COLLOQUIUM MATHEMATICUM

VOL. 86 2000 NO. 2

PERTURBATION OF ANALYTIC OPERATORS AND TEMPORAL REGULARITY OF DISCRETE HEAT KERNELS

BY

SÖNKE BLUNCK (CERGY-PONTOISE)

Abstract. In analogy to the analyticity condition $\|Ae^{tA}\| \le Ct^{-1}$, $t > 0$, for a continuous time semigroup $(e^{tA})_{t\ge0}$, a bounded operator T is called analytic if the discrete time semigroup $(T^n)_{n\in\mathbb{N}_0}$ satisfies $\|(T-I)T^n\| \le Cn^{-1}$, $n \in \mathbb{N}$. We generalize O. Nevanlinna's characterization of powerbounded and analytic operators T to the following perturbation result: if S is a perturbation of T such that $\|R(\lambda_0, T) - R(\lambda_0, S)\|$ is small enough for some $\lambda_0 \in \varrho(T) \cap \varrho(S)$, then the type ω of the semigroup $(e^{t(S-I)})$ also controls the analyticity of S in the sense that $\|(S-I)S^n\| \le C(\omega + n^{-1})e^{\omega n}$, $n \in \mathbb{N}$.

As an application we generalize and give a simple proof of a result by M. Christ on the temporal regularity of random walks T on graphs of polynomial volume growth. On arbitrary spaces Ω of at most exponential volume growth we obtain this regularity for any powerbounded and analytic operator T on $L_2(\Omega)$ with a heat kernel satisfying Gaussian upper bounds.

1. Introduction and main results.

Let X be a Banach space and $\mathcal{L}(X)$ the space of all bounded linear operators on X. Following [C-SC], an operator $T \in \mathcal{L}(X)$ is called *analytic* if there exists a constant $C > 0$ such that

$$\|(T-I)T^n\| \le Cn^{-1} \quad \text{for all } n \in \mathbb{N}.$$

This notion is a discrete time analogue of the property $\|Ae^{tA}\| \le Ct^{-1}$, $t > 0$, which characterizes the analyticity of a bounded semigroup $(e^{tA})_{t\ge0}$. The following characterization of analytic operators is due to O. Nevanlinna [N1, Thm. 4.5.4], [N2, Thm. 2.1].

THEOREM. *Let $T \in \mathcal{L}(X)$. Then the following are equivalent*:

(a) *T is powerbounded and analytic.*
(b) *$(e^{t(T-I)})$ is a bounded analytic semigroup and $\sigma(T) \subset \mathbb{D} \cup \{1\}$.*
(c) *$\|(\lambda-1)R(\lambda, T)\| \le C$ for all $|\lambda| > 1$.*

Here $\mathbb{D} \subset \mathbb{C}$ is the unit disk, $\sigma(T)$ is the spectrum of T and, for λ in the resolvent set $\varrho(T)$, we denote by $R(\lambda, T) := (\lambda - T)^{-1}$ the resolvent operator.

2000 *Mathematics Subject Classification*: 47A55, 60J10.

In particular, if T is powerbounded and analytic then $(e^{t(T-I)})$ is an analytic semigroup of type 0. If $S \in \mathcal{L}(X)$ is a perturbation of T such that $\|R(\lambda_0,T)-R(\lambda_0,S)\|$ is small enough for some $\lambda_0 \in \varrho(T)\cap\varrho(S)$ then the type ω of the analytic semigroup $(e^{t(S-I)})$ also controls the analyticity of S.

More precisely, in the first part of this paper we will prove the following perturbation theorem for analytic operators.

THEOREM 1.1. *Let $T \in \mathcal{L}(X)$ be powerbounded and analytic. Fix $\lambda_0 \in \varrho(T)$ and $C,\delta > 0$. Then there exist $\omega_0, d, D > 0$ such that for all $S \in \mathcal{L}(X)$ with $\|S\| \le C$ and all $\omega \in [0,\omega_0]$ the two conditions*

$$\|\lambda R(\lambda, S-I-\omega)\| \le C \quad \text{for all } \lambda \in \Sigma_\delta, \tag{1}$$

$$\lambda_0 \in \varrho(S) \quad \text{and} \quad \|R(\lambda_0,T)-R(\lambda_0,S)\| \le d \tag{2}$$

imply $\|S^n\| \le De^{\omega n}$ and $\|(S-I)S^n\| \le D(\omega+n^{-1})e^{\omega n}$ for all $n \in \mathbb{N}_0$.

Here and in what follows Σ_δ denotes the open sector $\{z : |\arg(z)| < \delta+\pi/2\}$ and for bounds of the type "for all $n \in \mathbb{N}_0$" involving negative powers $n^{-\alpha}$ we use the convention $0^{-\alpha} := 1$.

In the second part of the paper this perturbation result is applied to the problem of temporal regularity of discrete heat kernels which is the following.

Let (Ω,μ,d) be a σ-finite measure space equipped with a metric d and set $L_p := L_p(\Omega,\mu)$. Let $T \in \mathcal{L}(L_2)$ be a powerbounded and analytic operator whose powers T^n have integral kernels $p_n(x,y)$ satisfying the Gaussian bounds

$$|p_n(x,y)| \le C_0\, n^{-N/m} \exp\biggl(-b_0 \frac{d(x,y)^{m/(m-1)}}{n^{1/(m-1)}}\biggr) \quad \text{for all } n \in \mathbb{N} \tag{3}$$

and some $N, b_0 > 0, m > 1$. For $m = 2$, such estimates are quite common [H-SC]; for $m \ne 2$, they appear on the so-called graphical Sierpiński gaskets and related graphs with fractal structure [J], [BB].

The question arises under which conditions one can guarantee the following natural bound for the discrete time derivatives $Dp_n := p_{n+1} - p_n$:

$$|D^k p_n(x,y)| \le C_k n^{-N/m-k} \exp\biggl(-b_k \frac{d(x,y)^{m/(m-1)}}{n^{1/(m-1)}}\biggr) \quad \text{for all } n \in \mathbb{N}. \tag{4}$$

Let $T_\varrho \in \mathcal{L}(L_2)$, $\varrho \in \mathbb{R}$, denote the Davies perturbations of T (see Definition 3.1). Since $D^k p_n$ is the kernel of the operator $(T-I)^k T^n$, by Davies' perturbation method the estimate (4) is equivalent (see Lemma 3.2) to the ultracontractive estimates

$$\|(T_\varrho - I)^k T_\varrho^n\|_{1,\infty} \le C_k\, n^{-N/m-k} \exp(\omega_k |\varrho|^m n) \quad \text{for all } n \in \mathbb{N},\ \varrho \in \mathbb{R}. \tag{U_k}$$

Recall that our hypothesis (3) can be checked [H-SC] by establishing

$$(\widetilde{\mathrm{U}}_0) \qquad \|T_\varrho^n\|_{1,2}, \|T_\varrho^n\|_{2,\infty} \le C_0' n^{-N/2m} \exp(\omega_0'|\varrho|^m n) \quad \text{for all } n \in \mathbb{N},\ \varrho \in \mathbb{R}.$$

Hence the aim is to deduce (U_k) from $(\widetilde{\mathrm{U}}_0)$. This will be achieved by shifting the derivation from the $\|\cdot\|_{2,\infty}$-norm and the $\|\cdot\|_{1,2}$-norm to the $\|\cdot\|_{2,2}$-norm. More precisely, if we can verify the analyticity property

$$(\mathrm{A}_k) \qquad \|(T_\varrho - I)^k T_\varrho^n\|_{2,2} \le C_k'' n^{-k} \exp(\omega_k'|\varrho|^m n) \quad \text{for all } n \in \mathbb{N}_0,\ \varrho \in \mathbb{R}$$

then (U_k) follows easily from $(\widetilde{\mathrm{U}}_0)$, (A_k) and factorizations of the type

$$\|(T_\varrho - I)^k T_\varrho^{3n}\|_{1,\infty} \le \|T_\varrho^n\|_{1,2}\, \|T_\varrho^n (I-T)^k\|_{2,2}\, \|T_\varrho^n\|_{2,\infty}.$$

Intermediate steps in the proof of $(\widetilde{\mathrm{U}}_0)$ are often [H-SC] the verification of growth estimates for the semigroups $(T_\varrho^n)_{n\in\mathbb{N}_0}$ and $(e^{t(T_\varrho - I)})_{t\ge 0}$ in the form of (A_0) and of

$$(5) \qquad \|\lambda R(\lambda, T_\varrho - I - \omega_0'|\varrho|^m)\|_{2,2} \le C_0'' \quad \text{for all } \lambda \in \Sigma_\delta,\ |\varrho| \le 1.$$

Then we obtain (A_k) as a direct consequence of the following corollary to Theorem 1.1, whose conditions (c) and (d) correspond to (A_0) and (5).

COROLLARY 1.2. *Let X be a Banach space and $(S_\varrho)_{\varrho\in\mathbb{R}}$ a family of operators in $\mathcal{L}(X)$. Suppose there are constants $C, \omega, m, \delta > 0$ satisfying the following conditions*:

(a) $T := S_0$ *is powerbounded and analytic.*

(b) *There exist $\varrho_0 > 0$ and $\lambda_0 \in \mathbb{C}$ such that $\lambda_0 \notin \sigma(S_\varrho)$ for all $|\varrho| \le \varrho_0$ and*

$$\|R(\lambda_0, T) - R(\lambda_0, S_\varrho)\| \to 0 \quad \textit{as } \varrho \to 0.$$

(c) $\|S_\varrho^n\| \le Ce^{\omega|\varrho|^m n}$ *for all* $n \in \mathbb{N}_0,\ \varrho \in \mathbb{R}$.

(d) $\|\lambda R(\lambda, S_\varrho - I - \omega|\varrho|^m)\| \le C$ *for all* $\lambda \in \Sigma_\delta,\ |\varrho| \le 1$.

Then, for all $k \in \mathbb{N}$, there exist $\omega_k, C_k > 0$ such that

$$\|(S_\varrho - I)^k S_\varrho^n\| \le C_k n^{-k} e^{\omega_k|\varrho|^m n} \quad \textit{for all } n \in \mathbb{N}_0,\ \varrho \in \mathbb{R}.$$

The constants ω_k, C_k depend on the $(S_\varrho)_{\varrho\ne 0}$ only by the rate of convergence in (b).

Corollary 1.2 will be applied to $S_\varrho := T_\varrho$, the Davies perturbations of T. In this case, the resolvent convergence in (b) will be verified by means of the Gaussian kernel bounds (3). This requires the volume growth condition (6) on (Ω, μ, d) in the following result, which will be proved by the reasoning as just described.

PROPOSITION 1.3. *Let (Ω, μ, d) be a metric measure space of at most exponential volume growth*:

$$(6) \qquad \exists C, c > 0\ \forall r \ge 0,\ x \in \Omega: \quad |B(x,r)| \le Ce^{cr}.$$

Let $T \in \mathcal{L}(L_2)$ be a powerbounded and analytic operator whose Davies perturbations $(T_\varrho)_{\varrho\in\mathbb{R}}$ satisfy $(\widetilde{\mathrm{U}}_0)$, (A_0) *and* (5) *for some constants $C_0', C_0'', \omega_0', N > 0$ and $m > 1$. Furthermore, let T have an integral kernel $p \in L_\infty(\Omega^2)$ such that*

$$\exists C, b > 0 : \quad |p(x,y)| \le Ce^{-bd(x,y)^{m/(m-1)}}. \tag{7}$$

Then the T^n have integral kernels p_n such that for all $k \in \mathbb{N}_0$ there exist constants $C_k, b_k > 0$ with

$$|D^k p_n(x,y)| \le C_k n^{-N/m-k} \exp\left(-b_k \frac{d(x,y)^{m/(m-1)}}{n^{1/(m-1)}}\right) \quad \textit{for all } n \in \mathbb{N}.$$

Here $|B(x,r)|$ denotes the volume of the closed ball $B(x,r)$ with centre x and radius r. An application of Theorem 1.3 for $m = 2$ yields the following result on Markov chains.

THEOREM 1.4. *Let (Ω, μ, d) be a metric measure space of at most exponential volume growth as in* (6). *Let $N \in \mathbb{R}_+$ and $T \in \mathcal{L}(L_2)$ be the integral operator corresponding to a symmetric Markov kernel $p \in L_\infty(\Omega^2)$ satisfying*

$$\exists r_0 > 0\ \forall x \in \Omega : \quad \operatorname{supp}(p(x,\cdot)) \subset B(x, r_0), \tag{8}$$

$$\sup_{x,y} |p_n(x,y)| \le Cn^{-N/2} \quad \textit{for all } n \in \mathbb{N}, \tag{9}$$

$$-1 \notin \sigma(T) \tag{10}$$

where the p_n are the kernels of the T^n. Then for all $k \in \mathbb{N}_0$ we have

$$|D^k p_n(x,y)| \le C_k\, n^{-N/2-k} \exp\left(-b_k \frac{d(x,y)^2}{n}\right) \quad \textit{for all } n \in \mathbb{N}.$$

The estimate for $k = 0$ was shown by W. Hebisch and L. Saloff-Coste in [H-SC,Thm. 2.1] without restrictions on (Ω, μ, d) and $\sigma(T)$. And indeed, intermediate steps in their proof are the verification of $(\widetilde{\mathrm{U}}_0)$, (A_0) and (a slight weakening of) (5); see Lemmas 2.2, 2.3, 2.4 in [H-SC]. In fact, the arguments given there show (5) so that, in particular, the semigroup $(e^{t(T-I)})$ is bounded analytic. Hence, by Nevanlinna's theorem cited above and our additional assumption $-1 \notin \sigma(T)$, the selfadjoint operator T is powerbounded and analytic so that Proposition 1.3 easily implies Theorem 1.4 (see §4 below). We remark that if condition (9) is replaced by

$$|p_n(x,y)| \le C_0 f(n) \exp\left(-b_0 \frac{d(x,y)^2}{n}\right)$$

for all $n \in \mathbb{N}$ and some decreasing sequence $(f(n))_{n\in\mathbb{N}_0}$, then the proof of

Theorem 1.4 leads to the adapted conclusion

$$|D^k p_n(x,y)| \le C_k f(\lfloor (n+1)/3 \rfloor) n^{-k} \exp\left(-b_k \frac{d(x,y)^2}{n}\right) \quad \text{for all } n \in \mathbb{N}.$$

Theorem 1.4 was shown by M. Christ in [C] by a quite difficult proof for the special case when Ω is a connected graph equipped with the counting measure μ and its natural metric d. Moreover, in [C] the following additional assumptions are made:

- Ω is of polynomial volume dimension N, i.e.

$$\exists C > 0\ \forall r \ge 1,\ x \in \Omega : \quad C^{-1} r^N \le |B(x,r)| \le C r^N,$$

- $\exists \varepsilon > 0\ \forall x, y \in \Omega,\ d(x,y) = 1 \Rightarrow p(x,y) \ge \varepsilon.$

Note that every graph is at most of exponential volume growth provided that each node has a uniformly bounded number of neighbours.

2. Proof of Theorem 1.1 and Corollary 1.2. In Theorem 1.1 we consider perturbations $S \in \mathcal{L}(X)$ of a powerbounded and analytic operator $T \in \mathcal{L}(X)$ satisfying $\lambda_0 \in \varrho(S)$ for some fixed $\lambda_0 \in \varrho(T)$. For such perturbations S of T the characteristic resolvent estimate $\|R(\mu,T)\| \le C|\mu-1|^{-1}$ for all $|\mu| > 1$ remains valid at least for all $|\mu| > 1$ outside a circle around the singularity $\mu = 1$ of a radius proportional to $\|R(\lambda_0,T) - R(\lambda_0,S)\|$.

This is shown in the following lemma, whose proof is based on the continuity of the inversion map on $\{U \in \mathcal{L}(X) : U \text{ invertible}\}$.

LEMMA 2.1. *Let $T \in \mathcal{L}(X)$ satisfy $\|R(\mu,T)\| \le C|\mu-1|^{-1}$ for all $|\mu| > 1$. Let $\lambda_0 \in \varrho(T)$ and $\mu_0 > 0$. Then there exist $d_1, D_1 > 0$ such that for all operators $S \in \mathcal{L}(X)$ and for all $\mu_0 \ge |\mu| > 1$ we have*

$$\|R(\lambda_0,T) - R(\lambda_0,S)\| \le d_1|\mu-1| \ \Rightarrow\ \|R(\mu,S)\| \le D_1|\mu-1|^{-1}.$$

Proof. Obviously, it suffices to find $d_1, D_1 > 0$ such that

$$\|R(\lambda_0,T) - R(\lambda_0,S)\| \le d_1|\mu-1| \ \Rightarrow\ \|R(\mu,T) - R(\mu,S)\| \le D_1|\mu-1|^{-1}$$

for all S and μ as required. Recall that, for an invertible operator $U \in \mathcal{L}(X)$, any $V \in \mathcal{L}(X)$ with $\|U-V\| < \|U^{-1}\|^{-1}$ is invertible and

$$\|U^{-1} - V^{-1}\| \le \frac{\|U^{-1}\|\,\|U-V\|}{\|U^{-1}\|^{-1} - \|U-V\|}.$$

If $\|R(\lambda_0,T) - R(\lambda_0,S)\|$ is small enough we can apply this to $U := (\lambda_0-\mu)^{-1} - R(\lambda_0,T)$ and $V := (\lambda_0-\mu)^{-1} - R(\lambda_0,S)$ for $|\mu| > 1$. Indeed, since

$$\|U^{-1}\| = |\lambda_0-\mu|\,\|I + (\lambda_0-\mu)R(\mu,T)\| \le |\lambda_0-\mu| C_1 |\mu-1|^{-1}$$

we obtain $\mu \in \varrho(S)$ and

$$
\begin{aligned}
\|R(\mu,T) - R(\mu,S)\|\, |\lambda_0 - \mu| \\
&= \|U^{-1}R(\lambda_0,T) - V^{-1}R(\lambda_0,S)\| \\
&\le \|U^{-1}\|\, \|R(\lambda_0,T) - R(\lambda_0,S)\| + \|U^{-1} - V^{-1}\|\, \|R(\lambda_0,S)\| \\
&\le \|U^{-1}\|\, \|U - V\| \left(1 + \frac{\|R(\lambda_0,S)\|}{\|U^{-1}\|^{-1} - \|U - V\|}\right) \\
&\le |\lambda_0 - \mu| C_2\, |\mu - 1|^{-1} \frac{\|U - V\|}{c_3|\mu - 1| - \|U - V\|} \\
&\le |\lambda_0 - \mu| C_2\, |\mu - 1|^{-1}
\end{aligned}
$$

provided $\|U - V\| = \|R(\lambda_0,T) - R(\lambda_0,S)\| \le c_3|\mu - 1|/2$. ■

Proof of Theorem 1.1. This proof is motivated by the proof of the implication (c)⇒(a) in Nevanlinna's theorem cited in the Introduction as given in [N1, p. 102]. By hypothesis, T is powerbounded and analytic, hence due to [N2, Thm. 2.1] we can assume

$$\|(\mu - 1)R(\mu,T)\| \le C \quad \text{for all } |\mu| > 1.$$

Therefore, by Lemma 2.1, there exist $d_1, D_1 > 0$ satisfying for all $S \in \mathcal{L}(X)$ and for all $2C \ge |\mu| > 1$ the condition

$$\|R(\lambda_0,T) - R(\lambda_0,S)\| \le d_1|\mu - 1| \;\Rightarrow\; \|(\mu - 1)R(\mu,S)\| \le D_1. \tag{11}$$

One checks that for sufficiently small $t_0, c_0 > 0$ (depending only on δ) the map

$$[0,t_0] \to \mathbb{R}_+, \quad t \mapsto |1 + r + te^{i(\delta+\pi/2)}|\, |1 - c_0 t|^{-1},$$

is decreasing for all $r \in [0,1]$. Hence we have

$$|1 + r + te^{i(\delta+\pi/2)}| \le (1 + r)(1 - c_0 t) \quad \text{for all } t \in [0,t_0],\ r \in [0,1]. \tag{12}$$

Now we set $d := d_1 t_0/4$. Then, for all $S \in \mathcal{L}(X)$ satisfying $\|S\| \le C$ and (2), we deduce from (11) that $\varrho(S) \supset \{\mu : |\mu| > 1,\ |\mu - 1| \ge t_0/4\}$ and

$$\forall 2 \ge |\mu| > 1,\ |\mu - 1| \ge t_0/4 : \quad \|(\mu - 1)R(\mu,S)\| \le D_1.$$

Hence we find $r < 1$, $\widetilde{D}_1 > 0$ independent of the operator S with

$$\forall 2 \ge |\mu| \ge r,\ |\mu - 1| \ge t_0/2 : \quad \|(\mu - 1)R(\mu,S)\| \le \widetilde{D}_1.$$

By choosing a greater $r < 1$ or a smaller $\delta > 0$ if necessary, we derive for $\mathcal{M} := \{\mu : |\mu| = r,\ |\mu - 1| \ge t_0/2\}$ that

$$t_{\infty,0} := t_0/2 \;\Rightarrow\; 1 + t_{\infty,0}e^{i(\delta+\pi/2)} \in \mathcal{M}.$$

Hence there exist $\omega_0, n_0 > 0$ such that for all $n \ge n_0$, $\omega \in [0,\omega_0]$ we have

$$\exists t_{n,\omega} \in [0,t_0] : \quad 1 + \omega + n^{-1} + t_{n,\omega}e^{i(\delta+\pi/2)} \in \mathcal{M}.$$

For all such n and ω we construct a closed path $\Gamma_{n,\omega}$ as follows:

$$\begin{aligned}\Gamma_{n,\omega} &= \Gamma_{n,\omega,1} \cup \Gamma_{n,\omega,2} \cup \Gamma_{n,\omega,3},\\ \Gamma_{n,\omega,1} &:= 1+\omega+n^{-1}+[0,t_{n,\omega}]e^{i(\delta+\pi/2)},\\ \Gamma_{n,\omega,2} &\subset \mathcal{M},\\ \Gamma_{n,\omega,3} &:= 1+\omega+n^{-1}+[0,t_{n,\omega}]e^{-i(\delta+\pi/2)}.\end{aligned}$$

For all $S \in \mathcal{L}(X)$ satisfying $\|S\| \le C$, (1) and (2), $\Gamma_{n,\omega}$ is a path in $\varrho(S)$ around $\sigma(S)$ so that

$$(e^{-\omega}S)^n(S-I) = (2\pi i)^{-1} \int_{\Gamma_{n,\omega}} (e^{-\omega}\lambda)^n(\lambda-1)R(\lambda,S)\,d\lambda.$$

It remains to estimate the integrals over the $\Gamma_{n,\omega,j}$, $j=1,2,3$. Since $\Gamma_{n,\omega,1} \subset 1+\omega+\Sigma_\delta$ the hypothesis (1) and (12) yield

$$\begin{aligned}&\Big\| \int_{\Gamma_{n,\omega,1}} (e^{-\omega}\lambda)^n(\lambda-1)R(\lambda,S)\,d\lambda\Big\|\\ &\qquad \le Ce^{-\omega n} \int_{\Gamma_{n,\omega,1}} |\lambda|^n|\lambda-1|\,|\lambda-1-\omega|^{-1}\,|d\lambda|\\ &\qquad \le C\int_0^{t_0}(1+n^{-1})^n\,(1-c_0t)^n \frac{|\omega+n^{-1}+te^{i(\delta+\pi/2)}|}{|n^{-1}+te^{i(\delta+\pi/2)}|}\,dt\\ &\qquad \le Ce(1-\sin\delta)^{-1/2}(\omega n+1)\int_0^{t_0}(1-c_0t)^n\,dt\\ &\qquad \le C'(\omega+n^{-1}).\end{aligned}$$

For the integral over $\Gamma_{n,\omega,2}$ we have even exponential decay in n:

$$\Big\| \int_{\Gamma_{n,\omega,2}} (e^{-\omega}\lambda)^n(\lambda-1)R(\lambda,S)\,d\lambda\Big\| \le \int_{\mathcal{M}} r^n\widetilde{D}_1\,|d\lambda| \le 2\pi\widetilde{D}_1r^n.$$

Since the integral over $\Gamma_{n,\omega,3}$ is symmetric to $\Gamma_{n,\omega,1}$ we have shown

$$\|(e^{-\omega}S)^n(S-I)\| \le C_1(\omega+n^{-1}) \quad \text{for all } n \in \mathbb{N}_{\ge n_0}.$$

Hence the second assertion of Theorem 1.1 follows and it remains to show the powerboundedness of $e^{-\omega}S$. Since

$$(e^{-\omega}S)^n = (2\pi i)^{-1}\int_{\Gamma_{n,\omega}} (e^{-\omega}\lambda)^nR(\lambda,S)\,d\lambda$$

we can proceed as above. Indeed, using (1) again we get

$$\Big\| \int\limits_{\Gamma_{n,\omega,1}} (e^{-\omega}\lambda)^n R(\lambda,S)\,d\lambda \Big\| \le C\int\limits_0^{t_0} \frac{(1+n^{-1})^n(1-c_0t)^n}{|n^{-1}+te^{i(\delta+\pi/2)}|}\,dt$$
$$\le Ce\int\limits_0^{\infty} \frac{e^{-c_0tn}}{|n^{-1}+te^{i(\delta+\pi/2)}|}\,dt$$
$$= Ce\int\limits_0^{\infty} \frac{e^{-c_0\tau}}{|1+\tau e^{i(\delta+\pi/2)}|}\,d\tau < \infty.$$

For the integral over $\Gamma_{n,\omega,2}$ we obtain the bound

$$\Big\| \int\limits_{\Gamma_{n,\omega,2}} (e^{-\omega}\lambda)^n R(\lambda,S)\,d\lambda \Big\| \le \int\limits_{\mathcal{M}} \widetilde{D}_1|\lambda-1|^{-1}\,|d\lambda| \le \widetilde{D}_1(2/t_0)2\pi.$$

By symmetry of $\Gamma_{n,\omega,1}$ and $\Gamma_{n,\omega,3}$ we have established

$$\|(e^{-\omega}S)^n\| \le C_2 \quad \text{for all } n\in\mathbb{N}_{\ge n_0}. \ \blacksquare$$

Remark 2.2. Let T,λ_0,C,δ and ω_0,d be as in Theorem 1.1. Then for all $k\in\mathbb{N}_0$ there exists $D_k>0$ such that for all $S\in\mathcal{L}(X)$ with $\|S\|\le C$ and all $\omega\in[0,\omega_0]$ the two conditions (1) and (2) together imply

$$\|(S-I)^kS^n\| \le D_k(\omega+n^{-1})^ke^{\omega n} \quad \text{for all } n\in\mathbb{N}.$$

Proof of Corollary 1.2. Let $k\in\mathbb{N}_{\ge 2}$. It suffices to consider the case where $\omega|\varrho|^m$ is small since whenever $\omega|\varrho|^m\ge\varepsilon$ we have, by hypothesis (c),

$$\|(S_\varrho-I)^kS_\varrho^n\| \le C^{1+k}2^ke^{2k\omega|\varrho|^m n} \le C_{k,\varepsilon}n^{-k}e^{3k\omega|\varrho|^m n} \quad \text{for all } n\in\mathbb{N}.$$

But for $\omega|\varrho|^m$ small we obtain from Remark 2.2, applied to $S:=S_\varrho$,

$$\|(S_\varrho-I)^kS_\varrho^n\| \le D_k(\omega|\varrho|^m+n^{-1})^ke^{\omega|\varrho|^m n} \le D_kn^{-k}e^{(k+1)\omega|\varrho|^m n}. \ \blacksquare$$

3. Proof of Proposition 1.3. Let (Ω,μ,d) be a σ-finite measure space equipped with a metric d and define $L_p := L_p(\Omega,\mu)$. Fix some $\mathcal{A}\subset\{\psi\in L_\infty : \psi\ \mathbb{R}\text{-valued}\}$ such that

$$d(x,y) = \sup_{\psi\in\mathcal{A}} |\psi(x)-\psi(y)| \quad \text{for all } x,y\in\Omega.$$

For instance, one may choose $\mathcal{A} := \{d(x_0,\cdot)\wedge n : x_0\in\Omega,\ n\in\mathbb{N}\}$.

Definition 3.1. For any operator $T\in\mathcal{L}(L_2)$ we define its *Davies perturbations* $(T_\varrho)_{\varrho\in\mathbb{R}}$ by

$$T_\varrho := e^{\varrho\psi}Te^{-\varrho\psi} \quad \text{for } \varrho\in\mathbb{R},\ \psi\in\mathcal{A}.$$

Here we deliberately omit the dependence of T_ϱ on ψ so that the phrase "for all $\varrho\in\mathbb{R}$" has always to be read as "for all $\varrho\in\mathbb{R}$ and all $\psi\in\mathcal{A}$".

The following lemma is well known as "Davies' perturbation method".

LEMMA 3.2. *Let $K \in \mathcal{L}(L_2)$ have an integral kernel $k \in L_\infty(\Omega^2)$. Let $C, \omega > 0$, $m > 1$ and $b := \omega^{-1/(m-1)}(m-1)m^{-m/(m-1)}$. Then the following are equivalent*:

(a) $|k(x,y)| \le C\, e^{-b\, d(x,y)^{m/(m-1)}}$ *a.e.*
(b) $\|K_\varrho\|_{1,\infty} \le Ce^{\omega|\varrho|^m}$ *for all* $\varrho \in \mathbb{R}$.

Proof. Observe that $e^{\varrho\psi}Ke^{-\varrho\psi}$ has kernel $k(x,y)e^{\varrho(\psi(x)-\psi(y))}$. Hence if (b) holds then we have

$$|k(x,y)| \le Ce^{\omega|\varrho|^m}e^{-\varrho(\psi(x)-\psi(y))}$$

so that, for fixed x and y, approximating $d(x,y)$ by suitable $\psi \in \mathcal{A}$ and choosing $\varrho :=\mathrm{sgn}(\psi(x)-\psi(y))(d(x,y)/(\omega m))^{1/(m-1)}$ shows

$$|k(x,y)| \le Ce^{-bd(x,y)^{m/(m-1)}}.$$

For the converse we assume (a) and obtain

$$\begin{aligned}\|e^{\varrho\psi}Ke^{-\varrho\psi}\|_{1,\infty} &= \sup_{x,y}|k(x,y)|e^{\varrho(\psi(x)-\psi(y))}\\ &\le C\sup_{x,y} e^{-bd(x,y)^{m/(m-1)}}e^{|\varrho|\,d(x,y)}\\ &\le C\sup_{r\ge 0} e^{-br^{m/(m-1)}+|\varrho|r} = Ce^{\omega|\varrho|^m}. \ \blacksquare\end{aligned}$$

Proof of Proposition 1.3. Recall that, in addition to the assumptions of this section, our space (Ω,μ,d) is now of at most exponential volume growth:

$$\exists C, c > 0\ \forall r \ge 0,\ x \in \Omega: \quad |B(x,r)| \le Ce^{cr}.$$

In order to obtain (the desired estimates for) integral kernels of the operators $T^n(I-T)^k$, by Lemma 3.2 we have to estimate the $[(T-I)^kT^n]_\varrho = (T_\varrho - I)^kT_\varrho^n$ in the $\|\cdot\|_{1,\infty}$-norm. More precisely, we have to show for all $k \in \mathbb{N}_0$ that

$$(\mathrm{U}_k) \qquad \|(T_\varrho - I)^kT_\varrho^n\|_{1,\infty} \le C_kn^{-N/m-k}e^{\omega_k|\varrho|^mn} \quad \text{for all } n \in \mathbb{N},\ \varrho \in \mathbb{R}$$

where the relation between ω_k and b_k is $b_k = \omega_k^{-1/(m-1)}(m-1)m^{-m/(m-1)}$. By using factorizations of the type

$$\|(T_\varrho - I)^kT_\varrho^n\|_{1,\infty} \le \|T_\varrho^{\lceil n/3\rceil}\|_{1,2}\,\|(T_\varrho - I)^kT_\varrho^{\lfloor n/3\rfloor}\|_{2,2}\,\|T_\varrho^{\lfloor (n+1)/3\rfloor}\|_{2,\infty}$$

for $n \ge 2$ and the hypothesis (7) for $n = 1$ the estimate (U_k) is evident by the hypothesis $(\widetilde{\mathrm{U}}_0)$ once we establish the analyticity property

$$(\mathrm{A}_k) \qquad \|(T_\varrho - I)^kT_\varrho^n\|_{2,2} \le C_k''n^{-k}\exp(\omega_k'|\varrho|^mn) \quad \text{for all } n \in \mathbb{N}_0,\ \varrho \in \mathbb{R}.$$

Since by assumption we have (A_0) and (5), i.e.

$$\|\lambda R(\lambda, T_\varrho - I - \omega_0'|\varrho|^m)\|_{2,2} \le C_0'' \quad \text{for all } \lambda \in \Sigma_\delta,\ |\varrho| \le 1,$$

the property (A_k) follows directly from Corollary 1.2 if we can show

$$\|R(\lambda_0,T)-R(\lambda_0,T_\varrho)\|_{2,2}\to 0 \quad \text{as } \varrho\to 0 \tag{13}$$

for some λ_0. Now, if $\lambda_0\ge 1$ is large enough then

$$\|R(\lambda_0,T)-R(\lambda_0,T_\varrho)\|_{2,2}\le\sum_{n=1}^{\infty}\lambda_0^{-n-1}\|T^n-T_\varrho^n\|_{2,2}\quad \text{for all } |\varrho|\le 1.$$

By using (A_0) we have, for some constants $\overline{C},\overline{c}>0$ and all $M\in\mathbb{N}$, $|\varrho|\le 1$,

$$\sum_{n=M}^{\infty}\lambda_0^{-n-1}\|T^n-T_\varrho^n\|_{2,2}\le\sum_{n=M}^{\infty}\lambda_0^{-n-1}C_0''(1+e^{\omega_0'|\varrho|^m n})\le\overline{C}e^{-\overline{c}M}.$$

Since ($\widetilde{\mathrm{U}}_0$) and (A_0) are supposed to hold we have already seen that (U_0) holds, i.e. the T^n have integral kernels p_n such that

$$|p_n(x,y)|\le C_0 n^{-N/m}\exp\biggl(-b_0\frac{d(x,y)^{m/(m-1)}}{n^{1/(m-1)}}\biggr)\quad \text{for all } n\in\mathbb{N}.$$

Since T_ϱ^n has kernel $p_n(x,y)e^{\varrho(\psi(x)-\psi(y))}$ we can estimate as follows by applying Schur's Lemma in the first step:

$$\begin{aligned}\sum_{n=1}^{M}&\|T^n-T_\varrho^n\|_{2,2}\\ &\le\sum_{n=1}^{M}\sup_{x\in\Omega}\int_\Omega(|p_n(x,y)|\vee|p_n(y,x)|)(e^{|\varrho|d(x,y)}-1)\,dy\\ &\le C_0\sum_{n=1}^{M}\sup_{x\in\Omega}\sum_{k=1}^{\infty}|B(x,k)|\,\exp\biggl(-b_0\frac{(k-1)^{m/(m-1)}}{M^{1/(m-1)}}\biggr)(e^{|\varrho|k}-1)\\ &\le C_0CM\sum_{k=1}^{\infty}e^{c\,k}\exp\biggl(-b_0\frac{(k-1)^{m/(m-1)}}{M^{1/(m-1)}}\biggr)(e^{|\varrho|k}-1)\\ &\to 0\quad \text{as } \varrho\to 0,\ \text{by monotone convergence.}\end{aligned}$$

By letting $M\to\infty$ the convergence in (13) follows. ∎

Remark 3.3. If, in the situation of Proposition 1.3, the hypothesis ($\widetilde{\mathrm{U}}_0$) is replaced by

$$(\widehat{\mathrm{U}}_0)\qquad \|T_\varrho^n\|_{1,2},\|T_\varrho^n\|_{2,\infty}\le C_0'\sqrt{f(n)}\,\exp(\omega_0'|\varrho|^m n)\quad \text{for all } n\in\mathbb{N},\ \varrho\in\mathbb{R}$$

for some decreasing sequence $(f(n))_{n\in\mathbb{N}_0}$ then the above proof shows

$$|D^k p_n(x,y)|\le C_k f(\lfloor(n+1)/3\rfloor)n^{-k}\exp\biggl(-b_k\frac{d(x,y)^{m/(m-1)}}{n^{1/(m-1)}}\biggr)\qquad \text{for all } n\in\mathbb{N}.$$

4. Proof of Theorem 1.4. Let (Ω, μ, d) be again a σ-finite measure space equipped with a metric d and let $T \in \mathcal{L}(L_2)$ be the integral operator corresponding to a symmetric Markov kernel $p \in L_\infty(\Omega^2)$. Furthermore, let $(T_\varrho)_{\varrho \in \mathbb{R}}$ be the Davies perturbations of T defined with respect to

$$\mathcal{A} := \{d(x_0, \cdot) \wedge n : x_0 \in \Omega,\ n \in \mathbb{N}\}.$$

The following lemma is a slight modification of [H-SC, Lemma 2.3] with essentially the same proof. We give it for the sake of completeness.

LEMMA 4.1. *If the symmetric Markov kernel $p \in L_\infty(\Omega^2)$ satisfies the support-condition in Theorem* 1.4, *i.e.*

$$\exists r_0 > 0\ \forall x \in \Omega : \quad \operatorname{supp}(p(x, \cdot)) \subset B(x, r_0), \tag{14}$$

then, for all $\delta \in [0, \pi/2)$, there exists $\omega > 0$ such that

$$\operatorname{Re} e^{i\varphi} \langle (T_\varrho - I)g, g \rangle \le \omega \varrho^2 \|g\|_2^2 \quad \text{for all } g \in L_2(\Omega),\ |\varrho| \le 1,\ |\varphi| \le \delta.$$

Proof. We can adopt the arguments of the proof of [H-SC, Lemma 2.3] although in [H-SC] the Davies perturbations are defined with respect to $\widetilde{\mathcal{A}} = \{d(x_0, \cdot) : x_0 \in \Omega\}$. First we note that

$$|e^{\varrho\psi(x)} - e^{\varrho\psi(y)}| \le r_0 |\varrho| (e^{\varrho\psi(x)} + e^{\varrho\psi(y)}), \qquad x \in \Omega,\ y \in B(x, r_0), \tag{15}$$

for all $\varrho \in \mathbb{R}$ and $\psi \in \mathcal{A}$. Defining $f := e^{-\varrho\psi} g$ we have

$$\begin{aligned}
&4\langle (I - T_\varrho)g, g\rangle \\
&\quad = 4\langle (I - T)f, e^{2\varrho\psi} f\rangle \\
&\quad = 2\int (f(x) - f(y))\overline{(e^{2\varrho\psi(x)} f(x) - e^{2\varrho\psi(y)} f(y))}\, k(x, y)\, dx\, dy \\
&\quad = \int |f(x) - f(y)|^2 (e^{2\varrho\psi(x)} + e^{2\varrho\psi(y)}) k(x, y)\, dx\, dy \\
&\qquad + \int (f(x) - f(y))\overline{(f(x) + f(y))}(e^{2\varrho\psi(x)} - e^{2\varrho\psi(y)}) k(x, y)\, dx\, dy \\
&\quad = E_1 + E_2.
\end{aligned}$$

The first term E_1 is nonnegative. Using the Cauchy–Schwarz inequality, (14) and (15), we can estimate the second term E_2 by

$$\begin{aligned}
|E_2|^2 &\le \int |f(x) - f(y)|^2 (e^{\varrho\psi(x)} + e^{\varrho\psi(y)})^2 k(x, y)\, dx\, dy \\
&\qquad \times \int |f(x) + f(y)|^2 |e^{\varrho\psi(x)} - e^{\varrho\psi(y)}|^2 k(x, y)\, dx\, dy \\
&\le 8 r_0^2 E_1 \varrho^2 \int (|f(x)|^2 + |f(y)|^2)(e^{2\varrho\psi(x)} + e^{2\varrho\psi(y)}) k(x, y)\, dx\, dy \\
&\le C E_1 \varrho^2 \|e^{\varrho\psi} f\|_2^2 \qquad (|\varrho| \le 1) \\
&\le (\cos(\delta) E_1 + 4\omega\varrho^2 \|g\|_2^2)^2
\end{aligned}$$

for suitable $C,\omega>0$ independent of $|\varrho|\le 1$, $\psi\in\mathcal{A}$ and $g\in L_2(\Omega)$. This shows

$$\operatorname{Re} e^{i\varphi}\langle (I-T_\varrho)g,g\rangle \ge \cos(\varphi)E_1/4-|E_2/4|\ge -\omega\varrho^2\|g\|_2^2. \ \blacksquare$$

Proof of Theorem 1.4. Fix some $\delta\in(0,\pi/2)$, choose $\omega>0$ as in Lemma 4.1 and set $\omega_0'':=\cos(\delta)^{-1}\omega$. Employing Lemma 4.1 for functions $g\in L_2(\Omega)$ of the type $g=e^{te^{i\varphi}(T_\varrho-I-\omega_0''\varrho^2)}f$ shows

$$\frac{d}{dt}\|e^{te^{i\varphi}(T_\varrho-I-\omega_0''\varrho^2)}f\|_2^2\le 0\quad \text{for all } f\in L_2(\Omega),\ |\varrho|\le 1,\ |\varphi|\le\delta,\ t\ge 0.$$

Hence $\|e^{z(T_\varrho-I-\omega_0''\varrho^2)}\|_{2,2}\le 1$ for all $|\arg(z)|\le\delta$, which implies

$$\|\lambda R(\lambda,T_\varrho-I-\omega_0''\varrho^2)\|_{2,2}\le C\quad \text{for all } \lambda\in\Sigma_\delta,\ |\varrho|\le 1$$

by well known semigroup theory [P]. Now arguing as in the proof of [H-SC, Lemma 2.4] and of [H-SC, Lemma 2.2] yields the following two estimates:

$$\|T_\varrho^n\|_{2,2}\le Ce^{\omega_0'\varrho^2 n},\qquad \|T_\varrho^n\|_{1,2},\ \|T_\varrho^n\|_{2,\infty}\le Cn^{-N/4}e^{\omega_0'\varrho^2 n}$$

for all $n\in\mathbb{N}$, $\varrho\in\mathbb{R}$ and some $\omega_0'>0$. In particular, T is powerbounded so that its selfadjointness implies $\sigma(T)\subset[-1,1]$. Our additional assumption $-1\notin\sigma(T)$ thus ensures

$$\sigma(T)\subset(-1,1].$$

By Nevanlinna's Theorem [N1, Thm. 4.5.4] the operator T is analytic and Theorem 1.4 follows from Proposition 1.3. $\blacksquare$

Acknowledgements. The author was supported by a scholarship of the "Gemeinsames Hochschulsonderprogramm III von Bund und Ländern" by the German Academic Exchange Office (DAAD).

REFERENCES

[BB] M. T. Barlow and R. F. Bass, *Random walks on graphical Sierpiński carpets*, in preparation.

[C] M. Christ, *Temporal regularity for random walk on discrete nilpotent groups*, J. Fourier Anal. Appl., Kahane Special Issue (1995), 141–151.

[C-SC] T. Coulhon et L. Saloff-Coste, *Puissances d'un opérateur régularisant*, Ann. Inst. H. Poincaré Probab. Statist. 26 (1990), 419–436.

[H-SC] W. Hebisch and L. Saloff-Coste, *Gaussian estimates for Markov chains and random walks on graphs*, Ann. Probab. 21 (1993), 673–709.

[J] O. D. Jones, *Transition probabilities for the simple random walk on the Sierpiński graph*, Stochastic Process. Appl. 61 (1996), 45–69.

[N1] O. Nevanlinna, *Convergence of Iterations for Linear Equations*, Birkhäuser, Basel, 1993.

[N2] —, *On the growth of the resolvent operators for power bounded operators*, in: Linear Operators, J. Janas, F. H. Szafraniec and J. Zemánek (eds.), Banach Center Publ. 38, Inst. Math., Polish Acad. Sci., 1997, 247–264.

[P] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Appl. Math. Sci. 44, Springer, 1983.

Département de Mathématiques
Université de Cergy-Pontoise
2, avenue Adolphe Chauvin
95302 Cergy-Pontoise, France
E-mail: Soenke.Blunck@math.u-cergy.fr

Received 11 January 2000 (3874)