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ROOTS OF NAKAYAMA AND AUSLANDER–REITEN TRANSLATIONS

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Abstract. We discuss the roots of the Nakayama and Auslander–Reiten translations in the derived category of coherent sheaves over a weighted projective line. As an application we derive some new results on the structure of selfinjective algebras of canonical type.

Throughout this paper K will denote a fixed algebraically closed field. We work in the derived category $\mathrm{D}^{\mathrm{b}}(\mathbb{X})$ of the category $\mathrm{coh}\,\mathbb{X}$ of coherent sheaves on a weighted projective line $\mathbb{X}$ over K. We investigate whether, for a positive integer d, one of the automorphisms

$$\tau T^2, \quad \varrho\tau T^2, \quad \tau,$$

that is, the Nakayama translation, a twisted Nakayama translation or the Auslander–Reiten translation, respectively, has a dth root in the automorphism group of $\mathrm{D}^{\mathrm{b}}(\mathbb{X})$. Here, ϱ denotes a rigid automorphism, that is, an automorphism of $\mathrm{coh}\,\mathbb{X}$—identified with a member of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(\mathbb{X}))$—which preserves all Auslander–Reiten components and also the slope of indecomposable objects; further, T denotes the translation shift in the derived category $\mathrm{D}^{\mathrm{b}}(\mathbb{X})$. Let $\mathrm{Pic}_0\,\mathbb{X}$ denote the torsion group of the Picard group of $\mathbb{X}$, and let $\mathrm{Aut}\,\mathbb{X}$ denote the automorphism group of $\mathbb{X}$, identified with the group of all isomorphism classes of selfequivalences of the category $\mathrm{coh}\,\mathbb{X}$ fixing the structure sheaf. It then follows from [9] that the rigid automorphisms form a subgroup of $\mathrm{Pic}_0\,\mathbb{X} \rtimes \mathrm{Aut}\,\mathbb{X}$, and, moreover, this group is finite if $\mathbb{X}$ has at least three exceptional points.

Throughout the paper, by an *automorphism* we mean the isomorphism class of a selfequivalence of K-categories. When applied to a finite-dimensional basic K-algebra A, this means to identify automorphisms that differ by an inner automorphism. In particular, we say that an automorphism of A is *non-trivial* if it is not inner.

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Our interest in the above problem is motivated by recent investigations of the category $D^b(\mathbb{X})$ and by open problems in the representation theory of finite-dimensional selfinjective algebras. With each tilting sheaf Σ on $\mathbb{X}$ we may associate the finite-dimensional endomorphism algebra $B = \operatorname{End}\Sigma$ such that we have equivalences of the triangulated categories

$$\underline{\operatorname{mod}}\,\widehat{B} \cong D^b(\operatorname{mod} B) \cong D^b(\mathbb{X}),$$

where $\underline{\operatorname{mod}}\,\widehat{B}$ is the stable module category of the module category $\operatorname{mod}\widehat{B}$ of the repetitive algebra $\widehat{B}$ of B and $D^b(\operatorname{mod} B)$ is the derived category of bounded complexes in the module category $\operatorname{mod} B$ of B (see [5]). Then the automorphisms τT^2, $\varrho\tau T^2$, τ are induced by the Nakayama translation $\nu_{\widehat{B}}$, a twisted Nakayama translation $\varrho\nu_{\widehat{B}}$, the Auslander–Reiten translation $\tau_{\widehat{B}}$ on $\operatorname{mod}\widehat{B}$ on the stable level $\underline{\operatorname{mod}}\,\widehat{B}$. Moreover, for each admissible group G of K-linear automorphisms of $\widehat{B}$, the orbit algebra $\widehat{B}/G$ is a finite-dimensional selfinjective algebra whose representation theory is closely related to the representation theory of $\operatorname{mod}\widehat{B}$, and consequently to the sheaf theory on the weighted projective line $\mathbb{X}$. We show, in particular, that our study has applications leading towards the classification of selfinjective algebras of tubular canonical type.

1. Background. Let $\mathbb{X} = \mathbb{X}(\mathbf{p}, \underline{\lambda})$ be a weighted projective line in the sense of [4]. Roughly speaking, $\mathbb{X}$ is the projective line $\mathbb{P}_1(K)$ with a finite number $\lambda_1, \ldots, \lambda_t$ of marked points with attached positive integral weights $p_1, \ldots, p_t$, respectively. The category $\operatorname{coh}\mathbb{X}$ of coherent sheaves on $\mathbb{X}$ is an abelian category which is hereditary and noetherian, and which has a tilting object. Since $\operatorname{coh}\mathbb{X}$ is hereditary, each indecomposable object of the bounded derived category $D^b(\mathbb{X})$ has the form $X[n]$ with $X \in \operatorname{coh}\mathbb{X}$, and hence the structure of $D^b(\mathbb{X})$ is explicitly known to the same extent as $\operatorname{coh}\mathbb{X}$ is known.

The *weight type* of $\mathbb{X}$ is denoted by $\mathbf{p} = (p_1, \ldots, p_t)$, and we put $p = \operatorname{lcm}(p_1, \ldots, p_t)$. The *Picard group* $\operatorname{Pic}\mathbb{X}$ of $\mathbb{X}$ is isomorphic to the rank one abelian group $\mathbb{L}(\mathbf{p})$ on generators $\vec{x}_1, \ldots, \vec{x}_t$ with relations

$$\vec{c} := p_1\vec{x}_1 = \ldots = p_t\vec{x}_t.$$

The natural isomorphism $\sigma : \mathbb{L}(\mathbf{p}) \to \operatorname{Pic}\mathbb{X}$ maps each member $\vec{x}$ of $\mathbb{L}(\mathbf{p})$ to the associated line bundle shift $\sigma(\vec{x})$ sending X to $X(\vec{x})$. Hence the torsion group $\operatorname{Pic}_0\mathbb{X}$ of the Picard group is isomorphic to the torsion group $t\mathbb{L}(\mathbf{p})$ of $\mathbb{L}(\mathbf{p})$, which agrees with the kernel of the degree homomorphism $\delta : \mathbb{L}(\mathbf{p}) \to \mathbb{Z}$, given on the generators by $\delta(\vec{x}_i) = p/p_i$. Because $p\vec{x} = \delta(\vec{x})\vec{c}$ for each $\vec{x}$, an element $\vec{x} \in \mathbb{L}(\mathbf{p})$ is torsion if and only if the order of $\vec{x}$ is a divisor of p.

LEMMA 1.1. *The torsion group $t\mathbb{L}(\mathbf{p})$ of $\mathbb{L}(\mathbf{p})$ has order $p_1 \ldots p_t/p$.*

P r o o f. We put $\mathbb{L} = \mathbb{L}(\mathbf{p})$. Since the degree map $\delta : \mathbb{L} \to \mathbb{Z}$ is surjective there is an element $\vec{u}$ of degree one, and hence $\mathbb{L} = t\mathbb{L} \oplus \mathbb{Z}\vec{u}$. The formula $p\vec{x} = \delta(\vec{x})\vec{c}$ is valid for any $\vec{x} \in \mathbb{L}$, so $p\vec{u} = \vec{c}$. By means of δ the subgroups $\mathbb{Z}\vec{u}$ and $\mathbb{Z}\vec{c}$ are thus mapped isomorphically onto $\mathbb{Z}$ and its subgroup $p\mathbb{Z}$, respectively. Therefore, $[\mathbb{Z}\vec{u} : \mathbb{Z}\vec{c}] = p$. In view of the relations for $\mathbb{L}$ we get $\mathbb{L}/\mathbb{Z}\vec{c} = \prod_{i=1}^{t} \mathbb{Z}/p_i\mathbb{Z}$, hence

$$p_1 \ldots p_t = [\mathbb{L} : \mathbb{Z}\vec{c}] = [\mathbb{L} : \mathbb{Z}\vec{u}\,][\mathbb{Z}\vec{u} : \mathbb{Z}\vec{c}\,] = |t\mathbb{L}|p,$$

which implies the claim. ■

Let $\vec{\omega}$ denote the *dualizing element* $\vec{\omega} = (t-2)\vec{c} - \sum_{i=1}^{t} \vec{x}_i$. We note that the line bundle shift $\sigma(\vec{\omega})$ equals the Auslander–Reiten translation of $\mathrm{D^b}(\mathbb{X})$ [4, Corollary 2.3]. To emphasize the dependence on the weight type we also write $\delta(\mathbf{p})$ for the degree $\delta(\vec{\omega}) = (t-2)p - \sum_{i=1}^{t} p/p_i$ of the dualizing element, and call this integer the *discriminant* of $\mathbf{p}$.

2. The non-tubular case. We first investigate the non-tubular case, that is, assume $\delta(\mathbf{p}) \neq 0$. Let $\mathrm{K}_0(\mathbb{X})$ denote the Grothendieck group of the category $\operatorname{coh}\mathbb{X}$. The *rank* is the unique surjective linear form rank : $\mathrm{K}_0(\mathbb{X}) \to \mathbb{Z}$ which is non-negative on (classes of) objects from $\operatorname{coh}\mathbb{X}$. Let $\mathcal{O}$ denote the structure sheaf of $\mathbb{X}$. The *degree* deg : $\mathrm{K}_0(\mathbb{X}) \to \mathbb{Z}$ is a linear form which maps (the class of) each line bundle $\mathcal{O}(\vec{x})$ to $\delta(\vec{x})$, hence each indecomposable sheaf of rank zero to a positive integer. It follows [4] that each indecomposable object X in $\mathrm{D^b}(\mathbb{X})$ has a non-zero rank or a non-zero degree, and hence a well defined *slope* $\mu(X) = \deg X/\operatorname{rank} X \in \mathbb{Q} \cup \{\infty\}$. (By contrast, a decomposable non-zero object X in $\mathrm{D^b}(\mathbb{X})$ may have zero rank and degree, so that the slope of X is not defined.) Recall also that an automorphism ϱ of $\mathrm{D^b}(\mathbb{X})$ is called *rigid* if ϱ preserves Auslander–Reiten components and the slope of indecomposable objects.

The following is taken from [10]:

PROPOSITION 2.1. *Let $\mathbb{X}$ be non-tubular, that is, $\delta(\vec{\omega}) \neq 0$. Then for each automorphism α of $\mathrm{D^b}(\mathbb{X})$ there is an integer $d(\alpha)$, the degree of α, such that*

$$\mu(\alpha X) = \mu(X) + d(\alpha)$$

for each indecomposable object X of $\mathrm{D^b}(\mathbb{X})$. ■

In particular, each rigid automorphism ϱ has degree $d(\varrho) = 0$.

COROLLARY 2.2. *Let $d \geq 2$ and assume there exists an automorphism σ of $\mathrm{D^b}(\mathbb{X})$ such that σ^d equals a twisted Nakayama translation $\varrho\tau T^2$ for some rigid automorphism ϱ. Then d equals two and $\delta(\vec{\omega})$ is even.*

P r o o f. It follows from [9] that there is a unique representation $\sigma = \alpha T^n$ for some automorphism α of $\operatorname{coh}\mathbb{X}$ and $n \in \mathbb{Z}$. Since T belongs to the center of $\mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(\mathbb{X}))$ we get

$$\varrho\tau T^2 = \sigma^d = \alpha^d T^{dn},$$

hence $dn = 2$ and $\alpha^d = \varrho\tau$. Since $d \geq 2$, this implies $d = 2$ and $n = 1$. Moreover, passing to degrees in $\alpha^2 = \varrho\tau$ implies $2d(\alpha) = d(\varrho) + d(\tau)$. Since $d(\varrho) = 0$ and $d(\tau) = \delta(\vec{\omega})$ (see [4, Corollary 2.3]), this shows $2d(\alpha) = \delta(\vec{\omega})$, hence $\delta(\vec{\omega})$ is even. ■

The case of $\delta(\vec{\omega})$ even is easily characterized:

LEMMA 2.3. *Let* $\mathbf{p} = (p_1, \ldots, p_t)$. *The discriminant* $\delta(\mathbf{p}) = \delta(\vec{\omega})$ *is even if and only if for a fixed integer* $m \geq 1$ *we have*:

(i) *an even number of* p_i*'s has the form* $2^m q_i$ *for some odd number* q_i,

(ii) *the other* p_j*'s have the form* $2^l q_j$, *where* $0 \leq l < m$ *and the numbers* q_i *are odd.*

Proof. The discriminant $\delta(\mathbf{p}) = \sum_{i=1}^{t}(p_i - 1)p/p_i - 2p$ is even if and only if there are an even number of odd summands $(p_i - 1)p/p_i$. Now, $(p_i - 1)p/p_i$ is odd if and only if p_i is even and p/p_i is odd. The latter means that the 2-part of p_i equals the 2-part 2^m of p. Hence an even number of p_i's have this maximal 2-part 2^m of p. ■

We put $N(\mathbf{p}) = 2 + \sum_{i=1}^{t}(p_i - 1)$, which is the rank of the Grothendieck group $\mathrm{K}_0(\mathbb{X})$.

REMARKS. (i) If $\delta(\mathbf{p}) < 0$, which corresponds to domestic type, then $\delta(\mathbf{p})$ is even if and only if we are in one of the following two cases:

(a) $p = (2, 2, 2n + 1)$, in which case $N(\mathbf{p}) = 2(n + 2)$ is even,

(b) $p = (2^m p, 2^m q)$ with p and q odd, in which case $N(\mathbf{p}) = 2^m(p + q)$ is also even.

(ii) None of the minimal wild canonical types

$$(2, 3, 7), \quad (2, 4, 5), \quad (3, 3, 4), \quad (2, 2, 2, 3), \quad (2, 2, 2, 2, 2)$$

yields an even discriminant. In fact, all these cases yield discriminant one.

(iii) On the other hand, the weight sequence

$$\mathbf{p} = (2, \ldots, 2), \qquad \text{with } r \text{ entries},$$

has discriminant $\delta(\mathbf{p}) = r - 4$, hence it is even if and only if r is even, in which case $N(\mathbf{p}) = r + 2$ is also even.

(iv) For the weight sequence $\mathbf{p} = (2, 4, 2m + 1)$, $m \geq 1$, we have $N(\mathbf{p}) = 2(m + 3)$ and $\delta(\mathbf{p}) = 2m - 3$. Note that the $2(m + 3)$, $m \geq 1$, exhaust all even numbers ≥ 8, and the $2m - 3$, $m \geq 2$, exhaust all odd natural numbers.

(v) For the weight sequence $\mathbf{p} = (2m + 1, 2m + 1, 2m + 1)$, $m \geq 1$, we have $N(\mathbf{p}) = 2(3m + 1)$ and $\delta(\mathbf{p}) = 4m - 1$.

(vi) For the weight sequence $\mathbf{p} = (2,2,3,3)$ we have $N(\mathbf{p}) = 8$ and $\delta(\mathbf{p}) = 2$. Note that the category $\operatorname{coh}\mathbb{X}$ has wild representation type.

We next investigate when the Auslander–Reiten translation τ in $\mathrm{D}^{\mathrm{b}}(\mathbb{X})$ has a dth root. We interpret the Picard group $\operatorname{Pic}\mathbb{X}$ as the group of automorphisms of $\mathrm{D}^{\mathrm{b}}(\mathbb{X})$ induced by the shift automorphisms $X \mapsto X(\vec{x})$ with $\vec{x} \in \mathbb{L}(\mathbf{p})$.

PROPOSITION 2.4. *Assume that* $\mathbb{X}$ *has non-tubular weight type* $\mathbf{p}$. *Then*:

(i) *If* $\sigma^d = \tau$ *for some automorphism* σ, *then* d *is a divisor of* $\delta(\mathbf{p})$.

(ii) *Conversely*, *let* d *be a divisor of* $\delta(\mathbf{p})$ *and assume* d *and* $p_1 \dots p_t/p$ *are coprime*; *then there exists a unique* $\sigma \in \operatorname{Pic}\mathbb{X}$ *such that* $\sigma^d = \tau$.

Proof. Assertion (i) follows from Proposition 2.1 on passing to degrees of automorphisms. Concerning (ii) we consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \to & t\mathbb{L}(\mathbf{p}) & \to & \mathbb{L}(\mathbf{p}) & \xrightarrow{\delta} & \mathbb{Z} & \to & 0 \\
 & & \cong\downarrow\cdot d & & \downarrow\cdot d & & \downarrow\cdot d & & \\
0 & \to & t\mathbb{L}(\mathbf{p}) & \to & \mathbb{L}(\mathbf{p}) & \xrightarrow{\delta} & \mathbb{Z} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \to & C & \xrightarrow{\cong} & \mathbb{Z}/\mathbb{Z}d & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By the assumption on d the vertical map on the left side is an isomorphism and the remaining two are monomorphisms. Moreover, the dualizing element $\vec{\omega}$ is sent to zero under the composition $\mathbb{L}(\mathbf{p}) \to \mathbb{Z} \to \mathbb{Z}/\mathbb{Z}d$, hence is sent to zero under $\mathbb{L}(\mathbf{p}) \to C$. By exactness of the middle vertical column this yields a unique element $\vec{x}$ from $\mathbb{L}(\mathbf{p})$ with $d\vec{x} = \vec{\omega}$. Consequently, $\sigma(\vec{x})^d = \sigma(\vec{\omega}) = \tau$, and the claim follows. ■

COROLLARY 2.5. *Let* $\mathbf{p} = (2,\dots,2)$ *with* r *entries and* $\delta := \delta(\mathbf{p}) = r-4$. *Then* τ *has a* δ*th root if* δ, *or equivalently* r, *is odd. More generally*, *for* $\mathbf{p} = (q,\dots,q)$ *with* r *entries and* q *prime we get* $\delta(\mathbf{p}) = (q-1)r - 2q$ *and* $|t\mathbb{L}(\mathbf{p})| = q^{r-1}$. *Hence*, *if* $q \nmid r$ *then* τ *has a* $\delta(\mathbf{p})$*th root.* ■

3. The tubular case. We start with a general result and note that the automorphisms of $\mathrm{D}^{\mathrm{b}}(\mathbb{X})$ of finite order form the subgroup $\operatorname{Pic}_0\mathbb{X} \rtimes \operatorname{Aut}(\mathbb{X})$ (see [9]).

PROPOSITION 3.1. *Assume that* $\mathbb{X}$ *is tubular and let* σ *be a* d*th root*, $d \geq 1$, *of a twisted Nakayama translation* $\varrho(\tau T^2)$. *Then* $d \in \{1,2,3,4,6\}$. *Moreover*, *each rigid automorphism of* $\mathrm{D}^{\mathrm{b}}(\mathbb{X})$ *has finite order.*

Proof. Note that σ induces an automorphism of $\mathrm{K}_0(\mathbb{X}) = \mathrm{K}_0(\mathrm{D}^\mathrm{b}(\mathbb{X}))$, preserving the Euler form. Let R denote the radical of $\mathrm{K}_0(\mathbb{X})$, that is, the radical of the quadratic form attached to the Euler form. Then restriction to R yields an automorphism σ_R of R preserving the Euler form. Note that T induces the map (identity) on $\mathrm{K}_0(\mathbb{X})$. Moreover, each automorphism of finite order of $\mathrm{D}(\mathbb{X})$ induces the identity map on R (see [9, Theorem 6.3]). Hence $\varrho_R = 1$, $\tau_R = 1$, and our assumption on σ implies that $\sigma_R^d = 1$.

By tubularity of $\mathbb{X}$, $R \cong \mathbb{Z}^2$ and we get $\mathrm{Aut}(R) \cong \mathrm{SL}_2(\mathbb{Z})$, since the Euler form is skew-symmetric and non-degenerate on R (see [8]). Thus σ_R becomes an element of finite order of $\mathrm{SL}_2(\mathbb{Z})$ and it is well known and elementary to prove that the only possible orders are $1, 2, 3, 4$ and 6. Hence the order d' of σ_R belongs to the set $\{1, 2, 3, 4, 6\}$ and divides d, so that $d = d'd''$ for a positive integer d''. Since the automorphism $\sigma^{d'}$ induces the identity on R, it preserves the slope and thus $\sigma^{d'} = \alpha\beta T^r$, where $\alpha \in \mathrm{Pic}_0\,\mathbb{X}$, $\beta \in \mathrm{Aut}(\mathbb{X})$ and $r \in \mathbb{Z}$ (see [9, Proposition 4.4]). Since T induces -1_R on R and $\alpha_R = 1$, $\beta_R = 1$, it follows that T^r induces the identity on R, and r is even. Next it follows that

$$\varrho\tau T^2 = \sigma^d = (\alpha\beta)^{d''} T^{rd''},$$

which implies $rd'' = 2$, hence $r = 2$ and $d'' = 1$. Thus $d = d'$ and the first claim follows.

Concerning the second claim we recall from the first part of the proof that a rigid automorphism ϱ preserves the slope, hence induces the identity map on the radical R. Hence by the above argument ϱ has the form $\alpha\beta T^r$ for some $\alpha \in \mathrm{Pic}_0\,\mathbb{X}$, $\beta \in \mathrm{Aut}(\mathbb{X})$ and $r \in \mathbb{Z}$. Since ϱ preserves all Auslander–Reiten components it follows that $r = 0$. ∎

We now investigate—first on the stable level—whether actually a twisted Nakayama translation has a dth root. We invoke a result from [9]:

Proposition 3.2. *If $\mathbb{X}$ is tubular of weight type $\mathbf{p}$, then there are tubular mutations L and S of $\mathrm{D}^\mathrm{b}(\mathbb{X})$ such that $LSL = SLS$ and the subgroup $\langle L, S\rangle$ of the automorphism group of $\mathrm{D}^\mathrm{b}(\mathbb{X})$ is isomorphic to the braid group on three strands. Moreover*,

$$(LS)^3 = (\mathrm{SL})^3 = \begin{cases} \gamma T & \text{if } \mathbf{p} = (3,3,3), \\ \tau^{-3} T & \text{else.} \end{cases}$$

Here, γ exchanges the two exceptional points not invoked in the construction of the tubular mutation S, hence $\gamma^2 = 1$. ∎

The equivalences L and S play a central role in the classification of indecomposable bundles in the category $\mathrm{coh}\,\mathbb{X}$, or equivalently of indecomposable objects in the derived category $\mathrm{D}^\mathrm{b}(\mathbb{X})$; see [8] for further information.

We next discuss, separately for each tubular weight type $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$ and $(2,3,6)$, whether a (twisted) Nakayama translation can actually have a dth root for $d \in \{2,3,4,6\}$. Observe that τ and T are always central in the automorphism group of $\mathrm{D}^{\mathrm{b}}(\mathbb{X})$. Note also that in view of Proposition 3.2 and the braid group relations $SLS = LSL$ we have

$$(LSL)^4 = (LSLS)^3 = (LS)^6 = \tau^{-6}T^2.$$

Case $(2,2,2,2)$. Here, $\tau^2 = 1$. Hence $\sigma_1 = \tau(LS)^2$ is a 3rd root of the Nakayama translation τT^2. Further, $\sigma_2 = LS$ is a 6th root and $\sigma_3 = LSL$ is a 4th root of the twisted Nakayama translation $\tau^{-1}(\tau T^2)$. It is more difficult, in this framework, to establish a square root of the Nakayama translation itself: Recall that $\sigma(\vec{x})$ denotes the line bundle shift associated with $\vec{x} \in \mathbb{L}(\mathbf{p})$. Moreover, following [8] we have $S = \sigma(\vec{x}_4)$. Then

$$\phi = (L\sigma(\vec{x}_1 - \vec{x}_2)S)(L\sigma(\vec{x}_1 - \vec{x}_3)S)(L\sigma(\vec{x}_2 - \vec{x}_3)S)$$

is an automorphism with $\phi^2 = \tau T^2$. The proof is analogous to [9, Proposition 7.2]. Note that $\vec{x}_1 - \vec{x}_2$, $\vec{x}_1 - \vec{x}_3$ and $\vec{x}_2 - \vec{x}_3$ belong to the torsion group of $\mathbb{L}(\mathbf{p})$.

Case $(3,3,3)$. Here, $\tau^3 = 1$. Hence $\sigma_1 = \tau LSL$ is a 4th root and $\sigma_2 = \tau^2 T$, $\sigma_3 = \tau^2\gamma T$ are 2nd roots of the Nakayama translation τT^2. Finally, $\sigma_4 = (LS)^2$ is a 3rd root and $\sigma_5 = LS$ is a 2nd root of the twisted Nakayama translation $\varrho(\tau T^2)$, $\varrho = \tau^{-1}$.

Case $(2,4,4)$. Here, $\tau^4 = 1$. Then $\sigma_1 = \tau(LS)^2$ is a 3rd root of the Nakayama translation τT^2. Further, $\sigma_2 = LS$ is a 6th root and $\sigma_3 = LSL$ is a 4th root of the twisted Nakayama translation $\tau(\tau T^2)$.

Case $(2,3,6)$. For this weight type we refer to the next proposition.

Proposition 3.3. *Assume that* $\mathbb{X}$ *has tubular type* $(2,3,6)$. *Then each rigid automorphism of* $\mathrm{D}^{\mathrm{b}}(\mathbb{X})$ *is a power of* τ. *Moreover, the twisted stable Nakayama translation* $\tau^s(\tau T^2)$ *has a square root, a* 3*rd root, a* 4*th root or a* 6*th root exactly if* s *is a member of* $E_2 = \{1,3,5\}$, $E_3 = \{2,5\}$, $E_4 = \{1,3,5\}$ *or* $E_6 = \{5\}$, *respectively. In particular, the stable Nakayama translation* τT^2 *does not have a non-trivial root.*

Proof. For the type $(2,3,6)$ the torsion group $\mathrm{Pic}_0\,\mathbb{X}$ of the Picard group has order 6, hence it is the cyclic group generated by τ. Since the three weights are pairwise different, we moreover get $\mathrm{Aut}\,(\mathbb{X}) = 1$, hence $\mathrm{Pic}_0\,\mathbb{X} \rtimes \mathrm{Aut}\,(\mathbb{X}) = \langle\tau\rangle$. In particular, each rigid automorphism of $\mathrm{D}^{\mathrm{b}}(\mathbb{X})$ is of the form $\varrho = \tau^s$ for some $s = 0,\ldots,5$.

In view of Proposition 3.1 it suffices to investigate when there is a dth root of the twisted stable Nakayama translation $\varrho(\tau T^2)$, where $\varrho = \tau^s$ and $d \in \{2,3,4,6\}$. Now assume

$$\sigma^d = \varrho(\tau T^2) = \tau^{s+1}T^2.$$

Since $\mathrm{Pic}_0\,\mathbb{X} \rtimes \mathrm{Aut}(\mathbb{X}) = \langle\tau\rangle$, the automorphism σ has the form $\sigma = \tau^m w$ where w belongs to the braid group $\langle L, S\rangle$ (see [9, Theorem 6.3]). Therefore, $\tau^{s+1}T^2 = \sigma^d = \tau^{dm}w^d$.

Since τ and T are central in $\mathrm{Aut}(\mathrm{D^b}(\mathbb{X}))$ it follows that w^d is a central element of the braid group $\langle L, S\rangle$, and hence is a power of $(LS)^3$, say $w^d = (LS)^{3l}$. This last assertion is well known [3, p. 63] and follows from the easily established fact that $(LS)^3$ is central in $\langle L, S\rangle$ and the factor group $\langle L, S\rangle/\langle (LS)^3\rangle \cong P\,\mathrm{SL}_2(\mathbb{Z})$ has trivial center. Taking things together we obtain, by Proposition 3.2,

$$\tau^{s+1}T^2 = \tau^{dm}(LS)^{3l} = \tau^{dm-3l}T^l.$$

This in turn implies $l=2$, and further $dm \equiv s+1 \pmod 6$. For $d \in \{2,3,4,6\}$ this congruence yields the solution sets E_d listed above.

Conversely, the following list yields dth roots of the twisted Nakayama $\varrho(\tau T^2)$, where $\varrho = \tau^j$ with $j \in E_d$:

$\sigma_1 = \tau(LS)^3$ is a square root with $\varrho = \tau$,
$\sigma_2 = \tau^2(LS)^3$ is a square root with $\varrho = \tau^3$,
$\sigma_3 = (LS)^3$ is a square root with $\varrho = \tau^5$,
$\sigma_4 = \tau(LS)^2$ is a 3rd root with $\varrho = \tau^2$,
$\sigma_5 = (LS)^2$ is a 3rd root with $\varrho = \tau^5$,
$\sigma_6 = \tau^2 L\,\mathrm{SL}$ is a 4th root with $\varrho = \tau$,
$\sigma_7 = \tau L\,\mathrm{SL}\,2$ is a 4th root with $\varrho = \tau^3$,
$\sigma_8 = L\,\mathrm{SL}$ is a 4th root with $\varrho = \tau^5$,
$\sigma_9 = LS$ is a 6th root with $\varrho = \tau^5$. ■

4. Selfinjective algebras of canonical type. By an algebra we mean a finite-dimensional associative K-algebra with an identity, which we shall assume to be basic and connected. For an algebra A, we denote by $\mathrm{mod}\,A$ the category of finite dimensional (over K) right A-modules and by $D : \mathrm{mod}\,A \to \mathrm{mod}\,A^{\mathrm{op}}$ the standard duality $\mathrm{Hom}_K(-, K)$. An algebra A is called *selfinjective* if $A \cong D(A)$ in $\mathrm{mod}\,A$, that is, A_A is injective. Moreover, A is called *symmetric* if A and $D(A)$ are isomorphic as A-A-bimodules. An important class of selfinjective algebras is formed by the algebras of the form $\widehat{B}/G$, where $\widehat{B}$ is the *repetitive algebra* (locally finite-dimensional, without identity) [7]

$$\widehat{B} = \begin{bmatrix} \ddots & \ddots & & & & \\ & Q_{m-1} & B_{m-1} & & & \\ & & Q_m & B_m & & \\ & & & Q_{m+1} & B_{m+1} & \\ & & & & \ddots & \ddots \end{bmatrix}$$

of an algebra B, where $B_m = B$ and $Q_m = {}_BD(B)_B$ for all $m \in \mathbb{Z}$, the algebras B_m are placed on the main diagonal of $\widehat{B}$, all the remaining entries are zero, the matrices in $\widehat{B}$ have only finitely many non-zero elements, addition is the usual addition of matrices, multiplication is induced from the B-bimodule structure of $D(B)$ and the zero map $D(B) \otimes_B D(B) \to 0$, and G is an admissible group of K-linear automorphisms of $\widehat{B}$ (considered as the corresponding K-category). Recall that a group G of K-linear automorphisms of $\widehat{B}$ is called *admissible* if its action on the objects of $\widehat{B}$ is free and has finitely many orbits. Then the orbit algebra $\widehat{B}/G$ is a finite-dimensional selfinjective algebra. Denote by $\nu_{\widehat{B}}$ the *Nakayama automorphism* of $\widehat{B}$ shifting B_m to B_{m+1} and Q_m to Q_{m+1} for all $m \in \mathbb{Z}$. Then the infinite cyclic group $(\nu_{\widehat{B}})$ is admissible, and $\widehat{B}/(\nu_{\widehat{B}})$ is the trivial extension $B \ltimes D(B)$ of B by $D(B)$, and so it is symmetric. We note that if B has finite global dimension, then the stable module category $\underline{\operatorname{mod}}\,\widehat{B}$ is equivalent, as a triangulated category, to the derived category $\mathrm{D^b}(\operatorname{mod} B)$ of bounded complexes over $\operatorname{mod} B$ (see [5]).

Let $\mathbb{X} = \mathbb{X}(\mathbf{p}, \underline{\lambda})$ be a weighted projective line, depending on a weight sequence $\mathbf{p} = (p_1, \ldots, p_t)$ of positive integers, and a parameter sequence $\underline{\lambda}$ of pairwise distinct elements from the projective line over K. Then $\mathbb{X}$ has a tilting bundle whose endomorphism algebra is a canonical algebra $\Lambda = \Lambda(\mathbf{p}, \underline{\lambda})$ in the sense of [14], and in view of the equivalence $\mathrm{D^b}(\operatorname{mod} \Lambda) \cong \mathrm{D^b}(\operatorname{coh} \mathbb{X})$ (see [4]), the finite-dimensional representation theory of Λ is then completely determined by the sheaf theory on the weighted projective line $\mathbb{X}$. An algebra B is called *concealed-canonical* (respectively, *almost concealed-canonical*) of type Λ (or $\mathbb{X}$) if B is the endomorphism algebra of a tilting bundle (respectively, tilting sheaf) on the weighted projective line $\mathbb{X} = \mathbb{X}(\mathbf{p}, \underline{\lambda})$. More generally, B is called *derived canonical* of type Λ (or $\mathbb{X}$) if $\mathrm{D^b}(\operatorname{mod} B)$ is equivalent to $\mathrm{D^b}(\operatorname{coh} \mathbb{X}) \cong \mathrm{D^b}(\operatorname{mod} \Lambda)$.

Finally [11], keeping the notation above, by a *selfinjective algebra of canonical type* Λ (or $\mathbb{X}$) we mean an algebra of the form $\widehat{B}/G$, where B is a derived canonical algebra of type Λ, and G is an admissible torsion-free group of K-linear automorphisms of $\widehat{B}$. In fact, it is known that then $\widehat{B} = \widehat{B'}$ for an almost concealed-canonical algebra B' of type Λ and G is infinite cyclic (see [1, 11, 12, 15]). On the other hand, a complete understanding of the generators of G is strongly related to the problem whether there are certain roots of the twisted Nakayama automorphisms $\varrho\nu_{\widehat{B}}$ of $\widehat{B}$.

The representation type of a selfinjective algebra A of canonical type $\Lambda = \Lambda(\mathbf{p}, \underline{\lambda})$ is completely determined by the discriminant

$$\delta(\mathbf{p}) = p\Big((t-2) - \sum_{i=1}^{t} \frac{1}{p_i}\Big),$$

where $p = \text{l.c.m.}(p_1, \ldots, p_t)$. Namely, A is tame if and only if $\delta(\mathbf{p}) \le 0$.

More precisely, if $\delta(\mathbf{p}) < 0$ then A is a domestic selfinjective algebra of Euclidean type, while, for $\delta(\mathbf{p}) = 0$, A is a non-domestic polynomial growth selfinjective algebra of tubular type. We also note (see [15]) that the class of tame selfinjective algebras of canonical type coincides with the class of all representation-infinite polynomial growth selfinjective algebras which admit simply connected Galois coverings. We refer to [1, 12, 15] for the representation theory of this class of selfinjective algebras.

The structure of admissible (infinite) cyclic groups of automorphisms of the repetitive algebras of almost concealed-canonical algebras with negative discriminant (equivalently, tilted algebras of Euclidean type) is well understood [10, 15]. We shall now derive some consequences of the facts established in the previous sections to some other cases.

We have the following fact concerning selfinjective algebras of wild canonical type.

Theorem 4.1. *Let A be a selfinjective algebra of wild canonical type $\Lambda = \Lambda(\mathbf{p}, \underline{\lambda})$ such that $\delta(\mathbf{p})$ or $N(\mathbf{p})$ is odd. Then $A \cong \widehat{B}/(\varphi\nu_{\widehat{B}}^{m})$, where B is an almost concealed-canonical algebra of type Λ, m is a positive integer, and φ is a K-linear automorphism of $\widehat{B}$ induced by an isomorphism of B. In particular, the stable Auslander–Reiten quiver Γ_A^{s} of A has $2m$ $\mathbb{P}_1(K)$-families of stable tubes of tubular type $\mathbf{p}$.*

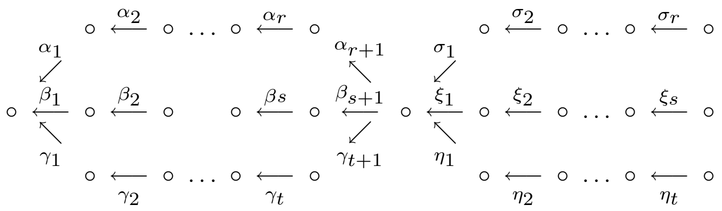
Proof. We know from [11, Theorem 3.7] that $A \cong \widehat{B}/G$ where B is an almost concealed-canonical algebra of type $\Lambda = \Lambda(\mathbf{p}, \underline{\lambda})$ and G is an infinite cyclic group of K-linear automorphisms of $\widehat{B}$. Moreover, if σ is a K-linear automorphism of $\widehat{B}$ such that $\sigma^d = \varrho\nu_{\widehat{B}}$ for some $d \geq 2$ and a rigid automorphism ϱ of $\widehat{B}$ then it follows from Corollary 2.2 that $d = 2$ and $\delta(\mathbf{p})$ is even. But if $\sigma^2 = \varrho\nu_{\widehat{B}}$ then $\mathrm{K}_0(B)$ is of even rank, and so $N(\mathbf{p})$ is even. Therefore, invoking [11, Lemma 3.6], we deduce that $G = (\varphi\nu_{\widehat{B}}^{m})$ for some positive integer m and a rigid automorphism φ of $\widehat{B}$. The final claim is a direct consequence of the facts that Γ_A is the quotient $\Gamma_{\widehat{B}}/(\varphi\nu_{\widehat{B}}^{m})$, the separating tubular families in the stable Auslander–Reiten quiver of $\widehat{B}$ have the same tubular type $\mathbf{p}$, and the stable Auslander–Reiten quiver of the trivial extension $B \ltimes D(B)$ has exactly two $\mathbb{P}_1(K)$-families of stable tubes (see [11, Section 3] for details). ∎

Applying [16, Theorem 3.2] and [17, Corollary 3.9] (see also [13, Theorem 2] we obtain the following consequence of the above theorem.

Corollary 4.2. *Let A be a selfinjective algebra of wild canonical type $\Lambda = \Lambda(\mathbf{p}, \underline{\lambda})$ such that $\delta(\mathbf{p})$ or $N(\mathbf{p})$ is odd. Then A is symmetric if and only if $A \cong B \ltimes D(B)$ for an almost concealed-canonical algebra B of type Λ.*

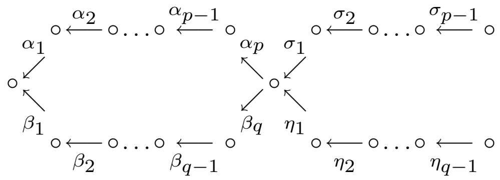
We end this section with some examples illustrating our considerations.

EXAMPLE 4.3. Let r, s, t be positive integers. Consider the algebra $B(r,s,t)$ given by the quiver



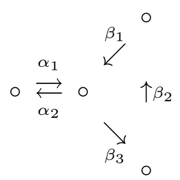
bound by $\alpha_{r+1}\alpha_r\ldots\alpha_1 + \beta_{s+1}\beta_s\ldots\beta_1 + \gamma_{t+1}\gamma_t\ldots\gamma_1 = 0$, $\sigma_1\alpha_{r+1} = 0$, $\xi_1\beta_{s+1} = 0$ and $\eta_1\gamma_{t+1} = 0$. Then $B = B(r,s,t)$ is an almost concealed-canonical algebra of type $(2r+1, 2s+1, 2t+1)$, and it is easy to see that $\widehat{B}$ admits a K-linear automorphism φ such that $\varphi^2 = \nu_{\widehat{B}}$.

EXAMPLE 4.4. Let $p, q \geq 2$ be two integers and a an element of $K \setminus \{0,1\}$. Consider the algebra $B(p,q,a)$ given by the quiver



bound by $\sigma_1\alpha_p\ldots\alpha_1 = \sigma_1\beta_q\ldots\beta_1$ and $\eta_1\alpha_p\ldots\alpha_1 = a\eta_1\beta_q\ldots\beta_1$. Then $B = B(p,q,a)$ is an almost concealed-canonical algebra of type (p,p,q,q), and it is easy to see that $\widehat{B}$ admits a K-linear automorphism φ such that $\varphi^2 = \nu_{\widehat{B}}$.

We note that if $B = B(r,s,t)$ or $B = B(p,q,a)$ then $A = \widehat{B}/(\varphi)$ is a symmetric algebra non-isomorphic to the trivial extension $B \ltimes D(B)$, because the rank of $K_0(A)$ is half the rank of $K_0(B \ltimes D(B))$. Moreover, the Auslander–Reiten quiver Γ_A of A has exactly one $\mathbb{P}_1(K)$-family of quasi-tubes (that is, the stable parts are tubes), and this family contains projective modules. For example, if $B = B(2,3,a)$ then $A = \widehat{B}/(\varphi)$ is the algebra given by the quiver



bound by the following relations: $\alpha_1\alpha_2\alpha_1 = \alpha_1\beta_3\beta_2\beta_1$, $\beta_1\alpha_2\alpha_1 = a\beta_1\beta_3\beta_2\beta_1$, $\alpha_2\alpha_1\alpha_2 = \beta_3\beta_2\beta_1\alpha_2$, $\alpha_2\alpha_1\beta_3 = a\beta_3\beta_2\beta_1\beta_3$, $\alpha_1\alpha_2\alpha_1\beta_3 = 0$, $\beta_1\beta_3\beta_2\beta_1\alpha_2 = 0$, and $\beta_2\beta_1\beta_3\beta_2\beta_1\beta_3\beta_2 = 0$.

We do not know any selfinjective algebra of wild canonical type whose Auslander–Reiten quiver has only one $\mathbb{P}_1(K)$-family of quasi-tubes, and all quasi-tubes in this family are stable tubes. It is equivalent to find a wild concealed-canonical algebra B such that $\widehat{B}$ admits a K-linear automorphism φ such that $\varphi^2 = \varrho\nu_{\widehat{B}}$ for some rigid automorphism ϱ of $\widehat{B}$. Note that for such an algebra B, the unique tubular family in Γ_B does not contain simple modules.

5. Selfinjective algebras of tubular type. Let B be an algebra and $e_1, \ldots, e_n$ be a complete set of primitive orthogonal idempotents of B such that $1 = e_1 + \ldots + e_n$. Denote by Q_B the (Gabriel) quiver of B with the set of vertices $1, \ldots, n$ corresponding to the set $e_1, \ldots, e_n$. For each vertex $i \in Q_B$, denote by $P_B(i)$ the indecomposable projective B-module e_iB and by $I_B(i)$ the indecomposable injective B-module $\mathrm{D}(Be_i)$. Then, for a sink $i \in Q_B$, the *reflection* $S_i^+ B$ *of* B *at* i is the quotient of the one-point extension $B[I_B(i)]$ by the two-sided ideal generated by e_i. The quiver $\sigma_i^+ Q_B$ of $S_i^+ B$ is called the *reflection of* Q_B *at* i. Observe that the sink i of Q_B is replaced in $\sigma_i^+ Q_B$ by a source i'. Moreover, we have

$$\widehat{B} \cong \widehat{S_i^+ B}.$$

A *reflection sequence of sinks* is a sequence $i_1, \ldots, i_t$ of vertices of Q_B such that i_s is a sink of $\sigma_{i_{s-1}}^+ \ldots \sigma_{i_1}^+ Q_B$ for $1 \le s \le t$ (see [7, (2.8)]). We have the following fact, proved in [12, Section 4], describing the relationship between tubular algebras with isomorphic repetitive algebras.

Theorem 5.1. *Let B be a tubular algebra with Q_B having n vertices. There is a sequence of natural numbers $1 \le t_1 < t_2 < \ldots < t_{r+1} = n$, uniquely determined by B, and a reflection sequence of sinks $i_1, \ldots, i_{t_1}$, $i_{t_1+1}, \ldots, i_{t_r}, i_{t_r+1}, \ldots, i_n$ in Q_B such that the following statements hold*:

(a) $S_{i_n}^+ \ldots S_{i_1}^+ B \cong \nu_{\widehat{B}}(B) \cong B$.

(b) $S_{i_{t_j}}^+ \ldots S_{i_1}^+ B$, $1 \le j \le r$, *are tubular algebras of the same tubular type as* B.

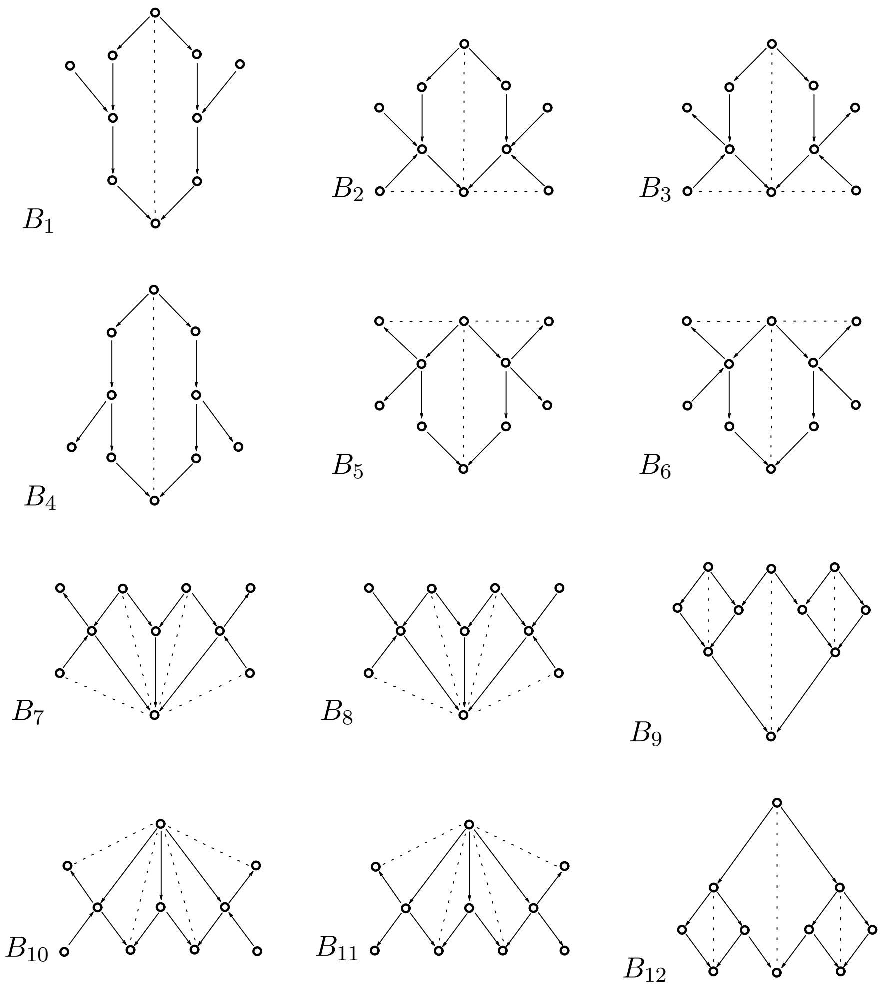
(c) *Every tubular algebra D with $\widehat{D} \cong \widehat{B}$ is isomorphic to $S_{i_{t_j}}^+ \ldots S_{i_1}^+ B$ for some $1 \le j \le r+1$.*

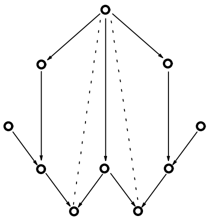
Following [15] the tubular algebra B is said to be *normal* if the tubular algebras $S_{i_{t_j}}^+ \ldots S_{i_1}^+ B$, $1 \le j \le r+1$, are pairwise non-isomorphic, or equivalently $B \ncong S_{i_{t_j}}^+ \ldots S_{i_1}^+ B$ for any $1 \le j \le r$. Otherwise, B is said to be *exceptional*. It follows from [15, Section 3] that B is exceptional if and only if there exists an automorphism φ of $\widehat{B}$ such that $\varphi^d = \varrho\nu_{\widehat{B}}$ for some $d \ge 2$ and a rigid automorphism ϱ of $\widehat{B}$ induced by an automorphism of B.

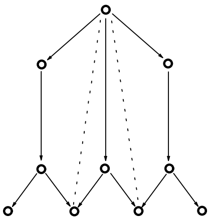
PROPOSITION 5.2. *If B is derived canonical of tubular type and φ is an automorphism of $\widehat{B}$ with $\varphi^d = \varrho\nu_{\widehat{B}}$, where $\nu_{\widehat{B}}$ is the Nakayama translation of $\widehat{B}$ and ϱ is a rigid automorphism of $\widehat{B}$, then $d \in \{1, 2, 3, 4, 6\}$ and d is a divisor of the rank of the Grothendieck group* $\mathrm{K}_0(\mathbb{X})$.

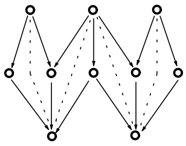
Proof. By Happel's theorem the stable category $\underline{\mathrm{mod}}\,\widehat{B}$ is equivalent to $\mathrm{D^b}(\mathbb{X})$ for a weighted projective line $\mathbb{X}$ of tubular type, and hence passage to the stable category yields the first result, by Proposition 3.1. For this we observe that in the tubular case each rigid automorphism of $\widehat{B}$ preserves the slope of indecomposable objects in the stable category of $\widehat{B}$. For the second assertion we refer to [15, Lemma 3.8]. ∎

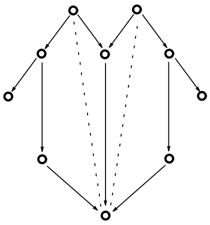
We shall give a complete description of all selfinjective algebras of tubular type $(2, 3, 6)$. Consider the following family of algebras:

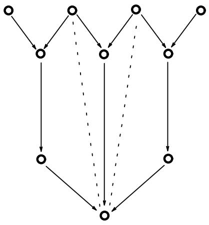


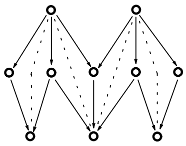

B_{13}

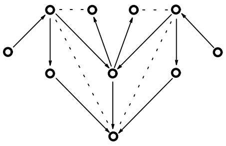

B_{14}

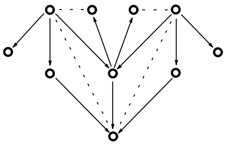

B_{15}

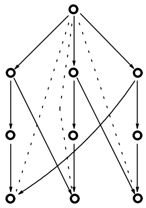

B_{16}

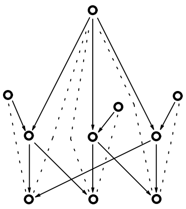

B_{17}

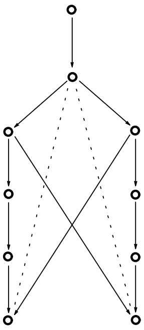

B_{18}


B_{19}


B_{20}


B_{21}


B_{22}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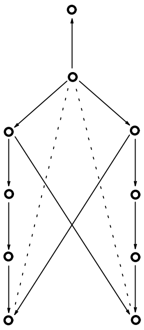
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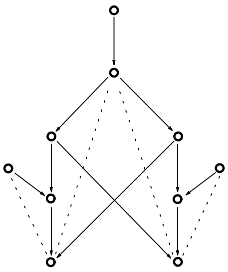
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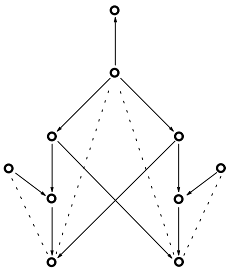
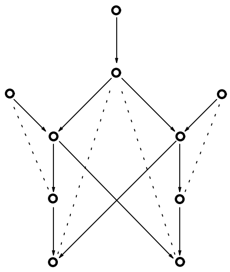
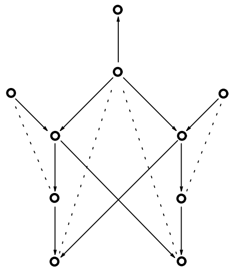
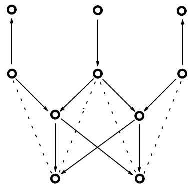
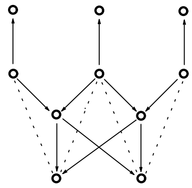
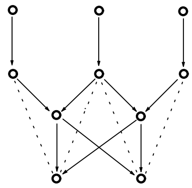

B_{26}

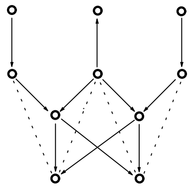

B_{27}

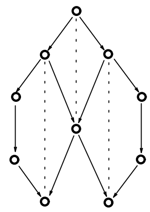

B_{28}

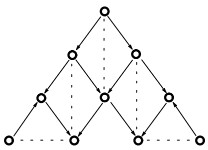

B_{29}

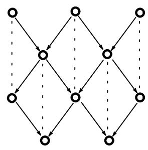

B_{30}


B_{31}


B_{32}


B_{33}


B_{34}


B_{35}

We note that the algebras $B_1, \dots, B_{35}$ are pairwise non-isomorphic, $B_4 \cong B_1^{\mathrm{op}}$, $B_5 \cong B_2^{\mathrm{op}}$, $B_6 \cong B_3^{\mathrm{op}}$, $B_{10} \cong B_7^{\mathrm{op}}$, $B_{11} \cong B_8^{\mathrm{op}}$, $B_{12} \cong B_9^{\mathrm{op}}$, $B_{16} \cong B_{13}^{\mathrm{op}}$, $B_{17} \cong B_{14}^{\mathrm{op}}$, $B_{18} \cong B_{15}^{\mathrm{op}}$, while $B_i \ncong B_j^{\mathrm{op}}$ for $i, j \in \{19, \dots, 35\}$.

THEOREM 5.3. (a) *The algebras $B_1, \dots, B_{35}$ are tubular algebras of type* $(2,3,6)$.

(b) *The repetitive algebras $\widehat{B}_1, \dots, \widehat{B}_{35}$ form a complete family of pairwise non-isomorphic repetitive algebras of tubular type* $(2,3,6)$ *having a non-trivial rigid twist.*

(c) B_{35} *and* B_{35}^{op} *are—up to isomorphism—the unique exceptional tubular algebras of type* $(2,3,6)$.

Proof. A straightforward checking shows that each of the algebras B_i, $1 \le i \le 35$, is a tubular extension or a tubular coextension, of tubular type $(2,3,6)$, of a tame concealed algebra of one of the Euclidean types $\widetilde{\mathbb{A}}_3$, $\widetilde{\mathbb{A}}_5$, $\widetilde{\mathbb{A}}_7$, $\widetilde{\mathbb{D}}_6$, or $\widetilde{\mathbb{D}}_8$, and consequently it is a tubular algebra of type $(2,3,6)$, by [14, Section 5]. Moreover, observe that the algebras B_{21} and B_{22} admit a natural automorphism of order 3, while the algebras B_i, $i \neq 21, 22$, admit a natural automorphism of order 2. Let B be a tubular algebra of type $(2,3,6)$ such that there exists a non-trivial automorphism φ of B. We shall prove that then $\widehat{B} \cong \widehat{B}_i$ for some $i \in \{1, \dots, 35\}$.

We know that B is a tubular extension of a (unique) tame concealed algebra C. Clearly, we then have $\varphi(C) = C$. We have the induced automorphisms $\varphi : \widehat{B} \to \widehat{B}$, $\varphi : \operatorname{mod} \widehat{B} \to \operatorname{mod} \widehat{B}$. Denote by ϱ the automorphism of $\underline{\operatorname{mod}}\, \widehat{B} = \mathrm{D}^{\mathrm{b}}(\operatorname{mod} B) = \mathrm{D}^{\mathrm{b}}(\mathbb{X})$ induced by φ on the stable level, where $\mathbb{X}$ is the weighted projective line of type $(2, 3, 6)$. Since ϱ is a rigid automorphism of $\mathrm{D}^{\mathrm{b}}(\mathbb{X})$ we know that $\varrho = \tau^s$ for some $s = 0, 1, \ldots, 5$.

Our next observation is that we may take C of Euclidean type different from $\widetilde{\mathbb{E}}_8$. Indeed, suppose C is of type $\widetilde{\mathbb{E}}_8$. Then B is a one-point extension of C by an indecomposable module lying on the mouth of the unique stable tube of rank 5 in Γ_C, and consequently φ fixes the extension vertex of this one-point extension. Next, a simple inspection of the Bongartz–Happel–Vossieck list [2, 6] of tame concealed algebras shows that, if φ fixes all vertices of Q_C, then in fact φ is trivial, a contradiction. Therefore, there are vertices $x \neq y$ in Q_C such that $\varphi(x) = y$. Since φ is a rigid automorphism of $\widehat{B}$, then the indecomposable projective $\widehat{B}$-modules $P_{\widehat{B}}(x)$ and $P_{\widehat{B}}(y)$ lie in the same $\mathbb{P}_1(K)$-family of quasi-tubes in $\Gamma_{\widehat{B}}$, and obviously φ shifts $P_{\widehat{B}}(x)$ to $P_{\widehat{B}}(y)$. Invoking now [12, Sections 3 and 4] we conclude that there exists a tubular extension D of a tame concealed algebra C' such that $\widehat{B} \cong \widehat{D}$, and hence D is a tubular algebra of type $(2, 3, 6)$, and x, y are vertices of Q_D but not of $Q_{C'}$. Clearly, C' is not of type $\widetilde{\mathbb{E}}_8$ and we are done. Therefore, we may assume that C is not of type $\widetilde{\mathbb{E}}_8$ and that there are two vertices x and y of Q_B but not of Q_C with $\varphi(x) = y$.

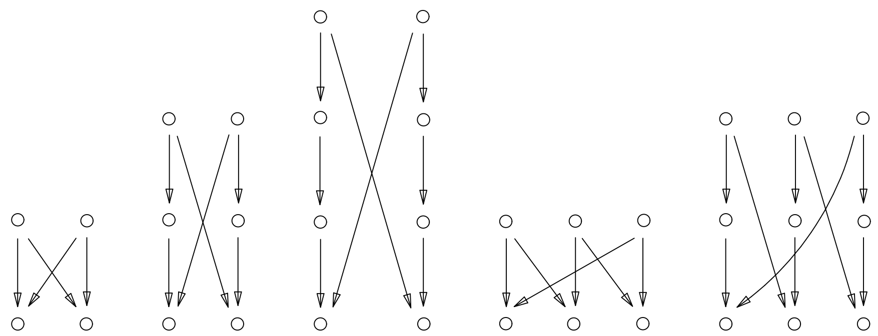
Assume that C is of type $\widetilde{\mathbb{E}}_7$. Then the set of vertices of Q_B consists of the vertices of Q_C and two extra vertices x and y such that the maximal C-submodules of the indecomposable projective B-modules $P_B(x)$ and $P_B(y)$ lie on the mouth of the unique stable tube of Γ_C of rank 4. Then it follows that the indecomposable $\widehat{B}$-modules $X = \operatorname{rad} P_{\widehat{B}}(x)$ and $Y = \operatorname{rad} P_{\widehat{B}}(y)$ lie in one of the stable tubes of rank 6 of the stable Auslander–Reiten quiver $\Gamma^{\mathrm{s}}_{\widehat{B}}$ of $\widehat{B}$. Since φ is a non-trivial automorphism of B, ϱ is a non-trivial rigid automorphism of $\underline{\operatorname{mod}}\, \widehat{B}$, and so $\varrho = \tau^s$ for some $s \in \{1, 2, 3, 4, 5\}$. On the other hand, ϱ acts on the set $\{X, Y\}$, and so $\varrho = \tau^3$ and $X = \varrho Y$, $Y = \varrho X$. Therefore, we conclude that B is a tubular extension of C by two non-isomorphic indecomposable modules lying on the mouth of the stable tube of rank 4. Moreover, a simple inspection of the Bongartz–Happel–Vossieck list shows that φ is an automorphism of order two.

Assume now that C is of type $\widetilde{\mathbb{E}}_6$. Then the set of vertices of Q_B consists of the vertices of Q_C and three extra vertices x, y, z such that the maximal C-submodules of $P_B(x)$, $P_B(y)$ and $P_B(z)$ lie on the mouth of one of the stable tubes of rank 3 in Γ_C. Then it follows that the indecomposable $\widehat{B}$-modules $X = \operatorname{rad} P_{\widehat{B}}(x)$, $Y = \operatorname{rad} P_{\widehat{B}}(y)$ and $Z = \operatorname{rad} P_{\widehat{B}}(z)$ lie in stable tubes of rank 6 of $\Gamma^{\mathrm{s}}_{\widehat{B}}$. Again, ϱ is a non-trivial rigid automorphism of $\underline{\operatorname{mod}}\, \widehat{B}$, and so $\varrho = \tau^s$ for some $s \in \{1,2,3,4,5\}$. Since φ acts on the set $\{x,y,z\}$, ϱ acts on the set $\{X,Y,Z\}$, and then we deduce that $\varrho = \tau^2$, that X, Y, Z lie in one stable tube of rank 6 in $\Gamma^s_{\widehat{B}}$, and that

$$Y = \tau^2 X, \quad Z = \tau^2 Y, \quad X = \tau^2 Z,$$

after a permutation of X, Y, Z. Hence, B is a tubular extension of C by all indecomposable modules lying on the mouth of one of the stable tubes of rank 3 in Γ_C. A simple inspection of the Bongartz–Happel–Vossieck list shows also that φ is an automorphism of B of order 3.

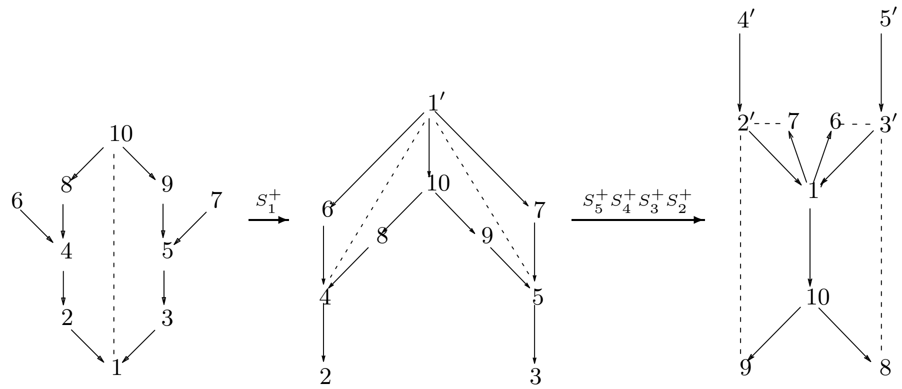
Finally, we note that if C is of type $\widetilde{\mathbb{A}}_m$, then C is one of the quivers

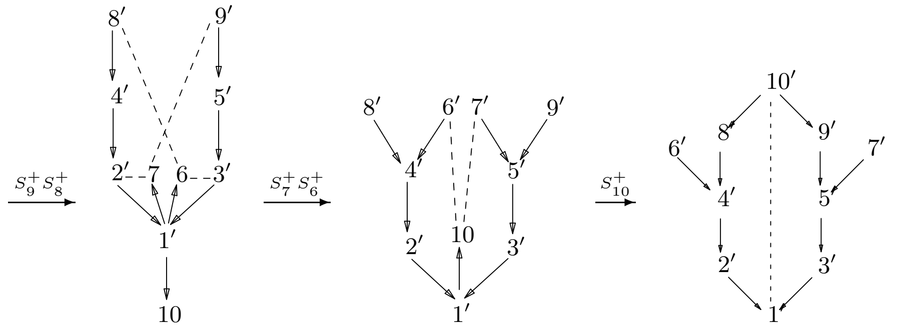


because B is a tubular extension of C of tubular type $(2,3,6)$ and φ is a non-trivial automorphism of B.

Next we calculate the tubular algebras whose repetitive algebra is isomorphic to one of $\widehat{B}_i$, $1 \le i \le 35$. According to Theorem 5.1, these tubular algebras are obtained by suitable reflections of the algebras $B_1,\dots,B_{35}$. This is a straightforward procedure, and we illustrate it only by three reflection sequences representing all situations which can occur.

For B_1 we have the following reflection sequence of tubular algebras:

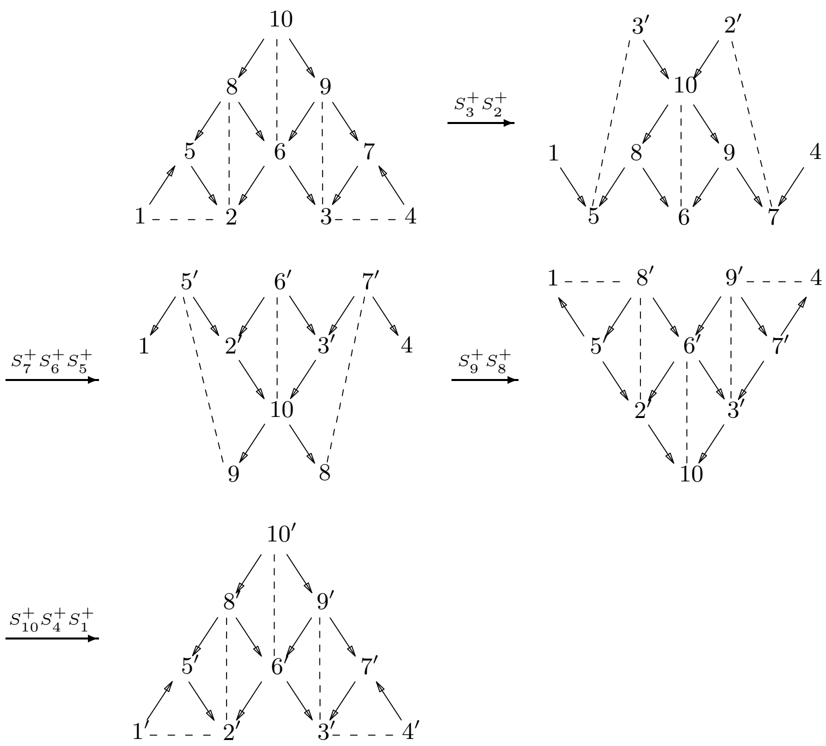




with $D_1 = S_1^+ B_1$, $D_2 = S_5^+ S_4^+ S_3^+ S_2^+ D_1$, $D_3 = S_9^+ S_8^+ D_2$, $D_4 = S_7^+ S_6^+ D_3$ and $B_1 = S_{10}^+ D_4$ forming a complete list of pairwise non-isomorphic tubular algebras of type $(2, 3, 6)$ whose repetitive algebra is isomorphic to $\widehat{B}_1$. Note that the opposite algebra $B_4 = B_1^{\mathrm{op}}$ does not occur in this sequence, and consequently $\widehat{B}_1 \not\cong \widehat{B}_4$. Clearly, the reflection sequence of tubular algebras produced by B_4 consists of B_4, D_4^{op}, D_3^{op}, D_2^{op} and D_1^{op}.

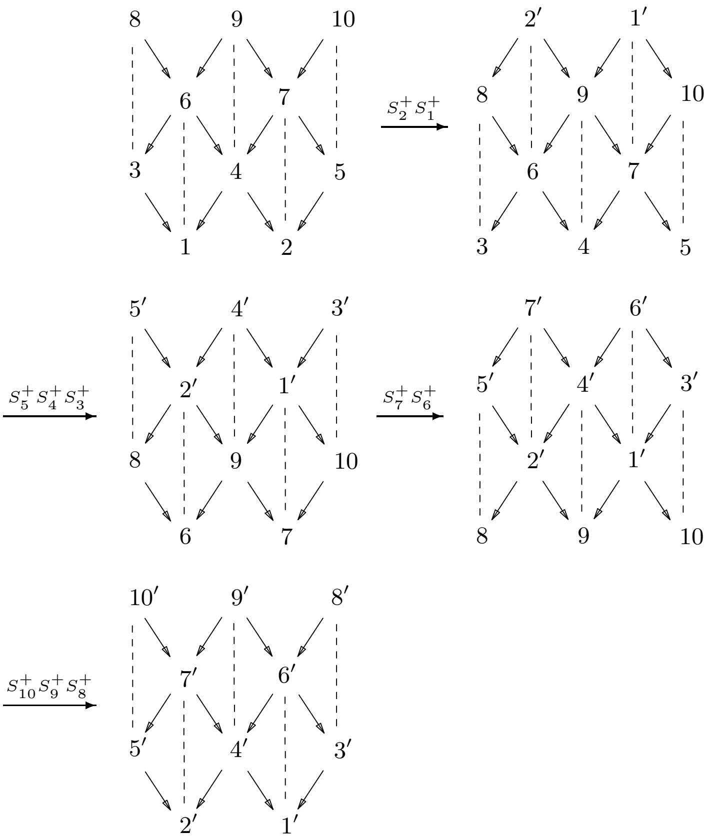
The same holds for the remaining dual pairs B_2 and B_5, B_3 and B_6, B_7 and B_{10}, B_8 and B_{11}, B_9 and B_{12}, B_{13} and B_{16}, B_{14} and B_{17}, B_{15} and B_{18}.

For B_{34} we have the following reflection sequence of tubular algebras:



with $E_1 = S_3^+ S_2^+ B_{34}$, $E_2 = S_7^+ S_6^+ S_5^+ E_1$, $E_3 = S_9^+ S_8^+ E_2$, $B_{34} = S_{10}^+ S_4^+ S_1^+ E_3$ forming a complete list of pairwise non-isomorphic tubular algebras of type $(2,3,6)$ whose repetitive algebra is isomorphic to $\widehat{B}_{34}$. Observe that $E_3 = B_{34}^{\mathrm{op}}$, and consequently $\widehat{B}_{34}^{\mathrm{op}} \cong \widehat{B_{34}^{\mathrm{op}}} \cong \widehat{B}_{34}$. In fact we have $\widehat{B}_i^{\mathrm{op}} \cong \widehat{B_i^{\mathrm{op}}} \cong \widehat{B}_i$ for $i = 19, \dots, 34$.

For B_{35} we have the following reflection sequence of tubular algebras:



Observe that $F_1 = S_2^+ S_1^+ B_{35} = B_{35}^{\mathrm{op}}$, $F_2 = S_5^+ S_4^+ S_3^+ F_1 = B_{35}$, $F_3 = S_7^+ S_6^+ F_2 = B_{35}^{\mathrm{op}}$, and $S_{10}^+ S_9^+ S_8^+ F_3 = B_{35}$. Therefore, B_{35} and B_{35}^{op} are exceptional tubular algebras of type $(2,3,6)$ and $\widehat{B}_{35}^{\mathrm{op}} \cong \widehat{B_{35}^{\mathrm{op}}} \cong \widehat{B}_{35}$. Hence any tubular algebra F with $\widehat{F} \cong \widehat{B}_{35}$ is isomorphic to B_{35} or B_{35}^{op}. Moreover, the shift $\varphi : \widehat{B}_{35} \to \widehat{B}_{35}$ induced by the isomorphism $B_{35} \to F_2 = S_5^+ S_4^+ F_1$ is an automorphism of $\widehat{B}$ such that $\varphi^2 = \varrho \nu_{\widehat{B}_{35}}$, where ϱ is the natural rigid automorphism of $\widehat{B}_{35}$ of order 2. Clearly, we also have $(\varrho\varphi)^2 = \varphi^2 = \varrho \nu_{\widehat{B}}$.

Calculating the reflection sequences of tubular algebras for all algebras $B_1, \dots, B_{35}$ we conclude that:

(α) For any two distinct members i and j from $\{1,\dots,35\}$, the reflection sequences of tubular algebras produced by B_i and B_j are disjoint.

(β) The algebras $B_1,\dots,B_{34}$ are normal.

(γ) The tubular algebras which are produced by all reflections from the algebras $B_1,\dots,B_{35}$ exhaust all possible tubular extensions, of tubular type $(2,3,6)$, of tame concealed algebras of Euclidean type $\widetilde{\mathbb{A}}_m$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ which admit a non-trivial automorphism.

Invoking now the first part of the proof, we conclude that $\widehat{B}_1,\dots,\widehat{B}_{35}$ form a complete family of pairwise non-isomorphic repetitive algebras of tubular type $(2,3,6)$ which admit a non-trivial rigid automorphism. Moreover, any non-trivial rigid automorphism of $\widehat{B}_i$, $1\le i\le 35$, is a natural automorphism of order 2 or 3. This shows (b).

For (c), suppose B is an exceptional tubular algebra of type $(2,3,6)$. Then there exists an automorphism φ of $\widehat{B}$ such that $\varphi^d=\varrho\nu_{\widehat{B}}$ for some proper divisor d of the rank of $\mathrm{K}_0(B)$ and a rigid automorphism ϱ of $\widehat{B}$. Since $\mathrm{K}_0(B)$ is of rank 10, we have $d=2$ or $d=5$. But $d=5$ is excluded by Proposition 3.1. If ϱ is trivial then $d=2$ is excluded by Proposition 3.3. Finally, if ϱ is non-trivial, then $\widehat{B}$ admits a non-trivial rigid twist. Therefore, $B\cong B_{35}$ or $B\cong B_{35}^{\mathrm{op}}$, and (c) follows. ∎

The following classification of selfinjective algebras of tubular type is now a direct consequence of the above theorem and [15, Proposition 3.9].

Theorem 5.4. *Let A be a selfinjective algebra. Then A is of tubular type* $(2,3,6)$ *if and only if A is isomorphic to one of the algebras*:

(a) $\widehat{B}/(\nu_{\widehat{B}}^m)$, *where B is a tubular algebra and m is a positive integer*,

(b) $\widehat{B}/(\varrho\nu_{\widehat{B}}^m)$, *where B is one of the algebras $B_1,\dots,B_{35}$, ϱ is the twist of $\widehat{B}$ induced by the corresponding twist of B of order* 2 *or* 3, *and m is a positive integer.*

(c) $\widehat{B}/(\varphi^m)$ *or* $\widehat{B}/(\varrho\varphi^m)$, *where $B=B_{35}$, ϱ is the twist of $\widehat{B}$ induced by the corresponding twist of B of order* 2 *such that $\varphi^2=\varrho\nu_{\widehat{B}}$, and m is an odd number.*

We obtain the following direct consequence of the above theorem.

Corollary 5.5. *Let A be a selfinjective algebra of tubular type* $(2,3,6)$. *Then A is symmetric if and only if*

$$A\cong B\ltimes \mathrm{D}(B)$$

for a tubular algebra B of type $(2,3,6)$.

In [15, (3.3), (3.4)] there are exhibited exceptional tubular algebras: Λ_1 of type $(2,2,2,2)$, Λ_2 of type $(2,2,2,2)$, and Λ_3 of type $(3,3,3)$ for which there exist K -linear automorphisms $\varphi_1:\widehat{\Lambda}_1\to\widehat{\Lambda}_1$, $\varphi_2:\widehat{\Lambda}_2\to\widehat{\Lambda}_2$ and

$\varphi_3 : \widehat{\Lambda}_3 \to \widehat{\Lambda}_3$ such that $\varphi_1^2 = \nu_{\widehat{\Lambda}_1}$, $\varphi_2^3 = \nu_{\widehat{\Lambda}_2}$ and $\varphi_3^4 = \nu_{\widehat{\Lambda}_3}$. We do not know whether there exists a tubular algebra B of type $(2,4,4)$ and a K-linear automorphism φ of $\widehat{B}$ such that $\varphi^3 = \varrho\nu_{\widehat{B}}$ for a rigid automorphism ϱ of $\widehat{B}$. Observe that on the stable level $\underline{\mathrm{mod}}\,\widehat{B}$, $\nu_{\widehat{B}} = \tau T^2$ admits a 3rd root, as shown in Section 3.

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