \tau_m) = h(\|\pi\tau_1 \ldots \tau_m\|).$$

Under the above hypotheses,

9.4. *R_0 is a realization of $\mathscr{L}$ in the set $\mathscr{T}$ of all terms and in the generalized algebra A. Moreover, for every formula α in $\mathscr{F}$ and every $\mathfrak{s}\colon V \to \mathscr{T}$,*

(11) $$\alpha_{R_0}(\mathfrak{s}) = h(\|\mathfrak{s}'\alpha\|).$$

In particular, for the identity valuation $\mathfrak{i}$,

(12) $$\alpha_{R_0}(\mathfrak{i}) = h(\|\alpha\|).$$

Let h_0 be the homomorphism from the $\mathcal{T}$-algebra F_T onto the Q-algebra F, defined by (3). The mapping k:

$$k(|a|) = h\big(h_1(h_0(|a|))\big) \quad \text{for} \quad a \text{ in } \mathcal{F}_{\mathcal{T}}$$

is a homomorphism from $F_{\mathcal{T}}$ into A. Let R be the realization (of $\mathcal{L}_{\mathcal{T}}$ in the set $\mathcal{T}_{\mathcal{T}}$ and the algebra A) defined by a) and b). It follows from (6) that

$$a_R(\mathfrak{s}) = k(|\mathfrak{s}'a|)$$

for every a in $\mathcal{L}$ and every mapping $\mathfrak{s}: V \to \mathcal{T}$. However,

$$h_0(|\mathfrak{s}'a|) = |\mathfrak{s}'a| \in F$$

by (5) since $\mathfrak{s}'a \in F$. Hence

$$a_R(\mathfrak{s}) = h(||\mathfrak{s}'a||)$$

by the definition of k and h_1.

To complete the proof of (11) it suffices to show that

(13) $$a_{R_0}(\mathfrak{s}) = a_R(\mathfrak{s}).$$

This follows from 8.4 where X is the set $\mathcal{T}_{\mathcal{T}}$, Y is the set $\mathcal{T}$ and f is the mapping (1) (see 9.1). As the realization R in 8.4 we take the realization R just defined, but restricted to the language $\mathcal{L}$, and as the realization R_0 we take the realization R_0 defined by A) and B).

Hypothesis § 8 (15) of 8.4 is satisfied by 9.1. Hypothesis § 8 (16) of 8.4 is also satisfied since, for any terms $\sigma_1, \ldots, \sigma_m \in \mathcal{T}_{\mathcal{T}}$ and any $\pi \in \Pi_m$, we have

$$\pi_{R_0}\big(f(\sigma_1), \ldots, f(\sigma_m)\big) = h\big(||\pi f(\sigma_1) \ldots f(\sigma_m)||\big)$$
$$= h\big(h_1(h_0(|\pi\sigma_1 \ldots \sigma_m|))\big) = k(|\pi\sigma_1 \ldots \sigma_m|) = \pi_R(\sigma_1, \ldots, \sigma_m)$$

by 9.2 (4). Hence, by 8.4, $a_{R_0}(f\mathfrak{s}) = a_R(\mathfrak{s})$. Since $\mathfrak{s}: V \to \mathcal{T}$, we have $f\mathfrak{s} = \mathfrak{s}$ by (2). This proves (13).

(12) follows immediately from (11).

Example 6. Let $\mathcal{L}$ be the formalized language described in § 5 Example 4, i. e. the language of a predicate calculus. Write $a \approx \beta$ if and only if both $\mathbf{I}a\beta$ and $\mathbf{I}\beta a$ are tautologies. Thus $A/\approx$ is an incomplete Boolean algebra with operations defined in a natural way. Hypotheses (9) and (10) are satisfied. Theorem 9.4 is the essential point in the algebraic proof of the Gödel completeness theorem and of other theorems on the existence of models for consistent theories ([5]).

REFERENCES

P. R. Halmos [1] *Polyadic algebras*, Proceedings of the National Academy of Sciences, USA, 40 (1954), p. 296-301.

— [2] *Algebraic logic, I. Monadic algebras*, Compositio Mathematica 12 (1955), p. 217-249; *II. Homogeneous locally finite polyadic algebras of infinite degree*, Fundamenta Mathematicae 43 (1956), p. 255-325; *III. Predicates, terms and operations in polyadic algebras*, Transactions of the American Mathematical Society 83 (1956), p. 430-470; *IV. Equality in polyadic algebras*, ibidem 86 (1957), p. 1-27.

— [3] *The basic concepts of algebraic logic*, American Mathematical Monthly 63 (1956), p. 365-387.

L. Henkin [1] *Boolean representation through propositional calculus*, Fundamenta Mathematicae 41 (1955), p. 89-96.

— [2] *The representation theorem for cylindrical algebras*, *Mathematical interpretation of formal systems*, Amsterdam 1955.

J. C. C. McKinsey [1] *A solution of the decision problem for the Lewis system S2 and S4 with an application to topology*, Journal of symbolic Logic 4 (1939), p. 115-158.

— and A. Tarski [1] *The algebra of topology*, Annals of Mathematics 45 (1944), p. 141-191.

— [2] *On closed elements in closure algebras*, ibidem 47 (1946), p. 122-162.

— [3] *Some theorems about the sentential calculi of Lewis and Heyting*, Journal of Symbolic Logic 13 (1948), p. 1-15.

J. Łoś [1] *The algebraic treatment of the methodology of elementary deductive systems*, Studia Logica 2 (1955), p. 151-212.

A. Mostowski [1] *Proofs of nondeducibility in intuitionistic logic*, Journal of Symbolic Logic 13 (1948), p. 204-207.

H. Rasiowa [1] *Algebraic treatment of the functional calculi of Heyting and Lewis*, Fundamenta Mathematicae 38 (1951), p. 99-126.

— [2] *Algebraic models of axiomatic theories*, ibidem 41 (1955), p. 291-310.

— [3] *On the ε-theorems*, ibidem 43 (1956); *Errata*, ibidem 44 (1957), p. 333.

H. Rasiowa and R. Sikorski [1] *A proof of the completeness theorem of Gödel*, ibidem 37 (1950), p. 193-200.

— [2] *A proof of the Skolem-Löwenheim theorem*, ibidem 38 (1951), p. 230-232.

— [3] *Algebraic treatment of the notion of satisfiability*, ibidem 40 (1953), p. 62-96.

— [4] *On existential theorems in non-classical functional calculi*, ibidem 41 (1954), p. 21-28.

L. Rieger [1] *On the lattice theory of Brouwerian propositional logic*, Spisy Vydávané Přírodovědeckou Fakultou University Karlovy 189 (1949), p. 1-40.

— [2] *On countable generalised σ-algebras, with a new proof of Gödel's completeness theorem*, Czechoslovak Mathematical Journal 1 (1951), p. 29-40.

A. Robinson [1] *Théorie mathématique des idéaux*, Paris 1955.

R. Sikorski [1] *Some applications of interior mappings*, Fundamenta Mathematicae 45 (1958), p. 200-212.

M. H. Stone [1] *Topological representation of distributive lattices and Brouwerian logics*, Časopis pro pěstovani Matematiky a Fysiky 67 (1937), p. 1-25.

A. Tarski [1] *A representation theorem for cylindrical algebras*, Bulletin of the American Mathematical Society 58 (1952), p. 65-66.

A. Tarski and F. B. Thompson [1] *Some general properties of cylindrical algebras*, ibidem 58 (1952), p. 65.

Reçu par la Rédaction le 5. 2. 1961

([5]) See Rasiowa and Sikorski [2], [3].