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CELL-LIKE RESOLUTIONS OF POLYHEDRA BY SPECIAL ONES

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Abstract. Suppose that P is a finite 2-polyhedron. We prove that there exists a PL surjective map $f : Q \to P$ from a fake surface Q with preimages of f either points or arcs or 2-disks. This yields a reduction of the Whitehead asphericity conjecture (which asserts that every subpolyhedron of an aspherical 2-polyhedron is also aspherical) to the case of fake surfaces. Moreover, if the set of points of P having a neighbourhood homeomorphic to the 2-disk is a disjoint union of open 2-disks, and every point of P has an arbitrarily small 2-dimensional neighbourhood, then we may additionally conclude that Q is a special 2-polyhedron.

A *resolution* of a space P is a pair (Q, f), where Q is a space (in a certain sense better than P) and $f : Q \to P$ is an onto map (in some sense good). In this paper we construct resolutions of polyhedra by maps with simple point-inverses (i.e. cell-like maps) by polyhedra with simple singularities and structure (i.e. fake surfaces and special polyhedra). We work in the PL category and use the notation and definitions from [RS72, HMS93]. All spaces considered are finite polyhedra and all maps are assumed to be PL.

A 2-polyhedron P is called a *fake surface* if each of its points has a closed neighbourhood homeomorphic to one of those in Figure 1: the 2-disk, the book with 3 pages or the cone over the 1-skeleton of the 3-simplex.

By P' we denote the *intrinsic* 1-*skeleton* of a polyhedron P, i.e. the subpolyhedron of P formed by points having no neighbourhood homeomorphic to a closed 2-disk. The *manifold set* of P is $P - P'$. By P'' we denote the *intrinsic* 0-*skeleton* of P (or of P'), i.e. the finite subset of P' consisting of all points having no neighbourhood in P', homeomorphic to a closed 1-disk.

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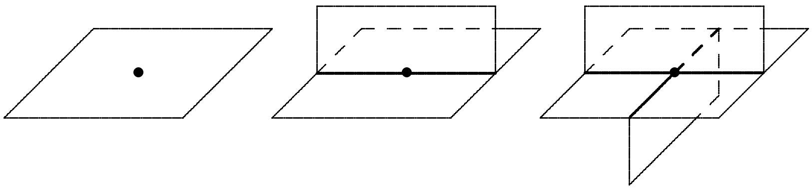

(a) (b) (c)

Fig. 1

If P is a fake surface, then P' is a graph whose vertices have degrees 1, 2 or 4, and P'' is the set of points of P having a neighbourhood shown in Figure 1(c). A fake surface P is called a *special* 2*-polyhedron* if its manifold set $P - P'$ and the manifold set $P' - P''$ of P' are *trivial*, i.e. they are a disjoint union of open 2- and 1-disks, respectively. An n-polyhedron P is *dimensionally homogeneous* if every point of P has arbitrarily small n-dimensional neighbourhoods.

THEOREM 1. (a) *Every* 2*-polyhedron has a resolution* (Q, f) *such that* Q *is a fake surface and the preimages of* f *are either points or arcs or* 2*-disks.*

(b) *Every dimensionally homogeneous* 2*-polyhedron* P *with a trivial manifold set has a resolution* (Q, f) *such that* Q *is a special* 2*-polyhedron and the preimages of* f *are either points or arcs or* 2*-disks.*

Theorem 1 is interesting because the resolutions obtained are special cases of *cell-like resolutions*, which play an important role in geometric topology (cf. [La77, Da86, MR88]). A polyhedron is said to be *cell-like* if and only if it is contractible. Note that this definition agrees with the standard one, since polyhedra are ANR's. An onto map is said to be *cell-like* if it is a proper surjective map with cell-like point-inverses.

It follows from Theorem 1(a) that in order to prove the Whitehead asphericity conjecture (which asserts that any subpolyhedron of an aspherical 2-polyhedron is also aspherical), it suffices to consider only fake surfaces. Indeed, by [La77] or [MR88], the restriction of f onto the preimage of every subpolyhedron is a homotopy equivalence, and the reduction follows. Theorem 1(b) has an analogous corollary.

Theorem 1 can possibly be applied to prove the following conjecture: *There exists a special* 2*-polyhedron* Q *which does not embed into* $\mathbb{R}^4$. This conjecture is interesting in connection with the well known fact that every 2-manifold embeds into $\mathbb{R}^4$. As a candidate we propose to take Q to be a resolution, given by Theorem 1(b), of the 2-skeleton P of the 6-dimensional simplex. The reason why this example might work is that if $Q \subset \mathbb{R}^4$, then by contracting in $\mathbb{R}^4$ the preimages of the resolution we are likely to obtain $\mathbb{R}^4$, in which P is embedded. The latter is well known to be impossible.

An n-polyhedron is called a *fake n-manifold* if each of its points has a closed neighbourhood homeomorphic to the product of I^{n-k} with a cone over the $(k-1)$-skeleton of the $(k+1)$-simplex for some $k = 0, \dots, n$ [Ma73, §4]. By P' we denote the *intrinsic* $(n-1)$-skeleton of P, i.e. the set of points of P having no neighbourhood homeomorphic to a closed n-disk. A fake n-manifold is called a *special n-polyhedron* (in the sense of Matveev [Ma73, §4]) if its manifold set $P - P'$ is trivial, i.e. is a disjoint union of open n-disks. We remark that this definition does not agree with that from above for $n = 2$, but a *connected* special 2-polyhedron in the sense of Matveev is not special in the sense of Casler [HMS93] if and only if it is one of the following three polyhedra:

(a) S^2 with a disk glued to the equator along the boundary with degree 1,

(b) $\mathbb{R}P^2$ with a disk glued to the "equator" along the boundary with degree 1, and

(c) the quotient space of the disk $D^2 = \{z \in \mathbb{C} : |z| \le 1\}$ under identification of the points $e^{i\phi}$, $e^{i(\phi+2\pi/3)}$ and $e^{i(\phi+4\pi/3)}$.

CONJECTURE 2. (a) *Every n-polyhedron P has a resolution (Q, f) such that Q is a fake n-manifold and the point-inverses of f are disks of dimensions $0, 1, \dots, n$.*

(b) *Every dimensionally homogeneous n-polyhedron P with a trivial manifold set has a resolution (Q, f) such that Q is a special n-polyhedron (in the sense of Matveev) and the point-inverses of f are disks of dimensions $0, 1, \dots, n$.*

The 1-dimensional case of Conjecture 2 is obvious (it is proved by blowing up the vertices into arcs, see Figure 3(a).

Observe that a point has a cell-like resolution by a special 2-polyhedron (e.g. the Bing house with two rooms [RS72]). Also, the 2-sphere S^2 has a cell-like resolution by a special 2-polyhedron (the union of a torus with two disks, attached to the longitude and the meridian of the torus, is mapped to S^2 by shrinking both disks to a point). For a connected 2-polyhedron, distinct from the point and S^2, the conditions of Theorem 1(b) are not only sufficient but also necessary for the existence of a cell-like resolution by a special polyhedron [Sa].

Note that by the Moore theorem [Da86] we cannot replace in Theorem 1(a) "fake surfaces" by "surfaces" (even if point-inverses of f are only assumed to be contractible). We conjecture that the singularities in Theorem 1(a) (Figure 1(a)–(c)) cannot be reduced to the first two (Figure 1(a)–(b)), or, equivalently, that $P = \operatorname{Con} K_4$ (Figure 1(c)) has no cell-like resolution by a 2-polyhedron Q such that every point of Q has a neighbourhood of Figure 1(a)–(b). Here K_4 is the complete graph with 4 vertices.

This conjecture is not as obvious as it may seem. For there exist two 2-polyhedra A and B with the same collections of links, but A can be cell-like mapped onto $\mathrm{Con}\, K_4$, and B can be cell-like resolved by a 2-polyhedron Q such that every point of Q has a neighbourhood shown in Figure 1(a)–(b). Here A is the mapping cylinder of the simplicial map $K_4 \to I$, mapping two vertices of K_4 to one end of I and two other vertices to the other end. Let G be a graph with vertices a, b, a_1, b_1 and edges aa_1, bb_1 of multiplicity 1 and ab, a_1b_1 of multiplicity 2. Then B is the mapping cylinder of the simplicial map $g : G \to I$, mapping a, a_1 to one end of I, and b, b_1 to the other.

Our proofs of Theorems 1(a) and 1(b) are an application and an extension of the "banana and pineapple" trick [HMS93, p. 36] (although we use a different language). Observe that the construction of a special polyhedron from a fake surface P [HMS93, p. 37] in fact gives a resolution of P. But certain fibres of this resolution are circles, hence this resolution is not *cell-like*.

Proof of Theorem 1(a) (see Figure 2). Take a triangulation T of P. For each vertex v of T take a 2-disk V and put $fV = v$. For each edge e of T take the rectangle $E = e \times [0, 1]$. Suppose that e joins the vertices v_1 and v_2.

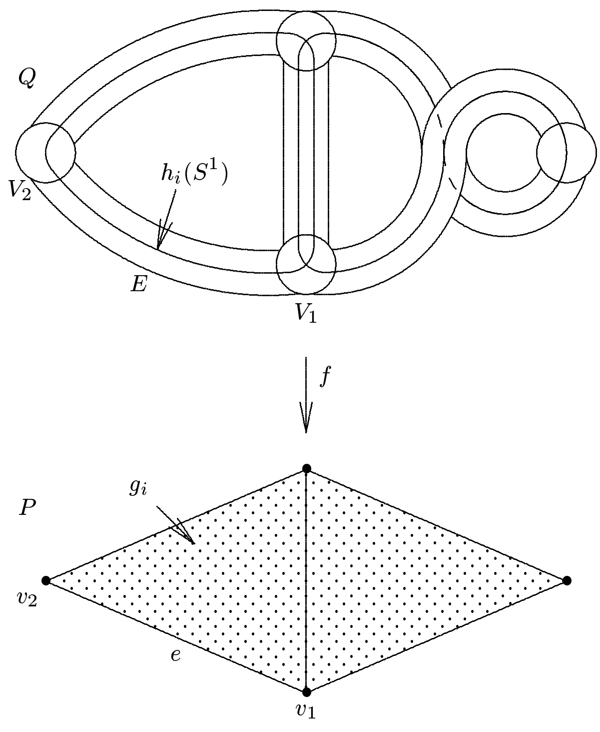

Fig. 2

Attach E to V_1 and V_2 so that $v_1 \times [0,1] \subset \partial V_1$ and $v_2 \times [0,1] \subset \partial V_2$. Clearly, it is possible to glue $\bigcup E_i$ to $\bigcup V_i$ so that $E_i \cap E_j = \emptyset$ for $i \neq j$.

For each point $c \in e$ and $x \in [0,1]$ put $f(c \times x) = c$. For each 2-simplex g_i of P take a general position embedding $h_i : S^1 \to (\bigcup V_i) \cup (\bigcup E_i)$ such that $f \circ h_i$ is the attaching map of g_i. By general position, $h_i(S^1)$ and $h_j(S^1)$ have only transversal intersections. For each g_i take a 2-disk $G_i \cong g_i$ and attach it to $(\bigcup V_i) \cup (\bigcup E_i)$ along $h_i(S^1)$. Let $Q = (\bigcup V_i) \cup (\bigcup E_i) \cup (\bigcup G_i)$. Extend the map $f : (\bigcup V_i) \cup (\bigcup E_i) \to P$ over $\bigcup G_i$ by the homeomorphism $G_i \cong g_i$. It is clear that (Q, f) is the required resolution. ∎

Proof of Theorem 1(b). The closure e of any connected component of $P' - P''$ is either an arc or a circle. Let a and b be the points of $e \cap P''$, possibly $a = b$ (if there are none, take any point $a = b \in e$). Then $U_e = R_P(ab, \{a, b\})$ is a suspension over a k-star, i.e. over a cone over a finite set with k elements (if $a = b$, then the vertices of the suspension are identified).

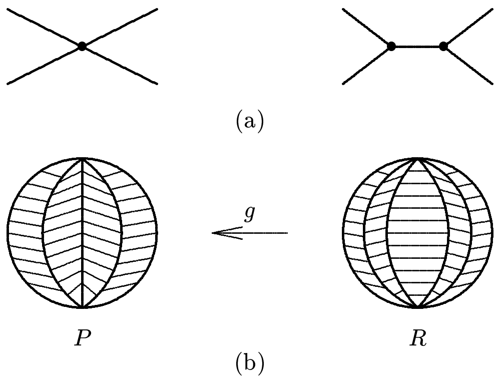

Fig. 3

Take a resolution of this star obtained by blowing up the vertices into arcs (Figure 3(a)) and extend it as a suspension to a resolution of U_e (Figure 3(b)). Define a resolution g in the same way over each U_e (we may assume that these U_e have disjoint interiors) and extend it identically over the rest of P to get a resolution $g : R \to P$. Clearly, each point of $R' - R''$ has a neighbourhood in R of type 1(b), the preimages of g are either points or arcs, R is dimensionally homogeneous and $R - R'$ is trivial.

Now, for each point $a \in R'$ the graph $\operatorname{lk} a$ is *cubic* (i.e., has degrees of vertices 1, 2 or 3). Take a general position map $u : \operatorname{lk} a \to \mathbb{R}^2$. By the "second Reidemeister moves" shown in Figure 4(a) we can modify u so that:

(1) the complement of the unbounded component of $\mathbb{R}^2 - u(\operatorname{lk} a)$ is a closed 2-disk D^2;

(2) any bounded component C of $\mathbb{R}^2 - u(\operatorname{lk} a)$ is an open disk and $\partial D^2 \cap \operatorname{Cl} C$ is connected;

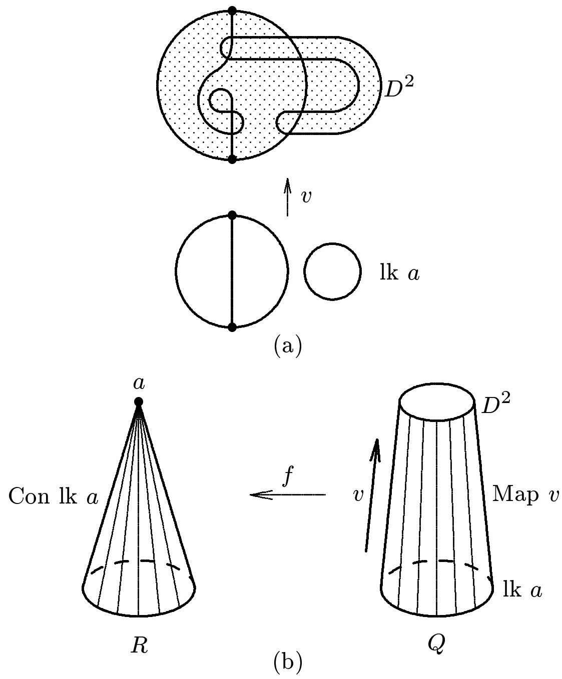

Fig. 4

(3) no connected component of $u(\operatorname{lk} a)$ is a circle;

(4) no interior of a path in $u(\operatorname{lk} a)$ with ends on D^2 is a connected component of $u(\operatorname{lk} a)$.

Denote by $v : \operatorname{lk} a \to D^2$ the restriction of the modified map u, and by $\operatorname{Map} v$ the mapping cylinder of v. Let $h : \operatorname{Map} v \to \operatorname{Con} \operatorname{lk} a$ be a contraction of $D^2 \subset \operatorname{Map} v$ to a point (Figure 4(b)). Then h is a resolution of sufficiently small $R_R(a) \cong \operatorname{Con} \operatorname{lk} a$.

Choose one point on each circle in R' that is a connected component of R'. Define h in the same way over small disjoint regular neighbourhoods of points of R'' and chosen points. Then extend it identically over the rest of R to get a resolution $h : Q \to R$.

Clearly, the preimages of h are either points or 2-disks. Since h is a homeomorphism over h-preimages of non-trivial preimages of g, it follows that the preimages of $f = h \circ g$ are either points or arcs or 2-disks. It is easy to check that Q is a fake surface. By (1)–(4) above, both $Q - Q'$ and $Q' - Q''$ are trivial, hence Q is indeed special. ∎

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