REPRESENTATION THEORY OF TWO-DIMENSIONAL
BRAUER GRAPH RINGS

BY

WOLFGANG RUMP (STUTTGART)

Abstract. We consider a class of two-dimensional non-commutative Cohen–Macaulay rings to which a Brauer graph, that is, a finite graph endowed with a cyclic ordering of edges at any vertex, can be associated in a natural way. Some orders $\Lambda$ over a two-dimensional regular local ring are of this type. They arise, e.g., as certain blocks of Hecke algebras over the completion of $\mathbb{Z}[q, q^{-1}]$ at $(p, q - 1)$ for some rational prime $p$. For such orders $\Lambda$, a class of indecomposable maximal Cohen–Macaulay modules (see introduction) has been determined by K. W. Roggenkamp. We prove that this list of indecomposables of $\Lambda$ is complete.

Introduction. For a rational prime $p$, and a finite Coxeter group $G$, let $\mathcal{H}_G$ denote the Hecke algebra of $G$ over $\mathbb{Z}[q, q^{-1}]$ (see [5, I, §11], [8], [9]), and $\mathcal{H} := \mathbb{Z}[q, q^{-1}]_m \otimes \mathcal{H}_G$ be its completion at the maximal ideal $m := (p, q - 1)$ of $\mathbb{Z}[q, q^{-1}]$. Then modulo $(q - 1)$, every block $B$ of $\mathbb{Z}_p G$ lifts to a block $B$ of $\mathcal{H}$. In particular, if $B$ is of cyclic defect with Brauer tree $T$, then $B$ is a tree order with respect to $T$ over the two-dimensional ring $\mathbb{Z}[q, q^{-1}]_m$ (cf. [17, Theorem 3.2]). A more general case, Brauer graph orders $\Lambda$ over a two-dimensional regular domain $R$, has been investigated in [16]. These $R$-orders $\Lambda$ can be described as follows. In case $\Lambda$ is basic, there is an overorder $\Gamma = \Gamma_1 \times \ldots \times \Gamma_r$ of $\Lambda$ with

\[
\Gamma_1 := \begin{pmatrix}
\Omega_1 & \Pi_1 & \ldots & \Pi_1 \\
& \ddots & \ddots & \\
& & \ddots & \Pi_1 \\
& & & \Omega_1 & \ldots & \Omega_1
\end{pmatrix}
\]

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where $\Omega_i$ are local Cohen–Macaulay orders, and $\Pi_i$ is an invertible ideal in $\Omega_i$ (cf. [19, §3]). Hence

\[
J_i := \begin{pmatrix}
\Pi_i & \cdots & \cdots & \Pi_i \\
\Omega_i & & & \\
& \ddots & & \\
& & \ddots & \\
\Omega_i & & & \Pi_i
\end{pmatrix}
\]

is an invertible ideal of $\Gamma_i$ such that $\Gamma_i/J_i = \Omega_i/\Pi_i \times \ldots \times \Omega_i/\Pi_i$. Now $\Lambda$ is determined by the inclusions $J := J_1 \times \ldots \times J_r \subseteq \Lambda \subseteq \Gamma$ with $\Lambda/J = \overline{A}_1 \times \ldots \times \overline{A}_s$, where each $\overline{A}_k$ is a diagonal in some $\Omega_i/\Pi_i \times \Omega_j/\Pi_j$ (only if $\Omega_i/\Pi_i \cong \Omega_j/\Pi_j$) such that $\Gamma/J$ has exactly 2s factors of the form $\Omega_i/\Pi_i$.

The Brauer graph of $\Lambda$ is obtained as follows. If $\Gamma_i$ are interpreted as vertices, and the (cyclically ordered) factors $\Omega_i/\Pi_i$ of each $\Gamma_i/J_i$ as germs of edges connected with $\Gamma_i$, then the edges are given by the diagonals $\overline{A}_k \subseteq \Omega_i/\Pi_i \times \Omega_j/\Pi_j$.

Let $K$ be the quotient field of $R$. In the case where $A$ is of prime defect type, i.e. when the $\Omega_i/\Pi_i$ are hereditary orders in a skew field, K. W. Roggenkamp [16] determined the indecomposable maximal Cohen–Macaulay modules $M$ over $\Lambda$ which admit a Cohen–Macaulay filtration. This means that there is a decomposition $KA = A_1 \times \ldots \times A_n$ of the $K$-algebra $KA = K \otimes_R A$ into blocks $A_i$ such that the factor modules $M_i/M_{i-1}$ of the corresponding filtration $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ of $M$ with $M_i := M \cap (A_1 \oplus \ldots \oplus A_i)$ are Cohen–Macaulay modules. The question remained open whether these indecomposables $M$ exhaust the totality of indecomposable maximal Cohen–Macaulay modules over $\Lambda$.

In the present note, we prove this statement (see Theorem of §3). Notice that $A$ does not represent an isolated singularity in the sense of Auslander [1]. In fact, in the above notation, the ideal $P \supseteq J$ of $A$ with $P/J = \overline{A}_1 \times \ldots \times \overline{A}_{s-1}$ is prime, and the localization $A_{(P \cap R)}$ is not hereditary. Hence there is no Auslander–Reiten quiver of all indecomposable maximal Cohen–Macaulay modules. Nevertheless, by a result of [3], it is possible to give a partial Auslander–Reiten quiver (see §4), consisting of the indecomposable maximal Cohen–Macaulay modules $M$ with $M_p$ projective or injective for all prime ideals $p$ of height one in $R$ (cf. [18]).

Regarding $A$ as a ring with Green walks (cf. [7]) as introduced in [19], our approach is slightly more general. Instead of orders over a regular local domain, we consider (non-commutative) Cohen–Macaulay rings in the sense of [20]. In analogy with the commutative case (see, e.g., [23, IV.B]) we define a two-dimensional Cohen–Macaulay ring as a left noetherian semilocal ring $A$ which has a pair $\{P, Q\}$ of invertible ideals with $P + Q \subseteq \text{Rad } A$ cofinite
in \( \Lambda \) such that \((P + Q)/P\) is invertible in \( \Lambda/P \). Such a system \( \{P, Q\} \) is called a \textit{Cohen–Macaulay system} for \( \Lambda \). The definition implies that \( \{Q, P\} \) is then also a Cohen–Macaulay system, and \( PQ = QP \). If in addition, \( P + Q = \text{Rad} \Lambda \) holds for some Cohen–Macaulay system \( \{P, Q\} \), then \( \Lambda \) is called \textit{regular}. For example, hereditary orders over a complete discrete valuation domain are regular in this sense.

In §2 we recall the definition of a \textit{ring} \( \Lambda \) with Green walks [19]. As in the case of Brauer graph orders, there is an over-ring \( \Gamma \) of \( \Lambda \) with an invertible ideal \( J \subseteq \Lambda \), and there is a combinatorial object \( H \) (see §2) associated with \( \Lambda \) which generalizes the Brauer graph. In this article we assume \( H \) to be a graph. Then \( \Lambda \) is called a \textit{Brauer graph ring} if \( \Lambda \) and \( \Gamma \) are Cohen–Macaulay rings. We say that \( \Lambda \) is of \textit{prime defect type} (cf. [17] when \( \Lambda \) is an order) if the invertible ideal \( J \) of \( \Gamma \) is part of a Cohen–Macaulay system \( \{J, Q\} \) with \( J + Q = \text{Rad} \Gamma \). In particular, \( \Gamma \) is a regular ring in this case.

The reader is referred to [1], [5], [12], [15] for basic terminology and facts concerning representation theory of orders over a regular ring, to [1], [3], [4], [14] for basic facts on Auslander–Reiten quivers, and to [23], [25] for the commutative theory of Cohen–Macaulay rings and modules. Rings are always assumed to be associative with unit element. Modules are assumed to be unitary left modules if not specified otherwise. The category of \( R \)-modules (resp. finitely generated \( R \)-modules) is denoted by \( R \text{-Mod} \) (resp. \( R \text{-mod} \)).

1. Cohen–Macaulay rings. First, we collect some results on non-commutative Cohen–Macaulay rings [20], [21] which will be used in the statement and proof of our main theorem in §3.

A (two-sided) ideal \( P \) of a ring \( R \) is said to be \textit{invertible} if \( RP_R \) belongs to the Picard group \( \text{Pic}(R) \) of \( R \) (cf. [5, II, §55]), i.e. if \( RP \) is a progenerator with \( \text{End}_R(P) = R^{op} \). The multiplicative semigroup of invertible ideals in \( R \) will be denoted by \( R^\circ \). Then we define the \textit{quotient ring} (see [20])

\[
\tilde{R} := \lim_{\rightarrow} \{P^{-1} \mid P \in R^\circ\}
\]

of \( R \), where \( P^{-1} := \text{Hom}_R(P, R) \). Thus \( R \) is a subring of \( \tilde{R} \), and for \( P \in R^\circ \),

\[
P \cdot P^{-1} = P^{-1} \cdot P = R.
\]

For an \( R \)-module \( M \), we consider the subsemigroup \( R^\circ(M) \) of \( R^\circ \) consisting of the \( P \in R^\circ \) such that \( Px \neq 0 \) for all non-zero elements \( x \in M \). These \( P \) are therefore called \textit{M-invertible}. For instance, if \( I \subseteq \text{Rad} R \) is an ideal of \( R \), and \( P \in R^\circ \), then the ideal \( P + I/I \) of \( R/I \) is invertible if and only if \( P \) is \( R(R/I) \)-invertible and \( (R/I)R \)-invertible [20, Proposition 5, and Corollary 2 of Proposition 4]. If \( R \) is noetherian with respect to two-sided ideals, then the second condition can be dropped [20, Corollary of Proposition 5]. We say that an \( R \)-module \( M \) is a \textit{lattice} if \( R^\circ(M) = R^\circ \). Equivalently, this means...
that the natural homomorphism
\[ M \rightarrow \tilde{R} \otimes_R M \]
(which maps \( x \in M \) to \( 1 \otimes x \)) is injective.

Recall that a commutative noetherian local ring \( R \) is said to be Cohen–Macaulay (see, e.g., [23]) if there exists a sequence \( a_1, \ldots, a_n \in \text{Rad} \ R \) such that \( a_i \) is a non-zerodivisor modulo \( Ra_1 + \ldots + Ra_{i-1} \) for \( i \in \{1, \ldots, n\} \), and \( R/(Ra_1 + \ldots + Ra_n) \) is of finite length. Similarly, we define a left noetherian semilocal ring \( R \) to be a Cohen–Macaulay ring if there are invertible ideals \( P_1, \ldots, P_n \subseteq \text{Rad} \ R \) with \( P_1 + \ldots + P_n \) cofinite in \( R \), and

\[ P_1 + \ldots + P_{i+1} \text{ is invertible modulo } P_1 + \ldots + P_i \text{ for } i \in \{1, \ldots, n-1\} \].

We proved (l.c.) that this definition is invariant under permutation of \( P_1, \ldots, P_n \), and that

\[ P_i P_j = P_j P_i \text{ for all } i,j \].

If in addition, \( P_1 + \ldots + P_n = \text{Rad} \ R \),

\[ R/(P_1 + \ldots + P_n) \text{ is length-finite} \]

then we speak of a regular ring \( R \).

In what follows, we assume that \( R \) is a Cohen–Macaulay ring. For \( M \in R\text{-mod} \), a set \( \mathcal{P} := \{P_1, \ldots, P_n\} \) of pairwise commuting invertible ideals in \( R \) is called a defining system for \( M \) if \( I := P_1 + \ldots + P_n \) is a defining ideal, i.e. if \( I \subseteq \text{Rad} \ R \), and \( M/IM \) is length-finite. Then the length function \( f(i) := l(M/IM) \) is a rational polynomial for large \( i \) [20, Theorem 1], and in case \( M \neq 0 \), the degree of \( f \) (which does not depend on \( I \)) is called the dimension of \( M \) and is denoted by \( \dim M \). In particular, \( \dim R := \dim(R) \). The defining system \( \mathcal{P} \) for \( M \) is called a Cohen–Macaulay system if \( \mathcal{P} \subseteq R^\circ(M) \), and

\[ P_{i+1} \in R^\circ(M/(P_1 + \ldots + P_i)M) \]

holds for \( i \in \{1, \ldots, n-1\} \). If such a \( \mathcal{P} \) exists, then \( n = \dim M \), and we call \( M \) a Cohen–Macaulay \( R \)-module. Here we are interested in maximal Cohen–Macaulay modules, i.e. Cohen–Macaulay modules \( M \) for which the equality \( \dim M = \dim R \) holds, including the zero module \( M = 0 \). By [20, Proposition 9 and Corollary 2] we get:

**Proposition 1.** Let \( R \) be a Cohen–Macaulay ring, and \( M \) an \( R \)-module.

(a) If \( M \) is a Cohen–Macaulay module, then \( \dim N = \dim M \) for every non-zero submodule \( N \) of \( M \).

(b) If \( M \) is a maximal Cohen–Macaulay module, then \( M \) is a lattice.

(c) If \( M \) is a lattice with \( \dim M = 1 \), then \( M \) is a Cohen–Macaulay module.

**Outline of proof.** (a) Let \( M \) be a Cohen–Macaulay module with Cohen–Macaulay system \( \{P_1, \ldots, P_n\} \). Suppose there is a non-zero submodule \( N \) of \( M \) with \( \dim N < n \). A slight generalization of [10, Corollary 12] implies (see [20, Proposition 6]) that \( P_1 \) has the full Artin–Rees property
(with respect to $N$), i.e. there exists an integer $m > 0$ with $N \cap P_i^m M = P_1^i (N \cap P_i^m M)$ for all $i \in \mathbb{N}$. We show first that $N' := N \cap P_i^m M$ is zero. Otherwise $N'/P_i^m N' = N'/P_i^m M \cong (N' + P_i^m M)/P_i^m M \subseteq M/P_i^m M$. Since $P_1$ is $M$-invertible, it can be shown (see [20, Lemma 1.1]) that $\dim M = 1 + \dim M/P_1 M$ and similarly, $\dim N' = 1 + \dim N'/P_i^m N'$. Moreover, it is easy to see that $\dim M/P_i^m M = \dim M/P_1 M$. Hence $\dim N'/P_i^m N' < \dim M/P_i^m M$. In order to apply induction, we have to show that $\{P_2, \ldots, P_n\}$ is a Cohen–Macaulay system for $M/P_i^m M$. This follows by a characterization of Cohen–Macaulay systems given in [20, Theorem 3 and Corollary 2 of Proposition 7]. Therefore we have $\dim N'/P_i^m N' = 0$, and thus $P_i^m N' = 0$ by Nakayama’s lemma. Hence $N' = 0$, a contradiction which proves (a).

(b) Next let $P \subseteq \mathrm{Rad} R$ be an invertible ideal of $R$. Using properties of the Hilbert–Samuel polynomial for $R$, it can be shown [20, Theorem 2] that $\dim R/P < \dim R$. For a maximal Cohen–Macaulay $R$-module $M$, this implies $\dim M/PM \leq R/P < \dim R = \dim M$. In order to show that $P$ is $M$-invertible, we make use of a generalized Fitting lemma [20, Lemma 6]: There exists an integer $m > 0$ with $P^m M \cap [P^m]M = 0$, where $[Q]M := \{x \in M \mid Qx = 0\}$. Hence $\dim [P]M \leq \dim [P^m]M \leq \dim M/P^m M < \dim M$. By the above, we infer $[P]M = 0$, i.e. $P \in R^0(M)$. If $P \in R^0$ does not satisfy $P \subseteq \mathrm{Rad} R$, it suffices to choose any invertible ideal $Q \subseteq \mathrm{Rad} R$. Then $QP \subseteq \mathrm{Rad} R$ is invertible, hence $M$-invertible, and thus $P$ is also $M$-invertible. This proves that $M$ is a lattice.

(c) Conversely, let $M$ be a one-dimensional lattice. Then $\dim R \geq 1$, and there exists an invertible ideal $P \subseteq \mathrm{Rad} R$ which is therefore $M$-invertible. As above, this implies $\dim M/PM = \dim M - 1 = 0$, i.e. $M/PM$ is of finite length. Hence $M$ is a Cohen–Macaulay module.

We need the following two results proved in [20] and [21].

**Proposition 2.** If $R$ is a regular ring with $\dim R \leq 2$, then $M \in R\text{-mod}$ is a maximal Cohen–Macaulay module if and only if $M$ is projective.

**Proof.** If $M$ is projective, then it is a direct summand of a (finitely generated) free $R$-module, hence a maximal Cohen–Macaulay module. Conversely, let $M \neq 0$ be a maximal Cohen–Macaulay module. In [20] we proved that $n := \dim R \leq 2$ implies that every Cohen–Macaulay system $\{P_1, \ldots, P_n\}$ for $R$ is also a Cohen–Macaulay system for $M$. Since $R$ is regular, we may assume $P_1 + \ldots + P_n = \mathrm{Rad} R$. Thus if $n = 0$, then $R$ is semisimple, whence $M$ is projective. Now we proceed by induction. For $n > 0$ it follows that $M/P_1 M$ is a maximal Cohen–Macaulay module over the regular ring $R/P_1$. Therefore we may assume that $M/P_1 M$ is projective over $R/P_1$. As $P_1 \subseteq \mathrm{Rad} R$ is invertible, a simple argument shows that $M$ is projective (see [21, Lemma 1.1]).

(a) Every defining system for $M$ is a Cohen–Macaulay system.

(b) A submodule $N$ of $M$ is Cohen–Macaulay if and only if there is no submodule $N'$ between $N$ and $M$ such that the module $N'/N$ is of finite length.

Outline of proof. (a) Let $\{P_1, P_2\}$ be a defining system for $M$. Then $1 \leq \dim M/P_1M \leq 1 + \dim M/(P_1 + P_2)M = 1$ implies $\dim (M/P_1M) = 1$. As in the proof of Proposition 1(b), we use the generalized Fitting lemma [20, Lemma 6] to show that $P_1$ is $M$-invertible. Moreover, the lemma yields $P_2^m(M/P_1M) \cap [P_2^m](M/P_1M) = 0$ for some (large) integer $m$. Hence we obtain an embedding of $[P_2^m](M/P_1M)$ into $M/(P_1 + P_2^m)M$ which shows that $[P_2^m](M/P_1M)$ is of finite length $l$. We have to show that $l = 0$. To this end, let $\{Q_1, Q_2\}$ be any Cohen–Macaulay system for $M$. There is a submodule $N$ of $M$ with $[P_2^m](M/P_1M) = N/P_1M$. Hence $Q_1^rP_1^{-1}N/Q_1^rM$ is of length $l$, and we can choose $r \in \mathbb{N}$ such that $Q_1^rP_1^{-1}N \subseteq M$. Furthermore, there exists some $s \in \mathbb{N}$ with $Q_2^s \cdot Q_1^rP_1^{-1}N \subseteq Q_1^rM$. Hence $Q_2^sQ_1^rP_1^{-1}N \subseteq Q_1^rM \cap Q_2^sM = Q_2^sQ_1^rM$ since $\{Q_1^r, Q_2^s\}$ is a Cohen–Macaulay system for $M$ (see [20, Corollary 2 of Proposition 7]). Thus we obtain $P_1^{-1}N \subseteq M$, i.e. $l = 0$.

(b) Let $N$ be a submodule of $M$. For $N = 0$ the assertion follows by Proposition 1. Thus assume $N \neq 0$. Every Cohen–Macaulay system $\{P, Q\}$ for $M$ is a defining system for $N$. Hence $N$ is Cohen–Macaulay if and only if $PN \cap QN = PQN$, i.e. $N' := P^{-1}N \cap Q^{-1}N = N$. By a similar argument to the proof of (a), the statement follows since $N'/N$ is length-finite.

The following result is a two-dimensional version of [21, Proposition 1]:

Proposition 4. If $\dim R = 2$, and $M_2 \twoheadrightarrow M_1 \rightarrow M_0$ is an exact sequence in $R\text{-mod}$ with maximal Cohen–Macaulay modules $M_0, M_1$, then $M_2$ is a maximal Cohen–Macaulay module.

Proof. By Proposition 3, every Cohen–Macaulay system $\{P, Q\}$ for $R\text{-mod}$ is a Cohen–Macaulay system for $M_0$ and $M_1$. If $M$ is the image of $M_1 \rightarrow M_0$, then the short exact sequence $M_2 \twoheadrightarrow M_1 \rightarrow M$ yields a short exact sequence $PM_2 \twoheadrightarrow PM_1 \rightarrow PM$, and thus an induced short exact sequence $M_2/PM_2 \rightarrow M_1/PM_1 \rightarrow M/PM$. Now it follows immediately that $\{P, Q\}$ is a Cohen–Macaulay system for $M_2$.

Suppose that $R$ is commutative. Then an $R$-algebra $A$ which is finitely generated as an $R$-module is said to be an $R$-order if $R\text{-mod}$ is a lattice. The following result [20, Theorem 4] shows that the above concept of Cohen–Macaulay module generalizes the classical notion (see, e.g., [23, IV.B]).
Proposition 5. Let \( \Lambda \) be an order over a local Cohen–Macaulay domain \( R \). Then a \( \Lambda \)-module \( M \) is a Cohen–Macaulay module if and only if \( M \) is a classical Cohen–Macaulay module over the commutative ring \( R \).

2. Brauer graph rings. Recall from [19] the definition of a ring with Green walks. For a module \( M \) over a semiperfect ring \( S \), the proradical \( \text{Pro} M \) is defined to be the sum of the images of all homomorphisms \( E \to F \to M \) for each \( M \in S\text{-Mod} \). Moreover, \( S/\text{Pro} S \) is a product of matrix rings over local rings. If there exists an ideal \( P \in S^\circ \) with \( \bigcap P_i = 0 \) and \( \text{Pro} S \subseteq P \subseteq \text{Rad} S \), then \( S \) is called prohereditary (with respect to \( P \)). Note that the condition \( \bigcap P_i = 0 \) is automatic whenever \( S \) is left noetherian (since \( P \) has then the Artin–Rees property [11, 4.2]).

A semiperfect subring \( R \) of \( S \) is called a Bäckström ring (with respect to \( S \) and \( P \)) if \( \text{Pro} R \subseteq P \subseteq R \), and \( S/P \) is projective as a left \( R/P \)-module. For example, if \( \Lambda \) is a Bäckström order over a complete discrete valuation domain (see, e.g., [15]), then \( \Lambda \) has a unique hereditary over-order \( \Gamma \) with \( J := \text{Rad} \Gamma = \text{Rad} \Lambda \). Thus \( \Lambda \) is a Bäckström ring with respect to \( \Gamma \) and \( J \). A Bäckström ring \( R \) with respect to \( S \) and \( P \) is said to be a ring with Green walks [19] if each indecomposable projective \( S/P \)-module remains indecomposable over \( R/P \). Then the structure of \( R \) is essentially given by its underlying cycle hypergraph \( H^S_R \) (see [19, §4]) which yields the projective resolution of \( R/P \) by walking around \( H^S_R \) under Green’s celebrated rule [7].

For the present purpose, consider the case where \( H^S_R \) is a graph (henceforth denoted by \( G_R \)), i.e., when for each indecomposable projective \( R \)-module \( E \), the \( S \)-module \( SE \) decomposes into exactly two non-isomorphic direct summands. If in this case, \( R \) and \( S \) are (left noetherian) Cohen–Macaulay rings, then \( R \) will be called a Brauer graph ring (with respect to \( S \) and \( P \)). If in addition, \( S \) has a Cohen–Macaulay system \( \{P_1, \ldots, P_n\} \) with \( P_1 = P \) and \( P_1 + \ldots + P_n = \text{Rad} S \), then we say that \( R \) is of prime defect type.

Note. All the above concepts are invariant under Morita equivalence. In particular, if \( E \) is a progenerator for a Brauer graph ring \( R \) as above, then \( R' := (\text{End}_R E)^{\text{op}} \) is a Brauer graph ring with respect to \( S' := (\text{End}_R SE)^{\text{op}} \) and \( P' := \text{Hom}_R(E, PE) \), and the Brauer graphs \( G_R \) and \( G_{R'} \) coincide. If \( R \) is of prime defect type, the same is true for \( R' \).

3. The main theorem. Before we state our main theorem which classifies the maximal Cohen–Macaulay modules over a two-dimensional Brauer graph ring of prime defect type, let us prove a preliminary result.
Lemma 1. Let $R$ be a two-dimensional Brauer graph ring with respect to $S$ and $P$ of prime defect type. Then $RS$ is a maximal Cohen–Macaulay module, $\dim S = 2$, and every invertible ideal $Q \in R^\circ$ commutes with $S$.

Proof. Since $S/P$ is finitely generated and projective over $R/P$, it follows that $RS$ is finitely generated, and thus $\dim RS = 2$. Then $\text{Rad} S = (\text{Rad} R)S$ implies $\dim S = 2$.

Let $E_1$ be an indecomposable projective $S$-module. Then there exists an indecomposable projective $R$-module $E$ with $SE = E_1 \oplus E_2$, and $E/PE$ is diagonally embedded into $E_1/PE_1 \oplus E_2/PE_2$. Hence each $R$-submodule $F$ of $E_1$ with $PE_1 \subseteq F$ is invariant under $S$. Furthermore, the natural projection $E \rightarrow E_2$ yields an exact sequence $PE_1 \hookrightarrow E \rightarrow E_2$. Since $E_2 \cong P^k E_2$ for a suitable $k \geq 1$, $E_2$ can be embedded into $E$. Therefore, $PE_1$ is a Cohen–Macaulay module by Proposition 4. Since multiplication by $P$ permutes the indecomposable projective $S$-modules, we infer that $RS$ is a Cohen–Macaulay module. Moreover, the $R$-module $F/SQE_1$ is projective. Hence $F/SQE_1$, and thus $PF/PQE_1$ is length-finite as an $R$-module. Since $F_{S}$ is Cohen–Macaulay, Proposition 3 yields $PF \subseteq QE_1 \subseteq F$. Therefore, the above shows that $QE_1 = SQE_1$. Moreover, Proposition 3 implies $QE_1 = SQE_1 = F$. Hence $QE_1$, and also $Q^{-1}E_1 (= Q^k E_1$ for some $k \geq 1$) are projective $S$-modules. Consequently, $SQ \subseteq QS$ and $SQ^{-1} \subseteq Q^{-1}S$, and thus $SQ = QS$. ■

In what follows, we assume that $R$ is a two-dimensional Brauer graph ring with respect to $S$ and $P$ of prime defect type. We fix a Cohen–Macaulay system $(P, Q)$ for $S$ with $P + Q = \text{Rad} S$. Let $E$ be an indecomposable projective $R$-module, with non-trivial decomposition $SE = E_1 \oplus E_2$, i.e. $E$ is an edge of $G_R$, and the blocks of $S$ containing $E_1$ and $E_2$, respectively, are the nodes connected by $E$ (see [19, §4]). For $m \in \mathbb{Z}$ we define the derived series of $E$ (cf. [16, §7]):

$$E^{(m)} := E \cap (Q^m E_1 \oplus Q^{-m} E_2).$$

By Lemma 1, $E' := E + (Q^m E_1 \oplus Q^{-m} E_2)$ is contained in a maximal Cohen–Macaulay $R$-module. Hence by Proposition 4, the exact sequence $E^{(m)} \rightarrow E \oplus (Q^m E_1 \oplus Q^{-m} E_2) \rightarrow E'$ implies that $E^{(m)}$ is a maximal Cohen–Macaulay module. Moreover, the $R$-modules (6) are indecomposable modulo $P$, hence indecomposable.

Lemma 2. Let $\Omega$ be a one-dimensional regular local ring, and $V$ a finite dimensional vector space over the skew field $\Omega$. Moreover, let $L_1, L_2$ be finitely generated $\Omega$-submodules of $V$ with $\Omega L_1 = \Omega L_2 = V$. Then $V$
has a decomposition \( V = V_1 \oplus \ldots \oplus V_n \) into one-dimensional subspaces with \( L_i = (L_i \cap V_1) + \ldots + (L_i \cap V_n) \), \( i \in \{1, 2\} \).

**Proof.** By assumption, \( J := \text{Rad} \Omega \) is invertible. Multiplying \( L_2 \) by a suitable power of \( J \), we may assume \( L_2 \subseteq L_1 \) and \( L_2 \not\subseteq JL_1 \). Take projective covers \( p_1: P_1 \to L_1/L_2 + JL_1 \) and \( p_2: P_2 \to L_2/L_2 \cap JL_1 \). Since \( L_1, L_2 \) are projective, \( p_1 \) and \( p_2 \) lift to embeddings \( P_i \to L_i \). Hence, \( P_1 + P_2 + JL_1 = P_1 + L_2 + JL_1 = L_1 \) implies \( P_1 + P_2 = L_1 \). Moreover, \( P_1 \cap P_2 \subseteq P_1 \cap (L_2 + JL_1) = JP_1 \). Hence \( P_1 \cap P_2 \subseteq JL_1 \cap P_2 = (L_2 \cap JL_1) \cap P_2 = JP_2 \), and thus \( P_1 \cap P_2 \subseteq JP_1 \cap JP_2 = J(P_1 \cap P_2) \) since \( J \) is invertible. Therefore, we obtain \( L_1 = P_1 \oplus P_2 \). Finally, \( (P_1 \cap L_2) + (P_2 \cap L_2) = (P_1 \cap L_2) + P_2 = L_2 \cap (P_1 + P_2) = L_2 \). The assertion now follows by induction. \( \blacksquare \)

Now we are ready to state our main result.

**Theorem.** Let \( R \) be a Brauer graph ring with respect to \( S \supseteq R \) and \( P \subseteq R \). Assume that \( R \) is of prime defect type. Then every indecomposable maximal Cohen–Macaulay \( R \)-module is either an indecomposable projective \( S \)-module, or of the form (6) with an indecomposable projective \( R \)-module \( E \).

**Proof.** Without loss of generality, we may assume \( R \) to be basic. Then \( S \) is also basic. Let \( \{P_1, P_2\} \) be a fixed Cohen–Macaulay system for \( R \). For an indecomposable maximal Cohen–Macaulay module \( _R M \), consider the largest \( S \)-submodule \( N \). By Proposition 3, \( _R N \) is a Cohen–Macaulay module. In fact, suppose there is an \( R \)-module \( N' \) between \( N \) and \( M \) such that \( N'/N \) is simple. Then \( (P_1 + P_2)SN' = S(P_1 + P_2)N' \subseteq N \) by Lemma 1, and thus \( SN'/N \) is of finite length. Hence \( SN' \subseteq M \), a contradiction. Therefore, \( _R N \) is a maximal Cohen–Macaulay module by Proposition 1, and Proposition 3 implies that \( \{P_1, P_2\} \) is a Cohen–Macaulay system for \( N \).

By Lemma 1, \( \{SP_1, SP_2\} \) is a Cohen–Macaulay system for the \( S \)-module \( N \), whence \( S \)-module \( N \) is projective. Now \( S/P = S_1 \times \ldots \times S_{2r} \) with local rings \( S_i \), and \( R/P = R_1 \times \ldots \times R_r \) with diagonals \( R_i \to S_{2i-1} \times S_{2i} \). Hence \( P^{-1}N/N = N_1 \oplus \ldots \oplus N_{2r} \), with \( N_i \in S_{i}-\text{mod} \), and \( M/N = M_1 \oplus \ldots \oplus M_r \) with \( R_i \)-submodules \( M_i \) of \( N_{2i-1} \oplus N_{2i} \). Since \( P^{-1}N \) is projective, these decompositions lift to \( P^{-1}N = F_1 \oplus \ldots \oplus F_{2r} \) with \( M = (M \cap (F_1 \oplus F_2)) \oplus \ldots \oplus (M \cap (F_{2r-1} \oplus F_{2r})) \). Therefore, let us assume \( M \subseteq F_1 \oplus F_{2r} \) with \( PF_1 \oplus PF_{2r} = N \). Then \( PF_i = M \cap F_i \), and \( SM = F_i' \oplus F_{2r}' \), where \( F_i' \) is the projection of \( M \) into \( F_i \). Let \( F_i'' \subseteq F_i \) be such that \( F_i''/F_i' \) is the largest length-finite \( S_i \)-submodule of \( F_i/F_i' \). Then \( F_i/F_i'' \) is a lattice, hence projective by Proposition 1 and 2. Therefore, \( F_i''/PF_i \) is a direct summand of \( F_i/PF_i \). Thus if \( M \) is not a projective \( S \)-module, the indecomposability of \( M \) implies that \( F_i/F_i'' \) is length-finite.

Now \( R_1 \) is a one-dimensional regular local ring, and the diagonal \( R_1 \to S_1 \times S_2 \) yields an identification \( R_1 = S_1 = S_2 \). Moreover, the diago-
M/N \rightarrow F'_1/_{PF_1} \oplus F'_2/_{PF_2} \text{ yields an isomorphism } F'_1/_{PF_1} \sim F'_2/_{PF_2} \text{ which extends to an isomorphism of vector spaces } \omega : \tilde{R}_1 \otimes_{R_1} F'_1/_{PF_1} \sim \tilde{R}_1 \otimes_{R_2} F'_2/_{PF_2} \text{ over the skew field } \tilde{R}_1. \text{ Let us regard } \omega \text{ as an identification, and define } V := \tilde{R}_1 \otimes_{R_1} F'_i/_{PF_i}. \text{ Then } L := F'_i/_{PF_i} \text{ is a lattice in } V, \text{ and } M/N \text{ can be viewed as the diagonal } L \hookrightarrow L \oplus L \subseteq V \oplus V. \text{ Moreover, the } F_i/_{PF_i} \text{ correspond to lattices } L_i \text{ in } V, i \in \{1, 2\}. \text{ We claim that } L_1 \cap L_2 = L. \text{ In fact, there is an } R\text{-submodule } M' \text{ of } F_1 \oplus F_2 \text{ such that } M'/N \text{ corresponds to the diagonal } L' \hookrightarrow L' \oplus L' \subseteq V \oplus V \text{ with } L' := L_1 \cap L_2. \text{ As } M'/M \text{ is length-finite, Proposition 3 implies } M' = M \text{ and thus } L' = L.

Now Lemma 2 yields a decomposition \( V = V_1 \oplus \ldots \oplus V_n \) into one-dimensional subspaces which simultaneously decomposes \( L_1 \) and \( L_2 \). Equivalently, this yields a pair of decompositions \( F_i/_{PF_i} = H_{i1} \oplus \ldots \oplus H_{in} \) such that \( M/N \) is decomposed under \( (H_{11} \oplus H_{21}) \oplus \ldots \oplus (H_{1n} \oplus H_{2n}) \). Hence these decompositions lift to a decomposition \( F_1 \oplus F_2 = E_1 \oplus \ldots \oplus E_{n-1} \oplus (M \cap (E_{2n-1} \oplus E_{2n})) \). Since \( M \) is indecomposable, we conclude \( n = 1 \). Then \( M/N \rightarrow E_i/_{PE_i} \oplus E_2/_{PE_2} \), and the above intersection condition \( (L_1 \cap L_2 = L) \) implies that one of the projections \( M/N \rightarrow E_i/_{PE_i} \) is surjective. Consequently, \( M \) is of the desired form \( (6) \).

Remark. For a Brauer graph order of prime defect type over a two-dimensional complete regular local ring, the indecomposables given in the preceding theorem have been characterized by Roggenkamp [16] as indecomposable maximal Cohen–Macaulay modules which admit a Cohen–Macaulay filtration (see introduction). The theorem therefore implies that this list of indecomposables is complete.

4. A partial Auslander–Reiten quiver. Let \( R \) be a two-dimensional commutative regular local ring, and \( A \) a Cohen–Macaulay \( R \)-order. By Proposition 5 this means that \( A \) is an \( R \)-algebra which is finitely generated and free as an \( R \)-module. Proposition 5 also implies that \( M \in R\text{-mod} \) is a maximal Cohen–Macaulay module if and only if \( M \) is free over \( R \). The category of these modules \( M \) will be denoted by \( A\text{-CM} \). In [22] we show that \( A\text{-CM} \) is almost abelian, i.e. additive such that every morphism in \( A\text{-CM} \) has a kernel and a cokernel, and cokernels (kernels) are stable under pullback (pushout). A sequence

\[
0 \rightarrow N \xrightarrow{u} M \xrightarrow{v} C \rightarrow 0
\]

of modules in \( A\text{-CM} \) will be called exact if \( u = \ker v \) and \( v = \cok u \). Equivalently, this says that \( u \) is a kernel of \( v \) in \( A\text{-Mod} \), and \( v \) has a decomposition

\[
v : M \twoheadrightarrow M/\Im u \hookrightarrow (M/\Im u)^* \quad \text{in } (\_)^* := \Hom_R(\_, R). \text{ In particular, } v \text{ need not be surjective. Accordingly, the concept of Auslander–Reiten sequence has to be extended to}
\]
certain sequences (7) which are not exact in $\Lambda$-mod (cf. [13, 2.1]). Namely, we define (7) to be an Auslander–Reiten sequence if it is a non-split exact sequence with $N$ and $C$ indecomposable such that the usual factorization property holds: every morphism $E \to C$ in $\Lambda$-$\text{CM}$ which is not split epic factors through $v$. By a standard argument (see [6]), the dual factorization property follows: every morphism $N \to E$ in $\Lambda$-$\text{CM}$ which is not split monic factors through $u$. It is well known that up to isomorphism, an Auslander–Reiten sequence (7) is determined by $C$ (hence also by $N$).

We say that $\Lambda$-$\text{CM}$ has almost split sequences if an Auslander–Reiten sequence (7) exists for every indecomposable object $C$, and also for every indecomposable object $N$ in $\Lambda$-$\text{CM}$. (See Proposition 6 below. For $\dim R \neq 2$ the projectives $C$ and the modules $N$ with $N^* \in \Lambda^{\text{opp}}$-$\text{mod}$ projective have to be excluded.) Let $M = M_1 \oplus \ldots \oplus M_s$ be a decomposition into indecomposables in $\Lambda$-$\text{CM}$. For our present purpose it suffices to consider the case where $M_1, \ldots, M_s$ are pairwise non-isomorphic. Then (7) can be replaced by a diagram with (isomorphism classes of) indecomposables as vertices:

\[
\begin{array}{ccc}
N & \xleftarrow{v} & M_1 \\
\downarrow & & \downarrow \\
M_s & \xrightarrow{u} & C
\end{array}
\]

Thus if $\Lambda$-$\text{CM}$ has Auslander–Reiten sequences, the totality of all diagrams (8) constitutes a graph $\mathfrak{A}_\Lambda$ with an automorphism $\tau : \mathfrak{A}_\Lambda \to \mathfrak{A}_\Lambda$ defined by $\tau(C) := N$. The pair $(\mathfrak{A}_\Lambda, \tau)$ is said to be the Auslander–Reiten quiver of $\Lambda$.

Recall that $\Lambda$ is said to be an isolated singularity [1] if $\Lambda_p$ is regular for each non-maximal prime ideal $p$ of $R$. In other words: $\Lambda_p$ is hereditary for $p$ of height one, and semisimple for $p = 0$. Auslander [1] has shown that $\Lambda$-$\text{CM}$ has Auslander–Reiten sequences if and only if $\Lambda$ is an isolated singularity. More generally, let $\mathcal{P}_\Lambda$ denote the full subcategory of modules $M$ in $\Lambda$-$\text{CM}$ such that $C_p$ is a projective $\Lambda_p$-module for each non-maximal prime ideal $p$ of $R$, and $\mathcal{I}_\Lambda := \{N \in \Lambda$-$\text{CM} | N^* \in \mathcal{P}_{\Lambda^{\text{opp}}}\}$. The following proposition generalizes the main result of [3].

**Proposition 6.** Let $R$ be a two-dimensional commutative regular local ring, and $\Lambda$ a Cohen–Macaulay $R$-order. For an indecomposable $C \in \Lambda$-$\text{CM}$, there exists an Auslander–Reiten sequence (7) if and only if $C \in \mathcal{P}$. In this case, $N \in \mathcal{I}$.

**Proof.** By [3, Theorem 2.1], it remains to consider the case where $C$ is projective. The following argument, based on [2, Theorem 3.6], is due to O. Iyama. Let $R \to I_0 \to I_1 \to I_2$ be an injective resolution of $R$-modules, and $S := C/\text{Rad}C$. Then the $J_i := \text{Hom}_R(\text{Hom}_\Lambda(C, \Lambda), I_i)$ provide an in-
jective resolution of $N := \text{Hom}_A(C, A)^*$. Hence $\text{Ext}_A^2(S, N)$ is a cokernel of $\text{Hom}_A(S, J_1) \to \text{Hom}_A(S, J_2)$. Now $\text{Hom}_A(S, J_i) = \text{Hom}_R(\text{Hom}_A(C, A) \otimes_A S, I_i) = \text{Hom}_R(\text{End}_A(S), I_i)$. This implies $\text{Ext}_A^2(S, N) = \text{Ext}_R^2(\text{End}_A(S), R) \neq 0$, which yields a non-split exact sequence $N \to M \to \text{Rad} C$ of $A$-modules. Since $c(M^{**}) \subseteq C$, we get a commutative diagram with exact rows:

$$
\begin{array}{ccc}
N & \xrightarrow{i} & M \\
& \searrow & \searrow \\
& & \text{PB} \text{Rad} C
\end{array}
$$

The square PB is a pullback, and $N$ is $R$-free. Therefore, $i'$ induces an embedding of $N'/N$ into the length-finite $R$-module $M^{**}/M$, whence $g$ is an isomorphism. As $i$ is not split monic, the same is true for $i'$. Consequently, $C' = \text{Rad} C$, and thus $M = M^{**} \in A\text{-CM}$. Therefore we obtain a non-split exact sequence (7) in $A\text{-CM}$ with $N$ indecomposable. In order to prove the factorization property, consider a morphism $f : F \to C$ in $A\text{-CM}$. The pullback of (7) along $f$ gives a short exact sequence $N \to E \to F$ in $A\text{-mod}$. Since $N = \text{Hom}_A(C, A)^*$, this sequence splits.

For each Auslander–Reiten sequence (7) in $A\text{-CM}$, the sequences

$$
0 \to N_p \xrightarrow{\tau_p} M_p \xrightarrow{\tau_{-1}} C_p \to 0
$$

are exact for all prime ideals $p$ of height $\leq 1$. Hence (9) splits for these $p$, and the $M_1, \ldots, M_s$ in (8) belong to $\mathcal{P}_A \cup \mathcal{I}_A$. Therefore, the diagrams (8) make up a generalized Auslander–Reiten quiver with vertices in $\mathcal{P}_A \cup \mathcal{I}_A$, where $\tau$ is defined on $\mathcal{P}_A$, and $\tau^{-1}$ is defined on $\mathcal{I}_A$. Let us call this a partial Auslander–Reiten quiver $\mathfrak{A}_A$ of $A$.

Now let $A$ be a Brauer graph order with respect to $\Gamma$ and $J$. Assume that $A$ is of prime defect type. Then $\mathcal{P}_A = \mathcal{I}_A = (A\text{-CM}) \setminus (J\text{-CM})$, and the partial Auslander–Reiten quiver has connected components of the form

$$
\begin{array}{cccc}
E^{(-2)} & E^{(0)} & E^{(2)} & E^{(3)} \\
E^{(-3)} & E^{(-1)} & E^{(1)} & E^{(3)}
\end{array}
$$

with $E$ indecomposable projective, and $E^{(i)}$ defined by (6). For each $i \in \mathbb{Z}$, there is an Auslander–Reiten sequence

$$
0 \to E^{(i)} \to E^{(i-1)} \oplus E^{(i+1)} \to E^{(i)} \to 0
$$

which is exact in $A\text{-mod}$ if and only if $i \neq 0$. Thus for each indecomposable projective $A$-module $E$, the derived series (6) constitutes a connected component (10) of $\mathfrak{A}_A$ which has the structure of a double tube (see [14]).
REFERENCES


Mathematisches Institut B/3
University of Stuttgart
Pfaffenwaldring 57
D-70550 Stuttgart, Germany
E-mail: wolfgang.rump@ku-eichstaett.de

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