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REPRESENTATION THEORY OF TWO-DIMENSIONAL BRAUER GRAPH RINGS

BY

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Dedicated to Professor Klaus Roggenkamp
on the occasion of his 60th birthday

Abstract. We consider a class of two-dimensional non-commutative Cohen–Macaulay rings to which a *Brauer graph*, that is, a finite graph endowed with a cyclic ordering of edges at any vertex, can be associated in a natural way. Some orders Λ over a two-dimensional regular local ring are of this type. They arise, e.g., as certain blocks of Hecke algebras over the completion of $\mathbb{Z}[q, q^{-1}]$ at $(p, q-1)$ for some rational prime p. For such orders Λ, a class of indecomposable maximal Cohen–Macaulay modules (see introduction) has been determined by K. W. Roggenkamp. We prove that this list of indecomposables of Λ is complete.

Introduction. For a rational prime p, and a finite Coxeter group G, let $\mathcal{H}_G$ denote the Hecke algebra of G over $\mathbb{Z}[q, q^{-1}]$ (see [5, I, §11], [8], [9]), and $\mathcal{H} := \mathbb{Z}[q, q^{-1}]_{\mathfrak{m}} \otimes \mathcal{H}_G$ be its completion at the maximal ideal $\mathfrak{m} := (p, q-1)$ of $\mathbb{Z}[q, q^{-1}]$. Then modulo $(q-1)$, every block B of $\mathbb{Z}_p G$ lifts to a block $\mathcal{B}$ of $\mathcal{H}$. In particular, if B is of cyclic defect with Brauer tree T, then $\mathcal{B}$ is a tree order with respect to T over the two-dimensional ring $\mathbb{Z}[q, q^{-1}]_{\mathfrak{m}}$ (cf. [17, Theorem 3.2]). A more general case, *Brauer graph orders* Λ over a two-dimensional regular domain R, has been investigated in [16]. These R-orders Λ can be described as follows. In case Λ is basic, there is an overorder $\Gamma = \Gamma_1 \times \ldots \times \Gamma_r$ of Λ with

$$\Gamma_i := \begin{pmatrix} \Omega_i & \Pi_i & \cdots & \Pi_i \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \Pi_i \\ \Omega_i & \cdots & \cdots & \Omega_i \end{pmatrix}$$

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where Ω_i are local Cohen–Macaulay orders, and Π_i is an invertible ideal in Ω_i (cf. [19, §3]). Hence

$$J_i := \begin{pmatrix} \Pi_i & \cdots & \Pi_i \\ \Omega_i & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ \Omega_i & \cdots\ \Omega_i & \Pi_i \end{pmatrix}$$

is an invertible ideal of Γ_i such that $\Gamma_i/J_i = \Omega_i/\Pi_i \times \ldots \times \Omega_i/\Pi_i$. Now Λ is determined by the inclusions $J := J_1 \times \ldots \times J_r \subseteq \Lambda \subseteq \Gamma$ with $\Lambda/J = \overline{\Lambda}_1 \times \ldots \times \overline{\Lambda}_s$, where each $\overline{\Lambda}_k$ is a diagonal in some $\Omega_i/\Pi_i \times \Omega_j/\Pi_j$ (only if $\Omega_i/\Pi_i \cong \Omega_j/\Pi_j$) such that Γ/J has exactly $2s$ factors of the form Ω_i/Π_i. The *Brauer graph* of Λ is obtained as follows. If Γ_i are interpreted as vertices, and the (cyclically ordered) factors Ω_i/Π_i of each Γ_i/J_i as germs of edges connected with Γ_i, then the edges are given by the diagonals $\overline{\Lambda}_k \subseteq \Omega_i/\Pi_i \times \Omega_j/\Pi_j$.

Let K be the quotient field of R. In the case where Λ is of *prime defect type*, i.e. when the Ω_i/Π_i are hereditary orders in a skew field, K. W. Roggenkamp [16] determined the indecomposable maximal Cohen–Macaulay modules M over Λ which admit a *Cohen–Macaulay filtration*. This means that there is a decomposition $K\Lambda = A_1 \times \ldots \times A_n$ of the K-algebra $K\Lambda = K \otimes_R \Lambda$ into blocks A_i such that the factor modules M_i/M_{i-1} of the corresponding filtration $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ of M with $M_i := M \cap (A_1 \oplus \ldots \oplus A_i)M$ are Cohen–Macaulay modules. The question remained open whether these indecomposables M exhaust the totality of indecomposable maximal Cohen–Macaulay modules over Λ.

In the present note, we prove this statement (see Theorem of §3). Notice that Λ does not represent an isolated singularity in the sense of Auslander [1]. In fact, in the above notation, the ideal $P \supseteq J$ of Λ with $P/J = \overline{\Lambda}_1 \times \ldots \times \overline{\Lambda}_{s-1}$ is prime, and the localization $\Lambda_{(P\cap R)}$ is not hereditary. Hence there is no Auslander–Reiten quiver of all indecomposable maximal Cohen–Macaulay modules. Nevertheless, by a result of [3], it is possible to give a partial Auslander–Reiten quiver (see §4), consisting of the indecomposable maximal Cohen–Macaulay modules M with $M_{\mathfrak{p}}$ projective or injective for all prime ideals $\mathfrak{p}$ of height one in R (cf. [18]).

Regarding Λ as a *ring with Green walks* (cf. [7]) as introduced in [19], our approach is slightly more general. Instead of orders over a regular local domain, we consider (non-commutative) Cohen–Macaulay rings in the sense of [20]. In analogy with the commutative case (see, e.g., [23, IV.B]) we define a two-dimensional *Cohen–Macaulay ring* as a left noetherian semilocal ring Λ which has a pair $\{P, Q\}$ of invertible ideals with $P + Q \subseteq \operatorname{Rad}\Lambda$ cofinite in ${}_{\Lambda}\Lambda$ such that $(P+Q)/P$ is invertible in Λ/P. Such a system $\{P,Q\}$ is called a *Cohen–Macaulay system* for Λ. The definition implies that $\{Q,P\}$ is then also a Cohen–Macaulay system, and $PQ = QP$. If in addition, $P+Q = \operatorname{Rad}\Lambda$ holds for some Cohen–Macaulay system $\{P,Q\}$, then Λ is called *regular*. For example, hereditary orders over a complete discrete valuation domain are regular in this sense.

In §2 we recall the definition of a *ring Λ with Green walks* [19]. As in the case of Brauer graph orders, there is an over-ring Γ of Λ with an invertible ideal $J \subseteq \Lambda$, and there is a combinatorial object H (see §2) associated with Λ which generalizes the Brauer graph. In this article we assume H to be a graph. Then Λ is called a *Brauer graph ring* if Λ and Γ are Cohen–Macaulay rings. We say that Λ is of *prime defect type* (cf. [17] when Λ is an order) if the invertible ideal J of Γ is part of a Cohen–Macaulay system $\{J,Q\}$ with $J+Q = \operatorname{Rad}\Gamma$. In particular, Γ is a regular ring in this case.

The reader is referred to [1], [5], [12], [15] for basic terminology and facts concerning representation theory of orders over a regular ring, to [1], [3], [4], [14] for basic facts on Auslander–Reiten quivers, and to [23], [25] for the commutative theory of Cohen–Macaulay rings and modules. Rings are always assumed to be associative with unit element. Modules are assumed to be unitary left modules if not specified otherwise. The category of R-modules (resp. finitely generated R-modules) is denoted by R-**Mod** (resp. R-**mod**).

1. Cohen–Macaulay rings. First, we collect some results on non-commutative Cohen–Macaulay rings [20], [21] which will be used in the statement and proof of our main theorem in §3.

A (two-sided) ideal P of a ring R is said to be *invertible* if ${}_RP_R$ belongs to the Picard group $\operatorname{Pic}(R)$ of R (cf. [5, II, §55]), i.e. if ${}_RP$ is a progenerator with $\operatorname{End}_R(P) = R^{\mathrm{op}}$. The multiplicative semigroup of invertible ideals in R will be denoted by R°. Then we define the *quotient ring* (see [20])

$$\widetilde{R} := \varinjlim\{P^{-1} \mid P \in R^\circ\} \tag{1}$$

of R, where $P^{-1} := \operatorname{Hom}_R(P,R)$. Thus R is a subring of $\widetilde{R}$, and for $P \in R^\circ$,

$$P \cdot P^{-1} = P^{-1} \cdot P = R. \tag{2}$$

For an R-module M, we consider the subsemigroup $R^\circ(M)$ of R° consisting of the $P \in R^\circ$ such that $Px \neq 0$ for all non-zero elements $x \in M$. These P are therefore called *M-invertible*. For instance, if $I \subseteq \operatorname{Rad} R$ is an ideal of R, and $P \in R^\circ$, then the ideal $P+I/I$ of R/I is invertible if and only if P is ${}_R(R/I)$-invertible and $(R/I)_R$-invertible [20, Proposition 5, and Corollary 2 of Proposition 4]. If R is noetherian with respect to two-sided ideals, then the second condition can be dropped [20, Corollary of Proposition 5]. We say that an R-module M is a *lattice* if $R^\circ(M) = R^\circ$. Equivalently, this means that the natural homomorphism

$$M \to \widetilde{R} \otimes_R M \tag{3}$$

(which maps $x \in M$ to $1 \otimes x$) is injective.

Recall that a commutative noetherian local ring R is said to be *Cohen–Macaulay* (see, e.g., [23]) if there exists a sequence $a_1, \ldots, a_n \in \operatorname{Rad} R$ such that a_i is a non-zerodivisor modulo $Ra_1 + \ldots + Ra_{i-1}$ for $i \in \{1, \ldots, n\}$, and $R/(Ra_1 + \ldots + Ra_n)$ is of finite length. Similarly, we define a left noetherian semilocal ring R to be a *Cohen–Macaulay ring* if there are invertible ideals $P_1, \ldots, P_n \subseteq \operatorname{Rad} R$ with $P_1 + \ldots + P_n$ cofinite in ${}_R R$ such that $P_1 + \ldots + P_{i+1}$ is invertible modulo $P_1 + \ldots + P_i$, for $i \in \{1, \ldots, n-1\}$. We proved (l.c.) that this definition is invariant under permutation of $P_1, \ldots, P_n$, and that it implies $P_i P_j = P_j P_i$ for all i, j. If in addition,

$$P_1 + \ldots + P_n = \operatorname{Rad} R, \tag{4}$$

then we speak of a *regular* ring R.

In what follows, we assume that R is a Cohen–Macaulay ring. For $M \in R$-**mod**, a set $\mathcal{P} := \{P_1, \ldots, P_n\}$ of pairwise commuting invertible ideals in R is called a *defining system* for M if $I := P_1 + \ldots + P_n$ is a *defining ideal*, i.e. if $I \subseteq \operatorname{Rad} R$, and M/IM is length-finite. Then the length function $f(i) := l(M/I^i M)$ is a rational polynomial for large i [20, Theorem 1], and in case $M \neq 0$, the degree of f (which does not depend on I) is called the *dimension* of M and is denoted by $\dim M$. In particular, $\dim R := \dim({}_R R)$. The defining system $\mathcal{P}$ for M is called a *Cohen–Macaulay system* if $\mathcal{P} \subseteq R^\circ(M)$, and

$$P_{i+1} \in R^\circ(M/(P_1 + \ldots + P_i)M) \tag{5}$$

holds for $i \in \{1, \ldots, n-1\}$. If such a $\mathcal{P}$ exists, then $n = \dim M$, and we call M a *Cohen–Macaulay R-module*. Here we are interested in *maximal* Cohen–Macaulay modules, i.e. Cohen–Macaulay modules M for which the equality $\dim M = \dim R$ holds, including the zero module $M = 0$. By [20, Proposition 9 and Corollary 2] we get:

PROPOSITION 1. *Let R be a Cohen–Macaulay ring, and M an R-module.*

(a) *If M is a Cohen–Macaulay module, then* $\dim N = \dim M$ *for every non-zero submodule N of M.*

(b) *If M is a maximal Cohen–Macaulay module, then M is a lattice.*

(c) *If M is a lattice with* $\dim M = 1$, *then M is a Cohen–Macaulay module.*

Outline of proof. (a) Let M be a Cohen–Macaulay module with Cohen–Macaulay system $\{P_1, \ldots, P_n\}$. Suppose there is a non-zero submodule N of M with $\dim N < n$. A slight generalization of [10, Corollary 12] implies (see [20, Proposition 6]) that P_1 has the full Artin–Rees property (with respect to N), i.e. there exists an integer $m > 0$ with $N \cap P_1^{m+i}M = P_1^i(N \cap P_1^m M)$ for all $i \in \mathbb{N}$. We show first that $N' := N \cap P_1^m M$ is zero. Otherwise $N'/P_1^m N' = N'/(N' \cap P_1^{2m}M) \cong (N' + P_1^{2m}M)/P_1^{2m}M \subseteq M/P_1^{2m}M$. Since P_1 is M-invertible, it can be shown (see [20, Theorem 2]) that $\dim M = 1 + \dim M/P_1 M$ and similarly, $\dim N' = 1 + \dim N'/P_1^m N'$. Moreover, it is easy to see that $\dim M/P_1^{2m}M = \dim M/P_1 M$. Hence $\dim N'/P_1^m N' < \dim M/P_1^{2m}M$. In order to apply induction, we have to show that $\{P_2, \ldots, P_n\}$ is a Cohen–Macaulay system for $M/P_1^{2m}M$. This follows by a characterization of Cohen–Macaulay systems given in [20, Theorem 3 and Corollary 2 of Proposition 7]. Therefore we have $N'/P_1^m N' = 0$, and thus $P_1^m N \subseteq N' = 0$ by Nakayama's lemma. Hence $N = 0$, a contradiction which proves (a).

(b) Next let $P \subseteq \operatorname{Rad} R$ be an invertible ideal of R. Using properties of the Hilbert-Samuel polynomial for R, it can be shown [20, Theorem 2] that $\dim R/P < \dim R$. For a maximal Cohen–Macaulay R-module M, this implies $\dim M/PM \le R/P < \dim R = \dim M$. In order to show that P is M-invertible, we make use of a generalized Fitting lemma [20, Lemma 6]: There exists an integer $m > 0$ with $P^m M \cap [P^m]M = 0$, where $[Q]M := \{x \in M \mid Qx = 0\}$. Hence $\dim[P]M \le \dim[P^m]M \le \dim M/P^m M < \dim M$. By the above, we infer $[P]M = 0$, i.e. $P \in R^\circ(M)$. If $P \in R^\circ$ does not satisfy $P \subseteq \operatorname{Rad} R$, it suffices to choose any invertible ideal $Q \subseteq \operatorname{Rad} R$. Then $QP \subseteq \operatorname{Rad} R$ is invertible, hence M-invertible, and thus P is also M-invertible. This proves that M is a lattice.

(c) Conversely, let M be a one-dimensional lattice. Then $\dim R \ge 1$, and there exists an invertible ideal $P \subseteq \operatorname{Rad} R$ which is therefore M-invertible. As above, this implies $\dim M/PM = \dim M - 1 = 0$, i.e. M/PM is of finite length. Hence M is a Cohen–Macaulay module. ∎

We need the following two results proved in [20] and [21].

Proposition 2. *If R is a regular ring with* $\dim R \le 2$, *then $M \in R$-**mod** is a maximal Cohen–Macaulay module if and only if M is projective.*

Proof. If M is projective, then it is a direct summand of a (finitely generated) free R-module, hence a maximal Cohen–Macaulay module. Conversely, let $M \neq 0$ be a maximal Cohen–Macaulay module. In [20] we proved that $n := \dim R \le 2$ implies that every Cohen–Macaulay system $\{P_1, \ldots, P_n\}$ for R is also a Cohen–Macaulay system for M. Since R is regular, we may assume $P_1 + \ldots + P_n = \operatorname{Rad} R$. Thus if $n = 0$, then R is semisimple, whence M is projective. Now we proceed by induction. For $n > 0$ it follows that $M/P_1 M$ is a maximal Cohen–Macaulay module over the regular ring R/P_1. Therefore we may assume that $M/P_1 M$ is projective over R/P_1. As $P_1 \subseteq \operatorname{Rad} R$ is invertible, a simple argument shows that M is projective (see [21, Lemma 1.1]). ∎

PROPOSITION 3. *Let R be a Cohen–Macaulay ring and M a two-dimensional Cohen–Macaulay module over R.*

(a) *Every defining system for M is a Cohen–Macaulay system.*

(b) *A submodule N of M is Cohen–Macaulay if and only if there is no submodule N' between N and M such that the module N'/N is of finite length.*

Outline of proof. (a) Let $\{P_1, P_2\}$ be a defining system for M. Then $1 \le \dim M/P_1M \le 1 + \dim M/(P_1 + P_2)M = 1$ implies $\dim M/P_1M = 1$. As in the proof of Proposition 1(b), we use the generalized Fitting lemma [20, Lemma 6] to show that P_1 is M-invertible. Moreover, the lemma yields $P_2^m(M/P_1M) \cap [P_2^m](M/P_1M) = 0$ for some (large) integer m. Hence we obtain an embedding of $[P_2^m](M/P_1M)$ into $M/(P_1 + P_2^m)M$ which shows that $[P_2^m](M/P_1M)$ is of finite length l. We have to show that $l = 0$. To this end, let $\{Q_1, Q_2\}$ be any Cohen–Macaulay system for M. There is a submodule N of M with $[P_2^m](M/P_1M) = N/P_1M$. Hence $Q_1^r P_1^{-1} N/Q_1^r M$ is of length l, and we can choose $r \in \mathbb{N}$ such that $Q_1^r P_1^{-1} N \subseteq M$. Furthermore, there exists some $s \in \mathbb{N}$ with $Q_2^s \cdot Q_1^r P_1^{-1} N \subseteq Q_1^r M$. Hence $Q_2^s Q_1^r P_1^{-1} N \subseteq Q_1^r M \cap Q_2^s M = Q_2^s Q_1^r M$ since $\{Q_1^r, Q_2^s\}$ is a Cohen–Macaulay system for M (see [20, Corollary 2 of Proposition 7]. Thus we obtain $P_1^{-1} N \subseteq M$, i.e. $l = 0$.

(b) Let N be a submodule of M. For $N = 0$ the assertion follows by Proposition 1. Thus assume $N \neq 0$. Every Cohen–Macaulay system $\{P, Q\}$ for M is a defining system for N. Hence N is Cohen–Macaulay if and only if $PN \cap QN = PQN$, i.e. $N' := P^{-1}N \cap Q^{-1}N = N$. By a similar argument to the proof of (a), the statement follows since N'/N is length-finite. ∎

The following result is a two-dimensional version of [21, Proposition 1]:

PROPOSITION 4. *If $\dim R = 2$, and $M_2 \hookrightarrow M_1 \to M_0$ is an exact sequence in R-**mod** with maximal Cohen–Macaulay modules M_0, M_1, then M_2 is a maximal Cohen–Macaulay module.*

Proof. By Proposition 3, every Cohen–Macaulay system $\{P, Q\}$ for ${}_RR$ is a Cohen–Macaulay system for M_0 and M_1. If M is the image of $M_1 \to M_0$, then the short exact sequence $M_2 \hookrightarrow M_1 \twoheadrightarrow M$ yields a short exact sequence $PM_2 \hookrightarrow PM_1 \twoheadrightarrow PM$, and thus an induced short exact sequence $M_2/PM_2 \hookrightarrow M_1/PM_1 \twoheadrightarrow M/PM$. Now it follows immediately that $\{P, Q\}$ is a Cohen–Macaulay system for M_2. ∎

Suppose that R is commutative. Then an R-algebra Λ which is finitely generated as an R-module is said to be an *R-order* if ${}_R\Lambda$ is a lattice. The following result [20, Theorem 4] shows that the above concept of Cohen–Macaulay module generalizes the classical notion (see, e.g., [23, IV.B]).

Proposition 5. *Let Λ be an order over a local Cohen–Macaulay domain R. Then a Λ-module M is a Cohen–Macaulay module if and only if M is a classical Cohen–Macaulay module over the commutative ring R.*

2. Brauer graph rings. Recall from [19] the definition of a ring with Green walks. For a module M over a semiperfect ring S, the *proradical* $\operatorname{Pro} M$ is defined to be the sum of the images of all homomorphisms $E \to F \to M$ with non-isomorphic indecomposable projective S-modules E and F. Then $\operatorname{Pro} S := \operatorname{Pro}({}_S S)$ is an ideal of S, contained in $\operatorname{Rad} S$, with $\operatorname{Pro} M = (\operatorname{Pro} S)M$ for each $M \in S$-**Mod**. Moreover, $S/\operatorname{Pro} S$ is a product of matrix rings over local rings. If there exists an ideal $P \in S^\circ$ with $\bigcap P^i = 0$ and $\operatorname{Pro} S \subseteq P \subseteq \operatorname{Rad} S$, then S is called *prohereditary* (with respect to P). Note that the condition $\bigcap P^i = 0$ is automatic whenever S is left noetherian (since P has then the Artin–Rees property [11, 4.2]). A semiperfect subring R of S is called a *Bäckström ring* (with respect to S and P) if $\operatorname{Pro} R \subseteq P \subseteq R$, and S/P is projective as a left R/P-module. For example, if Λ is a Bäckström order over a complete discrete valuation domain (see, e.g., [15]), then Λ has a unique hereditary over-order Γ with $J := \operatorname{Rad} \Gamma = \operatorname{Rad} \Lambda$. Thus Λ is a Bäckström ring with respect to Γ and J. A Bäckström ring R with respect to S and P is said to be a *ring with Green walks* [19] if each indecomposable projective S/P-module remains indecomposable over R/P. Then the structure of R is essentially given by its underlying *cycle hypergraph* H_R^S (see [19, §4]) which yields the projective resolution of R/P by walking around H_R^S under Green's celebrated rule [7].

For the present purpose, consider the case where H_R^S is a graph (henceforth denoted by G_R), i.e. when for each indecomposable projective R-module E, the S-module SE decomposes into exactly two non-isomorphic direct summands. If in this case, R and S are (left noetherian) Cohen–Macaulay rings, then R will be called a *Brauer graph ring* (with respect to S and P). If in addition, S has a Cohen–Macaulay system $\{P_1, \ldots, P_n\}$ with $P_1 = P$ and $P_1 + \ldots + P_n = \operatorname{Rad} S$, then we say that R is of *prime defect type*.

Note. All the above concepts are invariant under Morita equivalence. In particular, if E is a progenerator for a Brauer graph ring R as above, then $R' := (\operatorname{End}_R E)^{\mathrm{op}}$ is a Brauer graph ring with respect to $S' := (\operatorname{End}_R SE)^{\mathrm{op}}$ and $P' := \operatorname{Hom}_R(E, PE)$, and the Brauer graphs G_R and $G_{R'}$ coincide. If R is of prime defect type, the same is true for R'.

3. The main theorem. Before we state our main theorem which classifies the maximal Cohen–Macaulay modules over a two-dimensional Brauer graph ring of prime defect type, let us prove a preliminary result.

LEMMA 1. *Let R be a two-dimensional Brauer graph ring with respect to S and P of prime defect type. Then ${}_RS$ is a maximal Cohen–Macaulay module*, $\dim S = 2$, *and every invertible ideal $Q \in R^\circ$ commutes with S.*

Proof. Since S/P is finitely generated and projective over R/P, it follows that ${}_RS$ is finitely generated, and thus $\dim {}_RS = 2$. Then $\operatorname{Rad} S = (\operatorname{Rad} R)S$ implies $\dim S = 2$.

Let E_1 be an indecomposable projective S-module. Then there exists an indecomposable projective R-module E with $SE = E_1 \oplus E_2$, and E/PE is diagonally embedded into $E_1/PE_1 \oplus E_2/PE_2$. Hence each R-submodule F of E_1 with $PE_1 \subseteq F$ is invariant under S. Furthermore, the natural projection $E \twoheadrightarrow E_2$ yields an exact sequence $PE_1 \hookrightarrow E \twoheadrightarrow E_2$. Since $E_2 \cong P^k E_2$ for a suitable $k \geq 1$, E_2 can be embedded into E. Therefore, PE_1 is a Cohen–Macaulay module by Proposition 4. Since multiplication by P permutes the indecomposable projective S-modules, we infer that ${}_RS$ is a Cohen–Macaulay module. Hence QE_1 is Cohen–Macaulay for each $Q \in R^\circ$, and by Proposition 3 there is a maximal Cohen–Macaulay module $F \in S$-**mod** with $SQE_1 \subseteq F \subseteq E_1$ and F/SQE_1 length-finite. By Proposition 2, ${}_SF$ is projective. Hence F/SQE_1, and thus PF/PQE_1 is length-finite as an R-module. Since $QE_1 \in R$-**mod** is Cohen–Macaulay, Proposition 3 yields $PF \subseteq QE_1 \subseteq F$. Therefore, the above shows that $QE_1 = SQE_1$. Moreover, Proposition 3 implies $QE_1 = SQE_1 = F$. Hence QE_1, and also $Q^{-1}E_1$ ($=Q^k E_1$ for some $k \geq 1$) are projective S-modules. Consequently, $SQ \subseteq QS$ and $SQ^{-1} \subseteq Q^{-1}S$, and thus $SQ = QS$. ∎

In what follows, we assume that R is a two-dimensional Brauer graph ring with respect to S and P of prime defect type. We fix a Cohen–Macaulay system $\{P, Q\}$ for S with $P + Q = \operatorname{Rad} S$. Let E be an indecomposable projective R-module, with non-trivial decomposition $SE = E_1 \oplus E_2$, i.e. E is an edge of G_R, and the blocks of S containing E_1 and E_2, respectively, are the nodes connected by E (see [19, §4]). For $m \in \mathbb{Z}$ we define the *derived series* of E (cf. [16, §7]):

$$E^{(m)} := E \cap (Q^m E_1 \oplus Q^{-m} E_2). \tag{6}$$

By Lemma 1, $E' := E + (Q^m E_1 \oplus Q^{-m} E_2)$ is contained in a maximal Cohen–Macaulay R-module. Hence by Proposition 4, the exact sequence $E^{(m)} \hookrightarrow E \oplus (Q^m E_1 \oplus Q^{-m} E_2) \to E'$ implies that $E^{(m)}$ is a maximal Cohen–Macaulay module. Moreover, the R-modules (6) are indecomposable modulo P, hence indecomposable.

LEMMA 2. *Let Ω be a one-dimensional regular local ring*, *and V a finite dimensional vector space over the skew field $\widetilde{\Omega}$. Moreover*, *let L_1, L_2 be finitely generated Ω-submodules of V with $\widetilde{\Omega} L_1 = \widetilde{\Omega} L_2 = V$. Then V*

has a decomposition $V = V_1 \oplus \ldots \oplus V_n$ *into one-dimensional subspaces with* $L_i = (L_i \cap V_1) + \ldots + (L_i \cap V_n)$, $i \in \{1,2\}$.

Proof. By assumption, $J := \operatorname{Rad} \Omega$ is invertible. Multiplying L_2 by a suitable power of J, we may assume $L_2 \subseteq L_1$ and $L_2 \not\subseteq JL_1$. Take projective covers p_1: $P_1 \twoheadrightarrow L_1/L_2 + JL_1$ and p_2: $P_2 \twoheadrightarrow L_2/L_2 \cap JL_1$. Since L_1, L_2 are projective, p_1 and p_2 lift to embeddings $P_i \hookrightarrow L_i$. Hence, $P_1 + P_2 + JL_1 = P_1 + L_2 + JL_1 = L_1$ implies $P_1 + P_2 = L_1$. Moreover, $P_1 \cap P_2 \subseteq P_1 \cap (L_2 + JL_1) = JP_1$. Hence $P_1 \cap P_2 \subseteq JL_1 \cap P_2 = (L_2 \cap JL_1) \cap P_2 = JP_2$, and thus $P_1 \cap P_2 \subseteq JP_1 \cap JP_2 = J(P_1 \cap P_2)$ since J is invertible. Therefore, we obtain $L_1 = P_1 \oplus P_2$. Finally, $(P_1 \cap L_2) + (P_2 \cap L_2) = (P_1 \cap L_2) + P_2 = L_2 \cap (P_1 + P_2) = L_2$. The assertion now follows by induction. ∎

Now we are ready to state our main result.

THEOREM. *Let* R *be a Brauer graph ring with respect to* $S \supseteq R$ *and* $P \subseteq R$. *Assume that* R *is of prime defect type. Then every indecomposable maximal Cohen–Macaulay* R*-module is either an indecomposable projective* S*-module*, *or of the form* (6) *with an indecomposable projective* R*-module* E.

Proof. Without loss of generality, we may assume R to be basic. Then S is also basic. Let $\{P_1, P_2\}$ be a fixed Cohen–Macaulay system for R. For an indecomposable maximal Cohen–Macaulay module ${}_RM$, consider the largest S-submodule N. By Proposition 3, ${}_RN$ is a Cohen–Macaulay module. In fact, suppose there is an R-module N' between N and M such that N'/N is simple. Then $(P_1 + P_2)SN' = S(P_1 + P_2)N' \subseteq N$ by Lemma 1, and thus SN'/N is of finite length. Hence $SN' \subseteq M$, a contradiction. Therefore, ${}_RN$ is a maximal Cohen–Macaulay module by Proposition 1, and Proposition 3 implies that $\{P_1, P_2\}$ is a Cohen–Macaulay system for N.

By Lemma 1, $\{SP_1, SP_2\}$ is a Cohen–Macaulay system for the S-module N, whence ${}_SN$ is projective. Now $S/P = S_1 \times \ldots \times S_{2r}$ with local rings S_i, and $R/P = R_1 \times \ldots \times R_r$ with diagonals $R_i \hookrightarrow S_{2i-1} \times S_{2i}$. Hence $P^{-1}N/N = N_1 \oplus \ldots \oplus N_{2r}$ with $N_i \in S_i$-**mod**, and $M/N = M_1 \oplus \ldots \oplus M_r$ with R_i-submodules M_i of $N_{2i-1} \oplus N_{2i}$. Since $P^{-1}N$ is projective, these decompositions lift to $P^{-1}N = F_1 \oplus \ldots \oplus F_{2r}$ with $M = (M \cap (F_1 \oplus F_2)) \oplus \ldots \ldots \oplus (M \cap (F_{2r-1} \oplus F_{2r}))$. Therefore, let us assume $M \subseteq F_1 \oplus F_2$ with $PF_1 \oplus PF_2 = N$. Then $PF_i = M \cap F_i$, and $SM = F_1' \oplus F_2'$, where F_i' is the projection of M into F_i. Let $F_i'' \subseteq F_i$ be such that F_i''/F_i' is the largest length-finite S_i-submodule of F_i/F_i'. Then F_i/F_i'' is a lattice, hence projective by Proposition 1 and 2. Therefore, F_i''/PF_i is a direct summand of F_i/PF_i. Thus if M is not a projective S-module, the indecomposability of M implies that F_i/F_i' is length-finite.

Now R_1 is a one-dimensional regular local ring, and the diagonal $R_1 \hookrightarrow S_1 \times S_2$ yields an identification $R_1 = S_1 = S_2$. Moreover, the diagonal $M/N \hookrightarrow F_1'/PF_1 \oplus F_2'/PF_2$ yields an isomorphism $F_1'/PF_1 \xrightarrow{\sim} F_2'/PF_2$ which extends to an isomorphism of vector spaces $\omega : \widetilde{R}_1 \otimes_{R_1} F_1'/PF_1 \xrightarrow{\sim} \widetilde{R}_1 \otimes_{R_1} F_2'/PF_2$ over the skew field $\widetilde{R}_1$. Let us regard ω as an identification, and define $V := \widetilde{R}_1 \otimes_{R_1} F_i'/PF_i$. Then $L := F_i'/PF_i$ is a lattice in V, and M/N can be viewed as the diagonal $L \hookrightarrow L \oplus L \subseteq V \oplus V$. Moreover, the F_i/PF_i correspond to lattices L_i in V, $i \in \{1, 2\}$. We claim that $L_1 \cap L_2 = L$. In fact, there is an R-submodule M' of $F_1 \oplus F_2$ such that M'/N corresponds to the diagonal $L' \hookrightarrow L' \oplus L' \subseteq V \oplus V$ with $L' := L_1 \cap L_2$. As M'/M is length-finite, Proposition 3 implies $M' = M$ and thus $L' = L$.

Now Lemma 2 yields a decomposition $V = V_1 \oplus \ldots \oplus V_n$ into one-dimensional subspaces which simultaneously decomposes L_1 and L_2. Equivalently, this yields a pair of decompositions $F_i/PF_i = H_{i1} \oplus \ldots \oplus H_{in}$ such that M/N is decomposed under $(H_{11} \oplus H_{21}) \oplus \ldots \oplus (H_{1n} \oplus H_{2n})$. Hence these decompositions lift to a decomposition $F_1 \oplus F_2 = E_1 \oplus \ldots \oplus E_{2n}$ with $M = (M \cap (E_1 \oplus E_2)) \oplus \ldots \oplus (M \cap (E_{2n-1} \oplus E_{2n}))$. Since M is indecomposable, we conclude $n = 1$. Then $M/N \hookrightarrow E_1/PE_1 \oplus E_2/PE_2$, and the above intersection condition ($L_1 \cap L_2 = L$) implies that one of the projections $M/N \to E_i/PE_i$ is surjective. Consequently, M is of the desired form (6). ∎

Remark. For a Brauer graph order of prime defect type over a two-dimensional complete regular local ring, the indecomposables given in the preceding theorem have been characterized by Roggenkamp [16] as indecomposable maximal Cohen–Macaulay modules which admit a Cohen–Macaulay filtration (see introduction). The theorem therefore implies that this list of indecomposables is complete.

4. A partial Auslander–Reiten quiver. Let R be a two-dimensional commutative regular local ring, and Λ a Cohen–Macaulay R-order. By Proposition 5 this means that Λ is an R-algebra which is finitely generated and free as an R-module. Proposition 5 also implies that $M \in R$-**mod** is a maximal Cohen–Macaulay module if and only if M is free over R. The category of these modules M will be denoted by Λ-**CM**. In [22] we show that Λ-**CM** is *almost abelian*, i.e. additive such that every morphism in Λ-**CM** has a kernel and a cokernel, and cokernels (kernels) are stable under pullback (pushout). A sequence

$$0 \to N \xrightarrow{u} M \xrightarrow{v} C \to 0 \tag{7}$$

of modules in Λ-**CM** will be called *exact* if $u = \ker v$ and $v = \operatorname{cok} u$. Equivalently, this says that u is a kernel of v in Λ-**Mod**, and v has a decomposition

$$v : M \twoheadrightarrow M/\operatorname{Im} u \hookrightarrow (M/\operatorname{Im} u)^{**},$$

where $(\quad)^* := \operatorname{Hom}_R(-, R)$. In particular, v need not be surjective. Accordingly, the concept of Auslander–Reiten sequence has to be extended to certain sequences (7) which are not exact in Λ-**mod** (cf. [13, 2.1]). Namely, we define (7) to be an *Auslander–Reiten sequence* if it is a non-split exact sequence with N and C indecomposable such that the usual factorization property holds: every morphism $E \to C$ in Λ-**CM** which is not split epic factors through v. By a standard argument (see [6]), the dual factorization property follows: every morphism $N \to E$ in Λ-**CM** which is not split monic factors through u. It is well known that up to isomorphism, an Auslander–Reiten sequence (7) is determined by C (hence also by N).

We say that Λ-**CM** *has almost split sequences* if an Auslander–Reiten sequence (7) exists for every indecomposable object C, and also for every indecomposable object N in Λ-**CM**. (See Proposition 6 below. For $\dim R \neq 2$ the projectives C and the modules N with $N^* \in \Lambda^{\mathrm{op}}$-**mod** projective have to be excluded.) Let $M = M_1 \oplus \ldots \oplus M_s$ be a decomposition into indecomposables in Λ-**CM**. For our present purpose it suffices to consider the case where $M_1, \ldots, M_s$ are pairwise non-isomorphic. Then (7) can be replaced by a diagram with (isomorphism classes of) indecomposables as vertices:

(8)
$$\begin{tikzcd} & M_1 \arrow[dr] \arrow[dd, dash, dotted] & \\ N \arrow[ur] \arrow[dr] & & C \\ & M_s \arrow[ur] & \end{tikzcd}$$

Thus if Λ-**CM** has Auslander–Reiten sequences, the totality of all diagrams (8) constitutes a graph $\mathfrak{A}_\Lambda$ with an automorphism $\tau : \mathfrak{A}_\Lambda \to \mathfrak{A}_\Lambda$ defined by $\tau(C) := N$. The pair $(\mathfrak{A}_\Lambda, \tau)$ is said to be the *Auslander–Reiten quiver* of Λ.

Recall that Λ is said to be an *isolated singularity* [1] if $\Lambda_\mathfrak{p}$ is regular for each non-maximal prime ideal $\mathfrak{p}$ of R. In other words: $\Lambda_\mathfrak{p}$ is hereditary for $\mathfrak{p}$ of height one, and semisimple for $\mathfrak{p} = 0$. Auslander [1] has shown that Λ-**CM** has Auslander–Reiten sequences if and only if Λ is an isolated singularity. More generally, let $\mathcal{P}_\Lambda$ denote the full subcategory of modules M in Λ-**CM** such that $C_\mathfrak{p}$ is a projective $\Lambda_\mathfrak{p}$-module for each non-maximal prime ideal $\mathfrak{p}$ of R, and $\mathcal{I}_\Lambda := \{N \in \Lambda\text{-}\mathbf{CM} \mid N^* \in \mathcal{P}_{\Lambda^{\mathrm{op}}}\}$. The following proposition generalizes the main result of [3].

PROPOSITION 6. *Let R be a two-dimensional commutative regular local ring, and Λ a Cohen–Macaulay R-order. For an indecomposable $C \in \Lambda$-**CM**, there exists an Auslander–Reiten sequence* (7) *if and only if $C \in \mathcal{P}$. In this case, $N \in \mathcal{I}$.*

Proof. By [3, Theorem 2.1], it remains to consider the case where C is projective. The following argument, based on [2, Theorem 3.6], is due to O. Iyama. Let $R \rightarrowtail I_0 \to I_1 \twoheadrightarrow I_2$ be an injective resolution of R-modules, and $S := C/\operatorname{Rad} C$. Then the $J_i := \operatorname{Hom}_R(\operatorname{Hom}_\Lambda(C, \Lambda), I_i)$ provide an injective resolution of $N := \mathrm{Hom}_\Lambda(C, \Lambda)^*$. Hence $\mathrm{Ext}^2_\Lambda(S, N)$ is a cokernel of $\mathrm{Hom}_\Lambda(S, J_1) \to \mathrm{Hom}_\Lambda(S, J_2)$. Now $\mathrm{Hom}_\Lambda(S, J_i) = \mathrm{Hom}_R(\mathrm{Hom}_\Lambda(C, \Lambda) \otimes_\Lambda S, I_i) = \mathrm{Hom}_R(\mathrm{Hom}_\Lambda(C, S), I_i) = \mathrm{Hom}_R(\mathrm{End}_\Lambda(S), I_i)$. This implies $\mathrm{Ext}^2_\Lambda(S, N) = \mathrm{Ext}^2_R(\mathrm{End}_\Lambda(S), R) \neq 0$, which yields a non-split exact sequence $N \rightarrowtail M \overset{c}{\twoheadrightarrow} \mathrm{Rad}\, C$ of Λ-modules. Since $c(M^{**}) \subseteq C$, we get a commutative diagram with exact rows:

$$\begin{tikzcd}
N \arrow[r, hook, "i"] \arrow[d, hook, "g"] \arrow[dr, phantom, "\mathrm{PB}"] & M \arrow[r, two heads] \arrow[d, hook] & \mathrm{Rad}\, C \arrow[d, hook] \\
N' \arrow[r, hook, "i'"] & M^{**} \arrow[r, two heads] & C' \subseteq C
\end{tikzcd}$$

The square PB is a pullback, and N is R-free. Therefore, i' induces an embedding of N'/N into the length-finite R-module M^{**}/M, whence g is an isomorphism. As i is not split monic, the same is true for i'. Consequently, $C' = \mathrm{Rad}\, C$, and thus $M = M^{**} \in \Lambda$-**CM**. Therefore we obtain a non-split exact sequence (7) in Λ-**CM** with N indecomposable. In order to prove the factorization property, consider a morphism $f : F \to C$ in Λ-**CM**. The pullback of (7) along f gives a short exact sequence $N \rightarrowtail E \twoheadrightarrow F$ in Λ-**mod**. Since $N = \mathrm{Hom}_\Lambda(C, \Lambda)^*$, this sequence splits. ∎

For each Auslander–Reiten sequence (7) in Λ-**CM**, the sequences

$$0 \to N_{\mathfrak{p}} \xrightarrow{u} M_{\mathfrak{p}} \xrightarrow{v} C_{\mathfrak{p}} \to 0 \tag{9}$$

are exact for all prime ideals $\mathfrak{p}$ of height ≤ 1. Hence (9) splits for these $\mathfrak{p}$, and the $M_1, \ldots, M_s$ in (8) belong to $\mathcal{P}_\Lambda \cup \mathcal{I}_\Lambda$. Therefore, the diagrams (8) make up a generalized Auslander–Reiten quiver with vertices in $\mathcal{P}_\Lambda \cup \mathcal{I}_\Lambda$, where τ is defined on $\mathcal{P}_\Lambda$, and τ^{-1} is defined on $\mathcal{I}_\Lambda$. Let us call this a *partial Auslander–Reiten quiver* $\mathfrak{A}_\Lambda$ of Λ.

Now let Λ be a Brauer graph order with respect to Γ and J. Assume that Λ is of prime defect type. Then $\mathcal{P}_\Lambda = \mathcal{I}_\Lambda = (\Lambda\text{-}\mathbf{CM}) \setminus (\Gamma\text{-}\mathbf{CM})$, and the partial Auslander–Reiten quiver has connected components of the form

$$\begin{tikzcd}[column sep=small]
\cdots & & E^{(-2)} \arrow[dr, shift left] \arrow[dl, shift left] & & E^{(0)} \arrow[dr, shift left] \arrow[dl, shift left] & & E^{(2)} \arrow[dr, shift left] \arrow[dl, shift left] & & \cdots \\
& E^{(-3)} \arrow[ur, shift left] & & E^{(-1)} \arrow[ur, shift left] \arrow[ul, shift left] & & E^{(1)} \arrow[ur, shift left] \arrow[ul, shift left] & & E^{(3)} \arrow[ul, shift left] &
\end{tikzcd} \tag{10}$$

with E indecomposable projective, and $E^{(i)}$ defined by (6). For each $i \in \mathbb{Z}$, there is an Auslander–Reiten sequence

$$0 \to E^{(i)} \to E^{(i-1)} \oplus E^{(i+1)} \to E^{(i)} \to 0$$

which is exact in Λ-**mod** if and only if $i \neq 0$. Thus for each indecomposable projective Λ-module E, the derived series (6) constitutes a connected component (10) of $\mathfrak{A}_\Lambda$ which has the structure of a double tube (see [14]).

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