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SOME SPECTRAL RESULTS ON $L^2(H_n)$ RELATED TO THE ACTION OF $U(p,q)$

BY

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Abstract. Let H_n be the $(2n+1)$-dimensional Heisenberg group, let p, q be two non-negative integers satisfying $p+q=n$ and let G be the semidirect product of $U(p,q)$ and H_n. So $L^2(H_n)$ has a natural structure of G-module. We obtain a decomposition of $L^2(H_n)$ as a direct integral of irreducible representations of G. On the other hand, we give an explicit description of the joint spectrum $\sigma(L, iT)$ in $L^2(H_n)$ where

$$L=\sum_{j=1}^{p}(X_j^2+Y_j^2)-\sum_{j=p+1}^{n}(X_j^2+Y_j^2),$$

and where $\{X_1, Y_1, \ldots, X_n, Y_n, T\}$ denotes the standard basis of the Lie algebra of H_n. Finally, we obtain a spectral characterization of the bounded operators on $L^2(H_n)$ that commute with the action of G.

1. Introduction. Let p, q a pair of non-negative integers such that $p+q=n$. Consider the Heisenberg group $H_n=\mathbb{C}^n\times\mathbb{R}$ with group law $(z,t)(z',t')=\big(z+z', t+t'-\frac{1}{2}\operatorname{Im}B(z,z')\big)$ where $B(z,w)=\sum_{j=1}^{p} z_j\overline{w}_j-\sum_{j=p+1}^{n} z_j\overline{w}_j$. For $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, we write $x=(x',x'')$ with $x'\in\mathbb{R}^p$, $x''\in\mathbb{R}^q$. So, $\mathbb{R}^{2n}$ can be identified with $\mathbb{C}^n$ via the map

$$\Psi(x',x'',y',y'')=(x'+iy', x''-iy''),\qquad x',y'\in\mathbb{R}^p,\ x'',y''\in\mathbb{R}^q.$$

This map identifies the form $-\operatorname{Im}B(z,w)$ with the standard symplectic form on $\mathbb{R}^{2(p+q)}$. Moreover, $(x,y,t)\mapsto(\Psi(x,y),t)$ provides a global coordinate system on H_n and the vector fields

$$X_j=-\frac{1}{2}y_j\frac{\partial}{\partial t}+\frac{\partial}{\partial x_j},\quad Y_j=\frac{1}{2}x_j\frac{\partial}{\partial t}+\frac{\partial}{\partial y_j},\quad j=1,\ldots,n,\quad\text{and}\quad T=\frac{\partial}{\partial t}$$

satisfy $[X_j,Y_j]=T$, $[X_j,T]=[Y_j,T]=0$, $1\le j\le n$. Thus H_n can be viewed as the usual Heisenberg group $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ via the isomorphism $(x,y,t)\mapsto(\Psi(x,y),t)$. From now on, we will use freely this identification.

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We note that $U(p,q)$ acts by automorphisms on H_n via the action

$$g\cdot(z,t)=(gz,t),\qquad g\in U(p,q),\ (z,t)\in H_n. \tag{1.1}$$

Observe that the above group law is not the usual one, but it is adapted to the action of $U(p,q)$, $q=n-p$.

In [St-2], R. Strichartz proposed to define harmonic analysis on H^n to be the joint spectral theory associated with the differential operators L_0 and iT where $L_0=\sum_{j=1}^{n}(X_j^2+Y_j^2)$. The relevance of the operators L_0 and iT is due to the fact that they are the generators of the algebra of the left invariant differential operators which are invariant under the natural action of $U(n)$ on H_n.

Let $L=\sum_{j=1}^{p}(X_j^2+Y_j^2)-\sum_{j=p+1}^{n}(X_j^2+Y_j^2)$. Since L and iT generate the algebra of left invariant and $U(p,q)$-invariant differential operators, it is a natural question to look for a spectral theory on $L^2(H_n)$ related to the operators L and iT. In [G-S] we prove that there exist tempered $U(p,q)$-invariant distributions $S_{\lambda,k}$, $\lambda\in\mathbb{R}-\{0\}$, $k\in\mathbb{Z}$, satisfying

$$LS_{\lambda,k}=-|\lambda|(2k+p-q)S_{\lambda,k},\qquad iTS_{\lambda,k}=\lambda S_{\lambda,k} \tag{1.2}$$

and such that for $f\in S(\mathbb{R}^n)$,

$$f=\sum_{k\in\mathbb{Z}}\int_{-\infty}^{\infty} f*S_{\lambda,k}|\lambda|^n\,d\lambda.$$

Moreover, the distributions $S_{\lambda,k}$ are explicitly computed and it is proved that the solution space in $S'(H_n)^{U(p,q)}$ of the system (1.2) is one-dimensional (see also [F-2] and [H-T]).

On the other hand, let $G=U(p,q)\ltimes H_n$ be the semidirect product of $U(p,q)$ with H_n with group law $(g,z,t)(g',z',t')=(gg',(z,t)\cdot(gz',t'))$ for $g,g'\in U(p,q)$ and $(z,t),(z',t')\in H_n$. Then G acts on H_n by $(g,z,t)(z',t')=(z,t)(gz',t')$. For $f:H_n\to C$ and $(g,z,t)\in G$, we set

$$\varrho(g,z,t)f(z',t')=f((g,z,t)^{-1}(z',t')). \tag{1.3}$$

Thus ϱ defines a unitary representation of G on $L^2(H_n)$ that, restricted to $H_n\subset G$, agrees with the left regular representation of H_n on $L^2(H_n)$.

Our aim in this paper is to give an explicit description of the joint spectrum in $L^2(H_n)$ of L and iT and to obtain the decomposition of $L^2(H_n)$ as a direct integral of irreducible representations of G. The last question was solved in [St-2], for $p=n$, $q=0$, using the weight theory for representations of compact Lie groups. In order to study the general case, we will follow a different approach, using the results in [G-S] instead of weights. Finally, we state a spectral characterization of the bounded operators on $L^2(H_n)$ that commute with the action ϱ.

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2. Preliminaries. Let us consider, for $\lambda \in \mathbb{R} - \{0\}$, the Schrödinger representation of $H_n = \mathbb{R}^{2n} \times \mathbb{R}$ on $L^2(\mathbb{R}^n)$ defined by

$$\pi_\lambda(x,y,t)u(\xi) = \exp\left[-i\left(\lambda t + \operatorname{sign}(\lambda)\sqrt{|\lambda|}\langle x,\xi\rangle + \frac{\lambda}{2}\langle x,y\rangle\right)\right]u(\xi + \sqrt{|\lambda|}\,y).$$

For $u, v \in L^2(\mathbb{R}^n)$, let $E_\lambda(u,v)$ be the matrix entry associated with π_λ corresponding to the vectors u, v given by $E_\lambda(u,v)(x,y,t) = \langle \pi_\lambda(x,y,t)u, v\rangle$.

Also, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, let h_α be the Hermite function defined by

$$h_\alpha(\zeta) = (2^{|\alpha|}\alpha!\sqrt{\pi})^{-n/2} e^{-|\zeta|^2/2} \prod_{j=1}^{n} H_{\alpha_j}(\zeta_j)$$

with $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\alpha! = \alpha_1! \ldots \alpha_n!$ and where

$$H_k(s) = (-1)^k e^{s^2} \frac{d^k}{ds^k}(e^{-s^2})$$

is the kth Hermite polynomial. For $(z,t) \in \mathbb{C}^n \times \mathbb{R}$ (see, for example, [F-1]), we can write $E_\lambda(h_\alpha, h_\alpha)(z,t)$ in terms of Laguerre polynomials as

$$E_\lambda(h_\alpha, h_\alpha)(z,t) = e^{-i\lambda t} e^{-|\lambda||z|^2/4} \prod_{j=1}^{n} L^0_{\alpha_j}\left(\frac{1}{2}|\lambda||z_j|^2\right).$$

We set $\|\alpha\| = \alpha_1 + \ldots + \alpha_p - (\alpha_{p+1} + \ldots + \alpha_n)$. Thus $\{h_\alpha\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ satisfying

$$\begin{aligned} LE_\lambda(h_\alpha, h_\alpha) &= -|\lambda|(2\|\alpha\| + p - q)E_\lambda(h_\alpha, h_\alpha), \\ iTE_\lambda(h_\alpha, h_\alpha) &= \lambda E_\lambda(h_\alpha, h_\alpha). \end{aligned} \tag{2.1}$$

We also set, for $f \in L^1(H_n)$,

$$\pi_\lambda(f) = \int_{H_n} f(x,y,t)\pi_\lambda(x,y,t)^{-1}\,dx\,dy\,dt.$$

Let $\mathbb{R}^* = \mathbb{R} - \{0\}$ and let us denote by $\mathrm{HS}(L^2(\mathbb{R}^n))$ the space of Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. Let $\mathcal{L}^2(\mathbb{R}^*)$ be the Hilbert space of functions $\Phi : \mathbb{R}^* \to \mathrm{HS}(L^2(\mathbb{R}^n))$ such that $\lambda \mapsto \langle \Phi(\lambda)u, v\rangle$ is measurable for each $u, v \in L^2(\mathbb{R}^n)$ and $\int_{-\infty}^{\infty} \|\Phi(\lambda)\|^2_{\mathrm{HS}}|\lambda|^n\,d\lambda = \|\Phi\| < \infty$. The Plancherel Theorem asserts (see e.g. [T]) that the Fourier transform $f \mapsto (2\pi)^{-(n+1)/2}\pi_\lambda(f)$, initially defined, say, in $S(H_n)$, extends to an isometry from $L^2(H_n)$ onto $\mathcal{L}^2(\mathbb{R}^*)$. Moreover, for $f \in S(H_n)$ we have the inversion formula

$$f(x,y,t) = \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{\infty} \operatorname{tr}(\pi_\lambda(f)\pi_\lambda(x,y,t))|\lambda|^n\,d\lambda.$$

Since, in this case, $\sum_\alpha \int_{-\infty}^{\infty} |f * E_\lambda(h_\alpha, h_\alpha)||\lambda|^n \, d\lambda < \infty$, a computation shows that the inversion formula reads

$$f(x,y,t) = \frac{1}{(2\pi)^{n+1}} \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \sum_{\|\alpha\|=k} f * E_\lambda(h_\alpha, h_\alpha)|\lambda|^n \, d\lambda. \tag{2.2}$$

For $k \in \mathbb{Z}$, $\lambda \in \mathbb{R} - \{0\}$ let $S_{\lambda,k}$ be defined by

$$\langle S_{\lambda,k}, f\rangle = \frac{1}{(2\pi)^{n+1}} \sum_{\|\alpha\|=k} \langle E_\lambda(h_\alpha, h_\alpha), f\rangle, \qquad f \in S(H_n).$$

Then $S_{\lambda,k}$ is a well defined element in $S'(H_n)$; moreover, $S_{\lambda,k}$ can be explicitly computed and it is the unique (up to a constant) tempered and $U(p,q)$-invariant solution of the system (1.2) (see e.g. [G-S]). Also, (2.2) gives the decomposition

$$f = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} f * S_{\lambda,k}|\lambda|^n \, d\lambda, \qquad f \in S(H_n). \tag{2.3}$$

We will also need to consider, for a fixed $\lambda \neq 0$, the quotient group $\overline{H}_n = H_n/N$ where $N = \{0\} \times (2\pi/\lambda)\mathbb{Z}$. For $(x,y,t) \in H_n$, let $[x,y,t]$ be its projection on $\overline{H}_n$. Note that for $\mu = \lambda m$, $m \in \mathbb{Z} - \{0\}$, π_μ induces in a natural way a unitary representation $\overline{\pi}_\mu$ of $\overline{H}_n$ with matrix entries $\overline{E}_\mu(u,v)$ given by $\overline{E}_\mu(u,v)([x,y,t]) = E_\mu(u,v)(x,y,t)$.

Moreover, each irreducible unitary representation of $\overline{H}_n$ is unitarily equivalent to one and only one of the following representations:

(1) the representations $\overline{\pi}_\mu$ corresponding to $\mu = \lambda m$, $m \in \mathbb{Z}$,

(2) the one-dimensional representations $\sigma_{a,b}(x,y,t) = e^{i(ax+by)}$, $(a,b) \in \mathbb{R}^n \times \mathbb{R}^n$.

Now, the Plancherel inversion formula for $\overline{H}_n$ says that, for $f \in S(\overline{H}_n)$,

$$(2\pi)^{n+1} f(x,y,t) = \sum_{m\neq 0} \sum_{k\in\mathbb{Z}} \sum_{\|\alpha\|=k} f * \overline{E}_{\lambda m}(h_\alpha, h_\alpha)|m|^n + \widehat{\Phi}(-x,-y) \tag{2.4}$$

with $\Phi(a,b) = \sigma_{a,b}(f)$. Moreover,

$$(2\pi)^{n+1}\|f\|^2_{L^2(\overline{H}_n)} = \sum_{m\in\mathbb{Z}-\{0\}} \|\pi_{\lambda m}(f)\|^2_{\mathrm{HS}}|m|^n + \int_{\mathbb{R}^n\times\mathbb{R}^n} |\sigma_{a,b}(f)|^2 \, da\, db.$$

The proofs of these facts follow the same lines as those related to H_n (see e.g. [T]).

For $k, m \in \mathbb{Z}$ and $f \in S(\overline{H}_n)$, we set

$$\langle \overline{S}_{\lambda m,k}, f\rangle = \frac{1}{(2\pi)^{n+1}} \sum_{\|\alpha\|=k} \langle \overline{E}_{\lambda m}(h_\alpha, h_\alpha), f\rangle.$$

Thus, as in [G-S], $\overline{S}_{\lambda m,k} \in S'(\overline{H}_n)$.

3. Some spectral facts. Let $\overline{H}_n$ be the reduced Heisenberg group, associated with a fixed λ, defined as above. Let $\overline{G} = U(p,q) \ltimes \overline{H}_n$ be the semidirect product of $U(p,q)$ and $\overline{H}_n$, so ϱ projects to a unitary representation $\overline{\varrho}$ of $\overline{G}$ on $L^2(\overline{H}_n)$. Also, L and T can be viewed, in a natural way, as differential operators on $\overline{H}_n$.

Let $P_k : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, $k \in \mathbb{Z}$, be the orthogonal projection onto the closed subspace of $L^2(\mathbb{R}^n)$ spanned by $\{h_\alpha\}_{\|\alpha\|=k}$. For each $k \in \mathbb{Z}$, the Plancherel theorem for $\overline{H}_n$ implies that there exists a unique bounded operator $\wp_k : L^2(\overline{H}_n) \to L^2(\overline{H}_n)$ defined by the conditions $\overline{\pi}_\lambda \wp_k(f) = P_k \overline{\pi}_\lambda(f)$, $\overline{\pi}_{\lambda m} \wp_k(f) = 0$ for $m \neq 1$, and $\sigma_{a,b} \wp_k(f) = 0$ for all $a, b \in \mathbb{R}^n$. By the Plancherel theorem again it is immediately seen that $\wp_k^2 = \wp_k$, $\wp_k^* = \wp_k$ and so $\wp_k$ is an orthogonal projection. Moreover, for $f \in S(\overline{H}_n), w \in \overline{H}_n$, the inversion formula gives

$$\wp_k f(w) = \operatorname{tr}(P_k \overline{\pi}_\lambda(f) \overline{\pi}_\lambda(w)) = \sum_{\|\alpha\|=k} (f * \overline{E}_\lambda(h_\alpha, h_\alpha))(w).$$

Thus

$$\wp_k f = f * \overline{S}_{\lambda,k} \tag{3.1}$$

and so $f * \overline{S}_{\lambda,k} \in L^2(\overline{H}_n)$.

Since $L(f * \overline{E}_\lambda(h_\alpha, h_\alpha)) = f * L\overline{E}_\lambda(h_\alpha, h_\alpha)$ in $S'(\overline{H}_n)$, and recalling (2.1), we see that $h \in \wp_k(S(\overline{H}_n))$ implies $Lh = -|\lambda|(2k + p - q)h$ and $iTh = \lambda h$.

Also, if $f \in L^2(\overline{H}_n)$ and $k \neq k'$, then $\pi_{\lambda m}(\wp_{k'} \wp_k f) = 0$ for $m \neq 1$ and $\pi_\lambda(\wp_{k'} \wp_k f) = P_{k'} P_k \pi_\lambda f = 0$. Thus, by the Plancherel theorem, $\wp_{k'} \wp_k = 0$ for $k \neq k'$.

PROPOSITION 3.2. *$\wp_k(L^2(\overline{H}_n))$ is a $\overline{\varrho}$-irreducible module.*

Proof. Since $\wp_k f = f * \overline{S}_{\lambda,k}$ for $f \in S(\overline{H}_n)$ and $\overline{S}_{\lambda,k}$ is $U(p,q)$-invariant, it follows that $\wp_k$ is a $\overline{\varrho}$-morphism. Now, we proceed by contradiction. Assume that there exists a $\overline{\varrho}$-invariant, non-zero and closed subspace W of $\wp_k(L^2(\overline{H}_n))$. Let $P : \wp_k(L^2(\overline{H}_n)) \to W$ be the orthogonal projection on W. Then P and $P\wp_k$ are $\overline{G}$-morphisms. Moreover, $P\wp_k : L^2(\overline{H}_n) \to L^2(\overline{H}_n)$ is a bounded operator that commutes with left translations, and hence there exists $\Phi \in S'(\overline{H}_n)$ such that $P\wp_k f = f * \Phi$ for $f \in S(\overline{H}_n)$. Since $P\wp_k$ also commutes with the action of $U(p,q)$, we conclude that Φ is $U(p,q)$-invariant. Furthermore, $L\Phi = -|\lambda|(2k+p-q)\Phi$ and $iT\Phi = \lambda\Phi$. Indeed, for $f \in S(\overline{H}_n)$,

$$\begin{aligned} \langle f, L\Phi \rangle &= (f * L\Phi)(0) = L(f * \Phi)(0) \\ &= -|\lambda|(2k + p - q)(f * \Phi)(0) = -|\lambda|(2k + p - q)\langle f, \Phi \rangle. \end{aligned}$$

The computation of $iT\Phi$ is analogous. Thus $\Phi = c\overline{S}_{\lambda,k}$ for some $c \in \mathbb{R} - \{0\}$, so $P\wp_k = \wp_k$ and then $\wp_k(L^2(\overline{H}_n)) \subset W$. ∎

For $f \in S(\overline{H}_n)$, a computation gives

$$f * \overline{E}_\lambda(h_\alpha, h_\alpha) = \sum_\beta \langle f, \overline{E}_\lambda(h_\alpha, h_\beta)\rangle \overline{E}_\lambda(h_\alpha, h_\beta)$$

and so

$$f * \overline{S}_{\lambda,k} = \sum_{\|\alpha\|=k} \sum_\beta \langle f, \overline{E}_\lambda(h_\alpha, h_\beta)\rangle \overline{E}_\lambda(h_\alpha, h_\beta). \tag{3.3}$$

In [St-1] it is proved that $\{\overline{E}_\lambda(h_\alpha, h_\beta)(\cdot, 0)\}_{\alpha,\beta}$ is an orthonormal set in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$. So, for $\|\alpha\| = k$, we have $\overline{E}_\lambda(h_\alpha, h_\beta) = \overline{E}_\lambda(h_\alpha, h_\beta) * \overline{S}_{\lambda,k}$ and then, by (3.1), $\overline{E}_\lambda(h_\alpha, h_\beta) \in \wp_k(L^2(\overline{H}_n))$. On the other hand, (3.1) also says that, for $f \in S(\overline{H}_n)$, $\wp_k(f)$ belongs to the closed subspace spanned by $\{\overline{E}_\lambda(h_\alpha, h_\beta) : \|\alpha\| = k,\ \beta \text{ arbitrary}\}$. Thus $\{\overline{E}_\lambda(h_\alpha, h_\beta) : \|\alpha\| = k,$ β arbitrary$\}$ is an orthonormal basis of $\wp_k(L^2(\overline{H}_n))$.

Following [St-2], we consider, for each $\lambda \in \mathbb{R}^*$, the Hilbert space H_λ of functions $f : H_n \to \mathbb{C}$ such that $f(x, y, t) = e^{-i\lambda t} F(x, y)$ with $F \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$ provided with the norm $\|f\| = \|f(\cdot, 0)\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}$. Note that each $E_\lambda(h_\alpha, h_\beta) \in H_\lambda$. We set $H_{\lambda,k} = \overline{\langle\{E_\lambda(h_\alpha, h_\beta)\} : \|\alpha\| = k,\ \beta \text{ arbitrary}\rangle}$, the closure taken in H_λ. Since

$$\varrho(g, x, y, t) E_\lambda(h_\alpha, h_\beta)(x', y', t') = \overline{\varrho}([g, x, y, t]) \overline{E}_\lambda(h_\alpha, h_\beta)([x', y', t'])$$

where $[g, x, y, t]$ denotes the projection of (g, x, y, t) on $\overline{G}$, and since

$$E_\lambda(h_\alpha, h_\beta)(x, y, t) = \overline{E}_\lambda(h_\alpha, h_\beta)([x, y, t]),$$

we see that $(H_{\lambda,k}, \varrho)$ is a unitary representation of G.

For $f \in S(H_n)$, since $\sum_{\|\alpha\|=k} \sum_\beta |\langle f, E_\lambda(h_\alpha, h_\beta)\rangle|^2 = \|P_k \pi_\lambda(f)\|^2$, the Plancherel identity says that

$$(2\pi)^{n+1} \|f\|^2_{L^2(H_n)} = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \|f * S_{\lambda,k}\|^2_{H_{\lambda,k}} |\lambda|^n \, d\lambda. \tag{3.4}$$

Moreover, the following analogue of (3.3) holds:

$$f * S_{\lambda,k} = \sum_{\|\alpha\|=k} \sum_\beta \langle f, E_\lambda(h_\alpha, h_\beta)\rangle E_\lambda(h_\alpha, h_\beta);$$

thus $f * S_{\lambda,k} \in H_{\lambda,k}$ for a.e. $\lambda \in \mathbb{R}^*$.

Let $\Phi : \mathbb{R}^* \times \mathbb{Z} \to \bigcup_{(\lambda,k)\in\mathbb{R}^*\times\mathbb{Z}} H_{\lambda,k}$ be such that $\Phi(\lambda, k) \in H_{\lambda,k}$ for a.e. $\lambda \in \mathbb{R}$. So

$$\Phi(\lambda, k) = \sum_{\|\alpha\|=k} \sum_\beta c_\lambda(\alpha, \beta) E_\lambda(h_\alpha, h_\beta)$$

with $\sum_{\|\alpha\|=k} \sum_\beta |c_\lambda(\alpha, \beta)|^2 < \infty$ for a.e. $\lambda \in \mathbb{R}$. We say that Φ is *measurable* if for every α, β the map $\lambda \mapsto c_\lambda(\alpha, \beta)$ is a measurable function. Let us consider the direct integral of Hilbert spaces

$$\sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} H_{\lambda,k}|\lambda|^n \, d\lambda,$$

i.e., the space of measurable functions Φ as above satisfying

$$\sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} \|\Phi(\lambda,k)\|^2_{H_{\lambda,k}} |\lambda|^n \, d\lambda < \infty.$$

We have

THEOREM 3.5. *Each $H_{\lambda,k}$ is an irreducible G-module*, $H_{\lambda,k} \ncong H_{\lambda',k'}$ *if* $(\lambda,k) \neq (\lambda',k')$ *and* $(L^2(H_n),\varrho)$ *is the direct integral of irreducible representations*

$$L^2(H_n) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} H_{\lambda,k}|\lambda|^n \, d\lambda.$$

Proof. Note that if $f \in H_\lambda$ then f is constant on each coset $[x,y,t] \in \overline{H}_n$ and so we can define $\overline{f} : \overline{H}_n \to \mathbb{C}$ by $\overline{f}([x,y,t]) = f(x,y,t)$. We consider the map $K_{\lambda,k} : H_{\lambda,k} \to \wp_k(L^2(\overline{H}_n))$ given by $K_{\lambda,k}f(x,y,t) = \overline{f}([x,y,t])$. Then $K_{\lambda,k}\varrho(\theta) = \overline{\varrho}([\theta])K_{\lambda,k}$, $\theta \in G$. Since $K_{\lambda,k}$ is a bijection and $\wp_k(L^2(\overline{H}_n))$ is $\overline{G}$-irreducible, we see that $H_{\lambda,k}$ is irreducible. Furthermore, $(H_{\lambda,k}, \varrho|_{H_n})$ is a primary H_n-module. Indeed, for fixed α, the map $h_\beta \mapsto E_\lambda(h_\alpha,h_\beta)$ extends to an injective H_n-morphism between $(\pi_\lambda, L^2(\mathbb{R}^n))$ and $(\varrho|_{H_n}, H_{\lambda,k})$. So, for $\lambda \neq \lambda'$ and $k,k' \in \mathbb{Z}$, $H_{\lambda,k} \ncong H_{\lambda',k'}$ as G-modules. In order to see that $H_{\lambda,k} \ncong H_{\lambda,k'}$ for $k \neq k'$, suppose that $U : H_{\lambda,k} \to H_{\lambda,k'}$ is a (bounded) G-isomorphism. Then $K_{\lambda,k'}UK^{-1}_{\lambda,k}\wp_k : L^2(\overline{H}_n) \to \wp_{k'}(L^2(\overline{H}_n))$ is a bounded operator on $L^2(\overline{H}_n)$ and a $\overline{G}$-morphism. We argue as in the proof of Proposition 3.2 to conclude that $K_{\lambda,k'}UK^{-1}_{\lambda,k}\wp_k = c\wp_{k'}$ for some constant c. Since $\wp_k\wp_{k'} = 0$ we obtain $U = 0$.

Finally, we note that by (3.4), the mapping $U : f \mapsto f * S_{\lambda,k}$ initally defined on $S(H_n)$ extends, up to a constant, to an isometry from $L^2(H_n)$ into the direct integral $\mathcal{H} = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} H_{\lambda,k}|\lambda|^n \, d\lambda$. On the other hand, for $\Phi \in \mathcal{H}$ we can write $\Phi(\lambda,k) = \sum_{\|\alpha\|=k,\,\beta} c_\lambda(\alpha,\beta)E_\lambda(h_\alpha,h_\beta)$ with $\sum_{\|\alpha\|=k,\,\beta} |c_\lambda(\alpha,\beta)|^2 < \infty$ for a.e. $\lambda \in \mathbb{R}$. Let $V(\Phi) \in L^2(H_n)$ be defined by $\langle \pi_\lambda(V(\Phi))h_\alpha, h_\beta\rangle = c_\lambda(\alpha,\beta)$. Thus V is, up to a constant, an isometry from $\mathcal{H}$ into $L^2(H_n)$ and $VU = I$. Since $\varrho(g)(U(f)(\lambda,k)) = (U(\varrho(g)(f)))(\lambda,k)$, the theorem follows. ∎

Our next step is to describe the joint spectrum of L and iT in $L^2(\mathbb{R}^n)$. This joint spectrum $\sigma(L,iT)$ is defined as the complement of the pairs $(\mu,\lambda) \in \mathbb{C}^2$ for which there exist bounded operators A,B on $L^2(H_n)$ such that $A(L-\mu I) + B(iT - \lambda I) = I$.

We recall the orthogonal decomposition

$$L^2(H_n) = \bigoplus_{m\in n+2\mathbb{Z}} (\mathrm{Ker}(L - \mathrm{i}m T) \cap L^2(H_n)), \tag{3.6}$$

the kernels taken in the distribution sense. Moreover, if for $m \in n + 2\mathbb{Z}$, we set $k_1(m) = (-m + q - p)/2$ and $k_2(m) = (m + q - p)/2$, then (see [G-S]) there exist orthogonal projections $P_{k_1(m)}, P_{k_2(m)} : L^2(H_n) \to L^2(H_n)$ given, for $f \in S(H_n)$, by $P_{k_1(m)} f = \int_0^\infty f * S_{\lambda,k_1(m)} |\lambda|^n \, d\lambda$ and $P_{k_2(m)} f = \int_{-\infty}^0 f * S_{\lambda,k_2(m)} |\lambda|^n \, d\lambda$ with $R(P_{k_1(m)}) \perp R(P_{k_2(m)})$ and satisfying

$$\mathrm{Ker}(L - \mathrm{i}m T) = R(P_{k_1(m)}) \oplus R(P_{k_2(m)}).$$

Now we set, for $\varepsilon = \pm 1$ and $k \in \mathbb{Z}$,

$$R_{k,\varepsilon} = \{(-\varrho(2k + p - q), \varepsilon\varrho) : \varrho > 0\}.$$

We also put $\mathbb{R}_0 = \{(0, \mu) : \mu \in \mathbb{R}\}$.

THEOREM 3.7. $\sigma(L, iT) = \mathbb{R}_0 \cup \bigcup_{k\in\mathbb{Z},\, \varepsilon=\pm1} R_{k,\varepsilon}$.

Proof. If $(\lambda, \mu) \in \mathbb{C}^2$, $\lambda \neq 0$ and $\mu \neq m\lambda$ for all $m \in n+2\mathbb{Z}$, then taking account of (3.6), we can define bounded operators $A, B : L^2(H_n) \to L^2(H_n)$ by

$$Af = \frac{1}{\lambda m - \mu} f, \quad Bf = \frac{-m}{\lambda m - \mu} f \quad \text{ for } f \in \mathrm{Ker}(L - \mathrm{i}m T).$$

So, we have $A(L-\mu I)+B(iT-\lambda I)=I$. Then $\sigma(L, iT) \subset \mathbb{R}_0 \cup \bigcup_{k\in\mathbb{Z},\varepsilon=\pm1} R_{k,\varepsilon}$.

Now we will see that every point $(m\lambda_0, \lambda_0)$ with $m \in n + 2\mathbb{Z}$ and $\lambda_0 \neq 0$ belongs to $\sigma(L, iT)$. We consider first the case $\lambda_0 > 0$ and $k_1(m) \geq 0$. Assume, by contradiction, that there exist bounded operators A, B on $L^2(H_n)$ such that

$$A(L - m\lambda_0 I) + B(iT - \lambda_0 I) = I. \tag{3.8}$$

Let φ_ε be an approximation to the identity centered at λ_0, i.e. $\varphi_\varepsilon(\lambda) = \varepsilon^{-1}\varphi(\varepsilon^{-1}(\lambda - \lambda_0))$ with $\varphi \in C^\infty(\mathbb{R})$, $\varphi \geq 0$, $\int \varphi = 1$, $\varphi(0) > 0$ and such that $\mathrm{supp}(\varphi) \subset (-1, 1)$. We set

$$f_\varepsilon(z, t) = \int_{-\infty}^{\infty} \varphi_\varepsilon(\lambda) E_\lambda(h_\alpha, h_\alpha)(z, t) \, d\lambda$$

where $\alpha = (k_1(m), 0, \ldots, 0)$, thus $\|\alpha\| = k_1(m)$ and $E_\lambda(h_\alpha, h_\alpha)(z, t) = e^{-i\lambda t} e^{-|\lambda||z|^2/4} L^0_{k_1(m)}(|\lambda||z_1|^2/2)$. In order to see that $f_\varepsilon \in L^2(H_n)$, we set

$$F_\varepsilon(z, t) = \varphi_\varepsilon(\lambda) e^{-|\lambda||z|^2/4} L^0_{k_1(m)}(|\lambda||z_1|^2/2).$$

Then $f_\varepsilon(z, t) = F_\varepsilon(z, \widehat{t})$, where $F_\varepsilon(z, \widehat{t})$ denotes the Fourier transform with respect to the second variable evaluated at t. The Plancherel theorem in

$\mathbb{R}^{2n+1} \cong \mathbb{C}^n \times \mathbb{R}$ says that

$$\|f_\varepsilon\|_{L^2(H_n)} = \|F_\varepsilon(\widehat{\xi}, \lambda)\|_{L^2(\mathbb{R}^{2n+1}, d\xi\, d\lambda)}.$$

Now, taking into account that, for $\varepsilon < \frac{1}{2}\lambda_0^{-1}$, φ_ε has a compact support contained in $(0, \infty)$ and using the usual formulas for the euclidean Fourier transform of the product of a polynomial by a Gaussian function, we find that $f_\varepsilon \in L^2(H_n)$.

Moreover, by (2.1) and as $(L - m\lambda_0)f_\varepsilon = mg_\varepsilon$ and $(iT - \lambda_0 I)f_\varepsilon = g_\varepsilon$ with

$$g_\varepsilon(z,t) = \int_{-\infty}^{\infty} (\lambda - \lambda_0)\varphi_\varepsilon(\lambda) E_\lambda(h_\alpha, h_\alpha)(z,t)\, d\lambda, \tag{3.9}$$

we obtain $A(L - m\lambda_0 I)f_\varepsilon + B(iT - \lambda_0 I)f_\varepsilon = (mA + B)g_\varepsilon$ and so (3.8) gives $f_\varepsilon = (mA + B)g_\varepsilon$. Since φ_ε is an approximation to the identity, it follows that $\lim_{\varepsilon\to 0} f_\varepsilon(z,t) = E_{\lambda_0}(h_\alpha, h_\alpha)(z,t)$ for each $(z,t) \in H_n$. Now, Fatou's Lemma gives

$$\begin{aligned} \|E_{\lambda_0}(h_\alpha, h_\alpha)\|_{L^2(H_n)} &\le \liminf_{\varepsilon\to 0} \|f_\varepsilon\|_{L^2(H_n)} \\ &\le \|mA + B\|_{\mathrm{Op}} \liminf_{\varepsilon\to 0} \|g_\varepsilon\|_{L^2(H_n)}. \end{aligned} \tag{3.10}$$

In order to obtain a contradiction we note that $g_\varepsilon(z,t) = G_\varepsilon(z, \widehat{t}\,)$ with

$$G_\varepsilon(z,t) = (\lambda - \lambda_0)\varphi_\varepsilon(\lambda) e^{-|\lambda||z|^2/4} L^0_{k_1(m)}(|\lambda||z_1|^2/2). \tag{3.11}$$

Also,

$$\|g_\varepsilon\|_{L^2(H_n)} = \|G_\varepsilon(\widehat{\xi}, \lambda)\|_{L^2(\mathbb{R}^{2n+1}, d\xi\, d\lambda)}.$$

Since $\lim_{\varepsilon\to 0} \|(\lambda - \lambda_0)\varphi_\varepsilon(\lambda)\|_{L^1(\mathbb{R}, d\lambda)} = 0$, a computation shows that $\lim_{\varepsilon\to 0} \|g_\varepsilon\|_{L^2(H_n)} = 0$. Taking account of (3.10) we obtain a contradiction, since $\|E_{\lambda_0}(h_\alpha, h_\alpha)\|_{L^2(H_n)} = \infty$. This ends the proof for the case $\lambda_0 > 0$ and $k_1(m) \ge 0$. The argument is the same for the other cases with $\lambda_0 \ne 0$. The case $\lambda_0 = 0$ follows by closure. ■

Finally we state

THEOREM 3.12. *Let A be a bounded operator on $L^2(H_n)$ that commutes with ϱ. Then there exists $m : \mathbb{R} \times \mathbb{Z} \to \mathbb{C}$ such that for $f \in S(H_n)$,*

$$Af(x,y,t) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} m(\lambda, k)(f * S_{\lambda,k})(x,y,t)|\lambda|^n\, d\lambda$$

with $\|A\| = \|m\|_\infty$. Conversely, if m is a measurable and bounded function on the joint spectrum $\sigma(L, iT)$, then the above integral operator extends to a bounded operator on $L^2(H_n)$ that commutes with ϱ.

Proof. We consider, for $f \in S(H_n)$, the integral decomposition given by (2.3):

$$f(x,y,t) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} f * S_{\lambda,k}(x,y,t)|\lambda|^n \, d\lambda.$$

By the Schwartz kernel theorem we know that $Af = f * K$ for some $K \in S'(H_n)$. Since A commutes with the action ϱ, we see that K is an $U(p,q)$-invariant distribution. Also, by the properties of the Fourier transform, we have $\pi_\lambda(Af) = \pi_\lambda(f)K_\lambda$ for a.e. λ, where each K_λ is a bounded operator on $L^2(\mathbb{R}^n)$ (see [S], p. 571). Moreover $\operatorname{ess\,sup}_\lambda \|K_\lambda\| < \infty$. Since K_λ commutes with the metaplectic representation ω restricted to $SU(p,q)$ we deduce that $K_\lambda P_k$ is a multiple $m_{\lambda,k} I_k$ where I_k is the identity on $\mathcal{H}_k = P_k(L^2(\mathbb{R}^n))$. Indeed, we recall that, for $k \in \mathbb{Z}$, $\mathcal{H}_k$ is the closed subspace of $L^2(\mathbb{R}^n)$ spanned by $\{h_\alpha\}_{\|\alpha\|=k}$ and that each $(\mathcal{H}_k, \omega)$ is an irreducible $SU(p,q)$-module (see [B-W], Ch. VIII). Also, since $\operatorname{ess\,sup}_\lambda \|K_\lambda\| < \infty$, we have $m \in L^\infty(\sigma(L, iT))$. Thus it is immediate to see that

$$\int_{-\infty}^{\infty} \operatorname{tr}(\pi_\lambda(Af))\pi_\lambda(x,y,t)|\lambda|^n \, d\lambda < \infty$$

for $f \in S(H_n)$ and $(x,y,t) \in H_n$. From this, the inversion formula says that, for $f \in S(H_n)$,

$$Af(x,y,t) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} m(\lambda,k)(f * S_{\lambda,k})(x,y,t)|\lambda|^n \, d\lambda$$

with $\sup_{\lambda\in\mathbb{R}-\{0\},\, k\in\mathbb{Z}} |m(\lambda,k)| < \infty$. Conversely, if m is a measurable and bounded function on the joint spectrum $\sigma(L, iT)$, each operator of this form extends to a bounded operator on $L^2(H_n)$ that commutes with ϱ. ■

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