COLLOQUIUM MATHEMATICUM

VOL. 86 2000 NO. 2

SOME EXAMPLES OF TRUE $F_{\sigma\delta}$ SETS

BY

MAREK BALCERZAK (ŁÓDŹ) AND
UDAYAN B. DARJI (LOUISVILLE, KY)

Abstract. Let $\mathcal{K}(X)$ be the hyperspace of a compact metric space endowed with the Hausdorff metric. We give a general theorem showing that certain subsets of $\mathcal{K}(X)$ are true $F_{\sigma\delta}$ sets.

Let (X,d) be a perfect compact metric space and for $x \in X$ and $\varepsilon > 0$ let $B(x,\varepsilon)$ denote the ball in X centered at x with radius ε. By $\mathcal{X} = \mathcal{K}(X)$ we denote the set of all nonempty closed subsets of X endowed with the Hausdorff metric

$$\delta(K,L) = \max\{\max_{x\in K} d(x,L), \max_{x\in L} d(x,K)\}$$

or equivalently with the Vietoris topology that is generated by the sets of the form

$$\{K \in \mathcal{X} : K \subseteq U\} \quad \text{and} \quad \{K \in \mathcal{X} : K \cap U \neq \emptyset\}.$$

where U is open in X. By $\overline{A}$ we denote the closure of $A \subseteq X$. A set is called *true* G_δ (respectively, *true* F_σ) if it is G_δ (respectively, F_σ) and is not F_σ (respectively, G_δ). True $F_{\sigma\delta}$ sets and true $G_{\delta\sigma}$ sets are defined analogously. Several examples of true $F_{\sigma\delta}$ sets in Polish spaces are given in [5, 23A–E]. In this paper we describe a class of new examples of true $F_{\sigma\delta}$ sets in the hyperspace $\mathcal{X}$. Note that some results on true $G_{\delta\sigma}$ subsets of the hyperspace were obtained in [8].

Let $\mathcal{I} \subseteq \mathcal{X}$ be such that

1. $\mathcal{I}$ is hereditary, i.e. if $A \in \mathcal{I}$, $B \subseteq A$ and $B \in \mathcal{X}$, then $B \in \mathcal{I}$,
2. if $F \subseteq X$ is finite then $F \in \mathcal{I}$,
3. if $F \in \mathcal{I}$ then F is nowhere dense in X, and
4. $\mathcal{I}$ is a G_δ subset of $\mathcal{X}$.

2000 *Mathematics Subject Classification*: 03E15, 54B20, 54H05.

Key words and phrases: Borel set, hyperspace.

Work on this paper was begun while the authors were at the Mathematische Forschungsinstitut Oberwolfach in May '98 during their Research in Pairs stay financed by the Volkswagen-Stiftung.

We observe that such an $\mathcal{I}$ is necessarily a true G_δ subset of $\mathcal{X}$ as $\mathcal{I}$ and $\mathcal{X} \setminus \mathcal{I}$ are dense in $\mathcal{X}$. We define

$$\mathcal{M} = \{K \in \mathcal{X} : (\forall U \subseteq X, U \text{ open})(K \cap U = \emptyset \text{ or } K \cap \overline{U} \notin \mathcal{I})\}.$$

A member of $\mathcal{M}$ will be called an *$\mathcal{I}$-perfect set.* This notion appeared in [7]. Natural examples of families $\mathcal{I}$ with Properties 1–4 can be produced from the respective subfamilies of $\mathcal{P}(X)$, the power set of X. For instance, if $\mathcal{N}$ is the σ-ideal of Lebesgue null sets in $X = [0,1]$ then $\mathcal{I} = \mathcal{N} \cap \mathcal{X}$ is good. Note that, if $\mathcal{J} \subseteq \mathcal{P}(X)$ is a σ-ideal with suitable properties then for $\mathcal{I} = \mathcal{J} \cap \mathcal{X}$, $\mathcal{I}$-perfect sets coincide with perfect sets in the so-called *topology generated by $\mathcal{J}$. (See [2] and [3].) In the measure case, the notion of an $\mathcal{I}$-perfect set is well known and considerably exploited in various contexts. For instance, it was used in [4] in the classification of Lebesgue null sets and called a self-supporting set. A recent application in real function theory is contained in [1]. In the category case, an $\mathcal{I}$-perfect set simply means a closed set whose nonempty intersection with an open set has a nonempty interior. We show the following result:

THEOREM 1. *Let X, $\mathcal{X}$ and $\mathcal{M}$ be as stated. Then, $\mathcal{M}$ is a true $F_{\sigma\delta}$ subset of $\mathcal{X}$.*

As applications, we obtain the following corollaries. A nonempty intersection of a closed set $K \subseteq X$ with an open set in X will be called a *portion* of K.

COROLLARY 1. *Let $n \geq 1$ be an integer and $X = [0,1]^n$. Let $\mathcal{M}$ consist of all $K \in \mathcal{X}$ such that every portion of K has positive n-dimensional Lebesgue measure. Then $\mathcal{M}$ is a true $F_{\sigma\delta}$ set.*

Proof. Apply Theorem 1 to the σ-ideal $\mathcal{I}$ of compact sets with n-dimensional Lebesgue measure zero. It is well known that $\mathcal{I}$ is a G_δ subset of $\mathcal{K}([0,1]^n)$ (for example see [5, 23.9]). ∎

COROLLARY 2. *Let $n \geq 1$ be an integer and $X = [0,1]^n$. Let*

$$\mathcal{M} = \{K \in \mathcal{X} : \textit{every portion of } K \textit{ has positive Hausdorff dimension}\}.$$

Then $\mathcal{M}$ is a true $F_{\sigma\delta}$ set.

Proof. All we need to observe is that

$$\mathcal{I} = \{M \in \mathcal{X} : \text{the Hausdorff dimension of } M \text{ is zero}\}$$

is a G_δ set. (The other requirements on $\mathcal{I}$ hold trivially.) Indeed, fix $0 < s \leq n$ and let $\mathcal{H}^s$ be the Hausdorff s-measure defined on $\mathcal{X}$. As $\mathcal{H}^s$ is upper semicontinuous (see [5, 30.15]), we see that $(\mathcal{H}^s)^{-1}(\{0\})$ is a G_δ subset of $\mathcal{X}$. As

$$\mathcal{I} = \bigcap_{j=1}^{\infty} (\mathcal{H}^{n/j})^{-1}(\{0\}),$$

we see that $\mathcal{I}$ is a G_δ subset of $\mathcal{X}$. ■

COROLLARY 3. *Let X be a perfect compact metric space. Let*

$$\mathcal{M} = \{K \in \mathcal{X} : \textit{every portion of } K \textit{ is nonmeager}\}.$$

Then $\mathcal{M}$ is a true $F_{\sigma\delta}$ set.

Proof. Again we apply Theorem 1 to the σ-ideal $\mathcal{I}$ of compact meager subsets of X. See [5, 23.9] for the fact that $\mathcal{I}$ is a G_δ subset of $\mathcal{X}$. ■

Let us make two remarks. First, in our applications each $\mathcal{I}$ satisfies the additional property of being a σ-ideal of compact sets. Indeed, a rather useful theorem of Kechris, Louveau and Woodin [6] states that a coanalytic σ-ideal of compact sets is either a true coanalytic set or a G_δ set. The second remark is that our set $\mathcal{M}$, the collection of $\mathcal{I}$-perfect sets, is Π^0_3 complete. (See [5, 22.10, 24.20].)

Proof of Theorem 1. For each positive integer n, we let $\mathcal{B}_n$ be a finite minimal collection of open balls with radius $1/n$ which covers X. Observe that $\bigcup_{n=1}^{\infty} \mathcal{B}_n$ forms a topological base of X. We first prove a simple lemma.

LEMMA 1. *For each positive integer n, let $\mathcal{M}_n = \{K \in \mathcal{X} : (\forall U \in \mathcal{B}_n)(K \cap U = \emptyset$ or $K \cap \overline{U} \notin \mathcal{I})\}$. Then each $\mathcal{M}_n$ is an F_σ set and $\mathcal{M} = \bigcap_{n=1}^{\infty} \mathcal{M}_n$.*

Proof. That $\mathcal{M} = \bigcap_{n=1}^{\infty} \mathcal{M}_n$ follows simply because $\bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a topological base of X and $\mathcal{I}$ is hereditary. To show that $\mathcal{M}_n$ is an F_σ set we will prove that for each open set U in X, $\mathcal{H}_U = \{K \in \mathcal{X} : K \cap U = \emptyset$ or $K \cap \overline{U} \notin \mathcal{I}\}$ is an F_σ set. First, the set $\{K \in \mathcal{X} : K \cap U = \emptyset\}$ is closed. The set

$$\{K \in \mathcal{X} : K \cap \overline{U} \notin \mathcal{I}\} = \{K \in \mathcal{X} : (\exists F \in \mathcal{X})(F \notin \mathcal{I} \text{ and } F \subseteq K \cap \overline{U})\}$$

is F_σ since it is the projection onto the first coordinate of the σ-compact set formed by the intersection of the closed set $\{(K, F) \in \mathcal{X}^2 : F \subseteq K\}$ and the F_σ set $\{(K, F) \in \mathcal{X}^2 : F \subseteq \overline{U}$ and $F \notin \mathcal{I}\}$. ■

We prove Theorem 1 by contradiction. Assume that $\mathcal{M} = \bigcup_{i=1}^{\infty} \mathcal{G}_i$ where $\mathcal{G}_i$ are G_δ sets. We will construct a sequence of closed sets $\mathcal{P}_i$ such that $\mathcal{P}_i \cap \mathcal{G}_i = \emptyset$ and $\bigcap_{i=1}^{\infty} \mathcal{P}_i$ contains an element of $\mathcal{M}$, yielding a contradiction.

We construct $\mathcal{P}_k$ by induction.

Let $k = 1$. Observe that $\mathcal{M}_1$ and $\mathcal{X} \setminus \mathcal{M}_1$ are dense in $\mathcal{X}$. Indeed, let $H \in \mathcal{X}$ and $\varepsilon > 0$. Let $F \subseteq H$ be a finite set such that $\delta(H, F) < \varepsilon/2$. Let K be the closed set formed by putting closed balls of radius $\varepsilon/4$ around each point of F. Then $\delta(H, F) < \varepsilon$, $F \notin \mathcal{M}_1$ and $\delta(H, K) < \varepsilon$, $K \in \mathcal{M}_1$.

As $\mathcal{M}_1$ is F_σ and $\mathcal{M}_1$ and $\mathcal{X} \setminus \mathcal{M}_1$ are dense in $\mathcal{X}$ we see that $\mathcal{G}_1$ is not dense in $\mathcal{X}$. If it were, we would have two disjont G_δ sets, $\mathcal{G}_1$ and $\mathcal{X} \setminus \mathcal{M}_1$, both dense in $\mathcal{X}$. This would contradict the fact that $\mathcal{X}$ is a Polish space. Hence $\mathcal{G}_1$ is not dense in $\mathcal{X}$. Using this fact, let $F = \{x_1, x_2, \ldots, x_{t_1}, p\}$ be a finite set and $\varepsilon \in (0,1)$ be such that the ball in $\mathcal{X}$ centered at F with radius ε misses $\mathcal{G}_1$. Now, let $\gamma_1 \in (0, \varepsilon)$ be such that no two points of F are within $4\gamma_1$ of each other. Let

$$\mathcal{P}_1 = \Big\{ K \in \mathcal{X} : \bigcup_{i=1}^{t_1} \overline{B(x_i, \gamma_1)} \cup \{p\} \subseteq K \subseteq \bigcup_{i=1}^{t_1} \overline{B(x_i, \gamma_1)} \cup \overline{B(p, \gamma_1)} \Big\}.$$

Then $\mathcal{P}_1$ is a closed set which misses $\mathcal{G}_1$.

Now suppose we are at stage k, $\mathcal{P}_k$ is a closed subset of $\mathcal{X}$ which misses $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k$ and there is a sequence of points $x_1, x_2, \ldots, x_{t_k}$, p in X and a sequence of positive numbers $r_1, r_2, \ldots, r_{t_k}$ and a real number $\gamma_k \in (0, 1/k)$ such that

- $\mathcal{P}_k = \{K \in \mathcal{X} : \bigcup_{i=1}^{t_k} \overline{B(x_i, r_i)} \cup \{p\} \subseteq K \subseteq \bigcup_{i=1}^{t_k} \overline{B(x_i, r_i)} \cup \overline{B(p, \gamma_k)}\}$,
- $\overline{B(x_1, r_1)}, \overline{B(x_2, r_2)}, \ldots, \overline{B(x_{t_k}, r_{t_k})}, \overline{B(p, \gamma_k)}$ are pairwise disjoint.

Let us construct P_{k+1} now. Let n be sufficiently large so that if $U \in \mathcal{B}_n$, then $\overline{U}$ intersects at most one of the sets

$$\overline{B(x_1, r_1)},\ \overline{B(x_2, r_2)}, \ldots, \overline{B(x_{t_k}, r_{t_k})},\ \overline{B(p, \gamma_k)}.$$

We can show in a fashion similar to the case $k = 1$ that $\mathcal{M}_n \cap \mathcal{P}_k$ and $\mathcal{P}_k \setminus \mathcal{M}_n$ are dense in $\mathcal{P}_k$. As $\mathcal{M}_n \cap \mathcal{P}_k$ is a dense F_σ subset of $\mathcal{P}_k$, and $\mathcal{P}_k \setminus \mathcal{M}_n$ is dense in $\mathcal{P}_k$ as well, we see that $\mathcal{G}_{k+1} \cap \mathcal{P}_k \subseteq \mathcal{M}_n \cap \mathcal{P}_k$ is not dense in $\mathcal{P}_k$. Notice that sets of the form $\bigcup_{i=1}^{t_k} \overline{B(x_i, r_i)} \cup F$, where $F \subset B(p, \gamma_k)$ is finite with $p \in F$, constitute a dense subfamily of $\mathcal{P}_k$. Thus we can choose a finite set $F \subseteq B(p, \gamma_k)$ containing p, and a number $\varepsilon \in (0, 1/(k+1))$ such that $\{K \in \mathcal{P}_k : \delta(K, \bigcup_{i=1}^{t_k} \overline{B(x_i, r_i)} \cup F) < \varepsilon\}$ misses $\mathcal{G}_{k+1}$. Now let $\gamma_{k+1} \in (0, \varepsilon)$ be such that no two points of F are within $4\gamma_{k+1}$ of each other and $B(x, \gamma_{k+1}) \subseteq B(p, \gamma_k)$ for $x \in F$. Now list points of $F \setminus \{p\}$ as $x_{t_k+1}, x_{t_k+2}, \ldots, x_{t_{k+1}}$ and let $r_{t_k+1} = r_{t_k+2} = \ldots = r_{t_{k+1}} = \gamma_{k+1}$. Let $\mathcal{P}_{k+1} = \{K \in \mathcal{X} : \bigcup_{i=1}^{t_{k+1}} \overline{B(x_i, r_i)} \cup \{p\} \subseteq K \subseteq \bigcup_{i=1}^{t_k} \overline{B(x_i, r_i)} \cup \overline{B(p, \gamma_{k+1})}\}$. Then $\mathcal{P}_{k+1} \subseteq \mathcal{P}_k$ and $\mathcal{P}_{k+1}$ misses $\mathcal{G}_{k+1}$. We also see that $\mathcal{P}_{k+1}$, $x_1, x_2, \ldots, x_{t_{k+1}}$, p, $r_1, r_2, \ldots, r_{t_{k+1}}$ and γ_{k+1} satisfy the required induction hypothesis.

Now let us observe that our sequence $\{x_j\}$ converges to p and $\bigcap_{i=1}^{\infty} \mathcal{P}_i$ is simply the set consisting of $K = \bigcup_{i=1}^{\infty} \overline{B(x_i, r_i)} \cup \{p\}$. Clearly, $K \in \mathcal{M}$, however, $K \notin \bigcup_{i=1}^{\infty} \mathcal{G}_i$, contradicting $\mathcal{M}$ being $G_{\delta\sigma}$. ∎

REFERENCES

[1] Z. Buczolich, *Cantor type sets of positive measure and Lipschitz mappings*, Real Anal. Exchange 17 (1991-92), 702–705.

[2] H. Hashimoto, *On the *topology and its application*, Fund. Math. 91 (1976), 5–10.

[3] D. Janković and T. R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly 97 (1990), 295–310.

[4] W. Just and C. Laflamme, *Classifying sets of measure zero with respect to their open covers*, Trans. Amer. Math. Soc. 321 (1990), 621–645.

[5] A. S. Kechris, *Classical Descriptive Set Theory*, Springer, New York, 1994.

[6] A. S. Kechris, A. Louveau and H. Woodin, *The structure of σ-ideals of compact sets*, Trans. Amer. Math. Soc. 301 (1987), 263–288.

[7] A. Louveau, *σ-idéaux engendrés par des ensembles fermés et théorèmes d'approximation*, ibid. 257 (1980), 143–169.

[8] E. Matheron, *How to recognize a true Σ_3^0 set*, Fund. Math. 158 (1998), 181–194.

Institute of Mathematics
Łódź Technical University
Politechniki 11, I-2
90-924 Łódź, Poland

and

Faculty of Mathematics
University of Łódź
Banacha 22
90-238 Łódź, Poland
E-mail: mbalce@krysia.uni.lodz.pl

Department of Mathematics
University of Louisville
Louisville, KY 40292-0001, U.S.A.
E-mail: darji@erdos.math.louisville.edu

Received 20 September 1999;
revised version 17 January 2000

(3832)