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VOLUME MEAN VALUES OF SUBTEMPERATURES

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Abstract. Several authors have found the characteristic mean value formula for temperatures over heat spheres. Those who derived a corresponding formula over heat balls have all chosen different mean values. In this paper we discuss an infinity of possible means over heat balls, and show that, in the wider context of subtemperatures, some are more desirable than others.

It is extremely well known that, if h is a harmonic function on a neighbourhood of a closed ball in $\mathbb{R}^n$, then the value of h at the centre of the ball is equal to both the average of h over the boundary of the ball and the average of h over the ball itself. Conversely, any continuous function on an open set which has either averaging property for every closed ball inside the set is harmonic. Averages over balls or their boundaries are also used to define subharmonic functions [3]. In that broader context they are very well behaved. If r is the radius, then either mean value is finite, an increasing function of r, a convex function of r^{2-n} or $\log(1/r)$ according to the value of n (cf. [6]), and there is constant κ such that the surface mean at radius κr is less than or equal to the volume mean at radius r (cf. [1]), which is less than or equal to the surface mean at radius r.

The corresponding situation for temperatures (solutions of the heat equation) and subtemperatures (the corresponding analogues of subharmonic functions) is much less well known and is not standardized. Several authors (Pini [5], Fulks [2], Smyrnélis [7], Kuptsov [4]) have shown how temperatures can be characterized in terms of weighted mean values over "heat spheres", which are smooth convex surfaces of the form

$$\{(x_0,t_0)\}\cup\left\{(y,s):\|x_0-y\|=\left(2n(t_0-s)\log\frac{c}{t_0-s}\right)^{1/2},\ t_0-c\le s<t_0\right\}$$

in $\mathbb{R}^{n+1}=\mathbb{R}^n\times\mathbb{R}$, where (x_0,t_0) and $c>0$ are given. By analogy with the harmonic case, (x_0,t_0) is called the "centre" and c the "radius" of the heat sphere. These authors all obtained the same formula, but not in the same

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form. Mean values over "heat balls", which are the domains

$$\Omega(x_0,t_0;c)=\left\{(y,s):\|x_0-y\|<\left(2n(t_0-s)\log\frac{c}{t_0-s}\right)^{1/2}\right\}$$

enclosed by heat spheres, have also been considered in several papers (Pini [5], Smyrnélis [7], Watson [8], Kuptsov [4], Watson [10]). However, all but the last two used different mean values. In this paper we study an uncountable family of possible volume means, and show that some are more desirable than others in the wider context of subtemperatures, because they have direct analogues of all the above-mentioned properties of the mean values of subharmonic functions over balls.

We need to establish more notation. For $x\in\mathbb{R}^n$ and $t>0$, we put

$$Q(x,t)=\|x\|^2(4\|x\|^2t^2+(\|x\|^2-2nt)^2)^{-1/2};$$

we also put $Q(0,0)=1$. The restriction to the heat sphere $\partial\Omega(x_0,t_0;c)$ of the function $(x,t)\mapsto Q(x_0-x,t_0-t)$ is continuous, and is positive except for a zero at $(0,c)$. If $\tau(c)=(4\pi c)^{-n/2}$ and σ denotes surface area, we put

$$\mathcal{M}(c)=\mathcal{M}(u;x_0,t_0;c)=\tau(c)\int\limits_{\partial\Omega(x_0,t_0;c)}Q(x_0-x,t_0-t)u(x,t)\,d\sigma$$

for any function u on $\partial\Omega(c)$ such that the integral exists. This is the surface mean that characterizes temperatures, in that a continuous function u on an open set E is a temperature if and only if $u(x_0,t_0)=\mathcal{M}(u;x_0,t_0;c)$ whenever $\overline{\Omega}(x_0,t_0;c)\subseteq E$.

Integrating this surface mean to obtain a volume mean, we have uncountably many possibilities to choose from. Perhaps the most natural ones take the form

$$\mathcal{V}_\beta(c)=\mathcal{V}_\beta(u;x_0,t_0;c)=\beta c^{-\beta}\int_0^c r^{\beta-1}\mathcal{M}(u;x_0,t_0;r)\,dr, \tag{1}$$

with $\beta>0$. In this notation, Pini (1954, case $n=1$) chose $\mathcal{V}_1(c^2)$, Smyrnélis (1969) chose $\mathcal{V}_{n(n+1)/2}(c^{2/n}/4\pi)$, Watson (1973) chose $\mathcal{V}_{n/2}(c)$, and Kuptsov (1981) chose $\mathcal{V}_{(n/2)+1}(c^2)$. Subsequently Watson (1990), while attempting to prove that volume means of subtemperatures can be as well behaved as those of subharmonic functions over balls, switched to $\mathcal{V}_{(n/2)+1}(c)$ because $\mathcal{V}_{n/2}(c)$ "is not the easiest to handle". We prove below that $\mathcal{V}_{n/2}$ does not have all the desired properties.

We can express the means $\mathcal{V}_\beta$ in rectangular coordinates using the function J, defined for all $x\in\mathbb{R}^n$ and $t>0$ by

$$J(x,t)=2nt\exp\left(-\frac{\|x\|^2}{2nt}\right)(4\|x\|^2t^2+(\|x\|^2-2nt)^2)^{-1/2}.$$

By [8, Lemma 3],

$$\int_0^c \Big(\int_{\partial\Omega(r)} f(y,s) J(x_0 - y, t_0 - s)\, d\sigma \Big) dr = \iint_{\Omega(c)} f(y,s)\, dy\, ds,$$

so that if $\alpha = \beta 2^{-n-1} n^{-1} \pi^{-n/2}$ then

$$\mathcal{V}_\beta(c) = \alpha c^{-\beta} \iint_{\Omega(c)} \frac{\|x_0 - y\|^2}{(t_0 - s)^{(n+4-2\beta)/2}} \times \exp\left(\frac{(2\beta - n)\|x_0 - y\|^2}{4n(t_0 - s)} \right) u(y,s)\, dy\, ds.$$

Thus the choice $\beta = n/2$ gives the simplest kernel in rectangular coordinates, while the choice $\beta = (n/2) + 1$ gives the kernel with the parabolic homogeneity associated with the heat equation, and makes $c^{-\beta}$ inversely proportional to the volume of $\Omega(c)$.

Subtemperatures were defined in terms of the means $\mathcal{M}$ in [8]. We give an equivalent formulation. If u is upper semicontinuous on an open set E, never takes the value ∞, is finite on a dense subset of E, and satisfies

$$u(x_0, t_0) \le \mathcal{M}(u; x_0, t_0; c)$$

whenever $\overline{\Omega}(x_0, t_0; c) \subseteq E$, then u is a *subtemperature*. It was proved in [8] that $\mathcal{M}$ can be replaced by $\mathcal{V}_{n/2}$. In fact, any $\mathcal{V}_\beta$ could be used.

In [11, p. 54], Watson proved that, whenever u is a subtemperature on a neighbourhood of (x_0, t_0), we have

$$r^{n/2} \mathcal{M}(u; x_0, t_0; r) \to 0 \quad \text{as } r \to 0.$$

Therefore $\mathcal{V}_\beta(c)$ is always finite if $\beta > n/2$. We show that $\mathcal{V}_\beta(c)$ may be infinite if $0 < \beta \le n/2$. Since $\mathcal{M}$ is an increasing function (see [8]), it follows from (1) that $\mathcal{V}_\beta(u; x_0, t_0; c)$ is either finite for all c, or is $-\infty$ for all c.

Example. Given (x_0, t_0), put $d = e^{-e}$, and let μ be the measure concentrated on $\{x_0\} \times [t_0 - d, t_0]$ with density

$$f(s) \;=\; \frac{1}{s \log(1/s)(\log\log(1/s))^2},$$

where $s = t_0 - t$. Then

$$\mu(\Omega(r)) = \int_0^r f(s)\, ds = \frac{1}{\log\log(1/r)}$$

whenever $0 < r \le d$. In particular, $\mu(\mathbb{R}^{n+1}) = 1$, so that the potential

$$u(x,t) = -\iint_{\mathbb{R}^{n+1}} \tau(t-s) \exp\Big(-\frac{\|x-y\|^2}{4(t-s)} \Big) d\mu(y,s)$$

is a subtemperature on $\mathbb{R}^{n+1}$, by [9, Theorem 16]. It therefore follows from [11, Theorem 1] that

$$\mathcal{M}(r) = \mathcal{M}(d) + \int_r^d \tau'(s)\mu(\Omega(s))\,ds$$

whenever $0 < r \le d$. Writing $\lambda_n = n^2 2^{-n-2}\pi^{-n/2}$, we have $\tau'(s) = -2n^{-1}\lambda_n s^{-(n/2)-1}$ and hence

$$\mathcal{V}_{n/2}(c) = \mathcal{M}(d) - \lambda_n c^{-n/2}\int_0^c r^{(n/2)-1}\Big(\int_r^d s^{-(n/2)-1}\mu(\Omega(s))\,ds\Big)dr$$

whenever $0 < c \le d$. By [10, Theorem 2], $\mathcal{M}(d)$ is finite. (In fact, using [10, Example] and Tonelli's theorem, we can show that $\mathcal{M}(d) = \tau(d)$.) We therefore need only prove that the iterated integral is infinite. Its value is at least

$$\begin{aligned}\int_0^c r^{(n/2)-1}\Big(\int_r^c s^{-(n/2)-1}\mu(\Omega(s))\,ds\Big)dr \\ &= \int_0^c s^{-(n/2)-1}\mu(\Omega(s))\Big(\int_0^s r^{(n/2)-1}dr\Big)ds \\ &= \frac{2}{n}\int_0^c \frac{1}{\log\log(1/s)}\cdot\frac{ds}{s} = \frac{2}{n}\int_{\log\log(1/c)}^{\infty} s^{-1}e^s\,ds = \infty.\end{aligned}$$

Hence $\mathcal{V}_{n/2}(c) = -\infty$ for all $c > 0$. It follows from (1) that $\mathcal{V}_\beta(c) = -\infty$ for any $\beta < n/2$.

General behaviour of the means $\mathcal{V}_\beta$. The good behaviour of $\mathcal{V}_{(n/2)+1}$ was established in [10], and the methods used there can also be applied to $\mathcal{V}_\beta$ for $\beta > (n/2)+1$, but not for $\beta < (n/2)+1$. However, if $n/2 < \beta < (n/2)+1$, then $\mathcal{V}_\beta$ still has all the desirable properties of $\mathcal{V}_{(n/2)+1}$. Furthermore, given any $\beta > 0$ and any point (x_0, t_0) such that $\mathcal{V}_\beta(u; x_0, t_0; c)$ is finite for all c, we can show that $\mathcal{V}_\beta$ is still an increasing function of c and a convex function of $\tau(c)$.

THEOREM. *Let $\beta > 0$, let u be a subtemperature on an open superset of $\overline{\Omega}(x_0, t_0; d)$, and suppose that $\mathcal{V}_\beta(u; x_0, t_0; c)$ is finite. Then*

(i) *$\mathcal{V}_\beta$ is increasing on $]0, d]$,*

(ii) *there is a convex function ψ_β such that $\mathcal{V}_\beta(c) = \psi_\beta(\tau(c))$ whenever $0 < c \le d$, and*

(iii) *$\mathcal{V}_\beta(c) \le \mathcal{M}(c)$.*

Furthermore, if $\beta > n/2$ *and* $\kappa_\beta = ((2\beta - n)/2\beta)^{2/n}$, *then*

$$\mathcal{M}(\kappa_\beta c) \le \mathcal{V}_\beta(c) \tag{2}$$

for all $c \in]0, d]$, *and no larger constant has the same property.*

P r o o f. (i) If $0 < b < c \le d$, then

$$\mathcal{V}_\beta(c) - \mathcal{V}_\beta(b) = \beta c^{-\beta} \int_0^c r^{\beta-1}(\mathcal{M}(r) - \mathcal{M}(br/c))\, dr \ge 0$$

because $\mathcal{M}$ is real-valued ([10, Theorem 2]) and increasing ([8, Theorem 12]).

(ii) Note that, if a, b, c, r are positive, then

$$\tau(b) - \tau(a) = \left(\frac{r}{c}\right)^{n/2}\left(\tau\left(\frac{br}{c}\right) - \tau\left(\frac{ar}{c}\right)\right).$$

Therefore, if $0 < a < b < c \le d$, then

$$\begin{aligned}
&\frac{\mathcal{V}_\beta(c) - \mathcal{V}_\beta(b)}{\tau(c) - \tau(b)} - \frac{\mathcal{V}_\beta(b) - \mathcal{V}_\beta(a)}{\tau(b) - \tau(a)} \\
&\quad = \beta c^{-\beta} \int_0^c \left(\frac{\mathcal{M}(r) - \mathcal{M}(br/c)}{\tau(c) - \tau(b)} - \frac{\mathcal{M}(br/c) - \mathcal{M}(ar/c)}{\tau(b) - \tau(a)}\right) r^{\beta-1}\, dr \\
&\quad = \beta c^{(n/2)-\beta} \int_0^c \left(\frac{\mathcal{M}(r) - \mathcal{M}(br/c)}{\tau(r) - \tau(br/c)} - \frac{\mathcal{M}(br/c) - \mathcal{M}(ar/c)}{\tau(br/c) - \tau(ar/c)}\right) r^{\beta-(n/2)-1}\, dr \\
&\quad \le 0
\end{aligned}$$

because $\mathcal{M}$ is a convex function of τ, by [10, Theorem 2].

(iii) Since $\mathcal{M}$ is increasing, this result follows from (1).

To prove (2), we use the convexity property of $\mathcal{M}$ and Jensen's inequality, as in the proof of [10, Theorem 3(iii)]. Thus, if $\mathcal{M} = \psi \circ \tau$ then

$$\begin{aligned}
\mathcal{V}_\beta(c) &= \beta c^{-\beta} \int_0^c r^{\beta-1} \psi(\tau(r))\, dr \\
&\ge \psi\Big(\beta c^{-\beta} \int_0^c r^{\beta-1} \tau(r)\, dr\Big) = \psi(\tau(\kappa_\beta c)) = \mathcal{M}(\kappa_\beta c).
\end{aligned}$$

To prove that κ_β is the largest such constant, choose a point (x_1, t_1) such that $t_1 < t_0$, and take

$$u(x,t) = \begin{cases} \tau(t - t_1)\exp(-\|x - x_1\|^2/(4(t - t_1))) & \text{if } t > t_1, \\ 0 & \text{if } t \le t_1. \end{cases}$$

Then $-u$ is a subtemperature on $\mathbb{R}^{n+1}$. Choose c_0 such that $u(x_0, t_0) =$

$\tau(c_0)$. By the example in [10],

$$\mathcal{M}(u; x_0, t_0; r) = \min\{u(x_0, t_0), \tau(r)\} = \begin{cases} u(x_0, t_0) & \text{if } 0 < r < c_0, \\ \tau(r) & \text{if } r > c_0. \end{cases}$$

Therefore, whenever $c > c_0/\kappa_\beta$,

$$\begin{aligned} \mathcal{V}_\beta(c) &= \beta c^{-\beta}\Big(\int_0^{c_0} + \int_{c_0}^{c}\Big) r^{\beta-1}\mathcal{M}(r)\,dr \\ &= u(x_0, t_0)(c_0/c)^\beta + \tau(\kappa_\beta)c^{-\beta}(c^{\beta-(n/2)} - c_0^{\beta-(n/2)}) \\ &= (u(x_0, t_0) - \tau(\kappa_\beta c_0))(c_0/c)^\beta + \tau(\kappa_\beta c) \\ &= (u(x_0, t_0) - \tau(\kappa_\beta c_0))(c_0/c)^\beta + \mathcal{M}(\kappa_\beta c). \end{aligned}$$

Since $\beta > n/2$, it follows that $\mathcal{V}_\beta(c) \sim \mathcal{M}(\kappa_\beta c)$ as $c \to \infty$, which proves the assertion.

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