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A DUALITY PRINCIPLE FOR STATIONARY RANDOM SEQUENCES

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Abstract. The paper is devoted to the study of stationary random sequences. A concept of dual sequences is discussed. The main aim of the paper is to establish a relationship between the errors of linear least squares predictions for sequences and their duals.

1. Preliminaries and notation. This paper is organized as follows. Section 1 collects together some basic facts and notation concerning stationary random sequences needed in what follows. In Section 2 a concept of dual sequences is discussed. In the last section a relationship between the errors of linear least squares predictions for sequences and their duals is established.

We suppose, as usual, that there is a probability measure defined on a σ-algebra of sets of some space Ω. Let M be the set of all complex-valued random variables whose squares are integrable. The set M is a Hilbert space under the inner product $(X,Y) = EX\overline{Y}$ where E stands for the expectation of random variables. Throughout this paper $\mathbb{Z}$ will denote the set of all integers.

A sequence $\mathbf{X} = \{X_n\}$ $(n \in \mathbb{Z})$ of random variables from M is said to be *stationary* if the inner product (X_{n+m}, X_m) does not depend on m. The function $R(n) = (X_{n+m}, X_m)$ $(n \in \mathbb{Z})$ is called the *covariance function* of the sequence in question. The Herglotz Theorem describes the covariance function as a Fourier transform

$$R(n) = \int_{-\pi}^{\pi} e^{inx}\,\mu(dx)$$

where the measure μ is concentrated on the interval $[-\pi,\pi)$. Of course, the correspondence $R \leftrightarrow \mu$ is one-to-one ([1], Chapter 10.3). The measure μ is called the *spectral measure* of the sequence $\mathbf{X}$.

For the empty set $\emptyset$ we put $[\mathbf{X},\emptyset] = \{0\}$. For a non-empty subset Q of $\mathbb{Z}$ we denote by $[X,Q]$ the closed linear manifold of M generated by the random variables X_n with $n \in Q$. For the sake of brevity we put $[\mathbf{X},\mathbb{Z}] = [\mathbf{X}]$.

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Each stationary sequence $\mathbf{X} = \{X_n\}$ $(n \in \mathbb{Z})$ induces a unitary operator T on $[\mathbf{X}]$ satisfying the condition $TX_n = X_{n+1}$ $(n \in \mathbb{Z})$ ([1], Chapter 10.1).

In what follows we shall use the notation

$$A_n = \{k : k < n\} \quad \text{and} \quad B_n = \{k : k > n\} \qquad (n \in \mathbb{Z}).$$

Two stationary sequences $\mathbf{X}$ and $\mathbf{Y}$ are said to be *retrospectively* or *progressively connected* if $[\mathbf{X}, A_n] = [\mathbf{Y}, A_n]$ for all $n \in \mathbb{Z}$ or $[\mathbf{X}, B_n] = [\mathbf{Y}, B_n]$ for all $n \in \mathbb{Z}$ respectively.

A stationary sequence $\mathbf{X}$ is called *deterministic* if

$$[\mathbf{X}, A_n] = [\mathbf{X}] \qquad (n \in \mathbb{Z}). \tag{1.1}$$

A stationary sequence $\mathbf{X}$ is called *completely non-deterministic* if

$$[\mathbf{X}] \neq \{0\} \quad \text{and} \quad \bigcap_{n \in \mathbb{Z}} [\mathbf{X}, A_n] = \{0\}.$$

Each non-deterministic stationary sequence $\mathbf{X}$ has a unique Wold decomposition $\mathbf{X} = \mathbf{X}' + \mathbf{X}''$ into two stationary sequences $\mathbf{X}'$ and $\mathbf{X}''$ where $\mathbf{X}'$ is completely non-deterministic, $\mathbf{X}''$ is deterministic,

$$[\mathbf{X}'] \perp [\mathbf{X}''] \tag{1.2}$$

and $[\mathbf{X}]$ is the orthogonal sum of $[\mathbf{X}']$ and $[\mathbf{X}'']$,

$$[\mathbf{X}] = [\mathbf{X}'] \oplus [\mathbf{X}''] \tag{1.3}$$

([1], Chapter 12.4). Moreover,

$$[\mathbf{X}, A_n] = [\mathbf{X}', A_n] \oplus [\mathbf{X}''] \qquad (n \in \mathbb{Z}), \tag{1.4}$$

which yields the equality

$$[\mathbf{X}, S] = [\mathbf{X}', S] \oplus [\mathbf{X}''] \tag{1.5}$$

whenever $S \supset A_k$ for some $k \in \mathbb{Z}$. For each subset Q of $\mathbb{Z}$ we have the inclusion

$$[\mathbf{X}, Q] \subset [\mathbf{X}', Q] \oplus [\mathbf{X}'', Q]. \tag{1.6}$$

The Hardy class H_p $(p > 0)$ consists of functions f analytic on $|z| < 1$ and satisfying the condition

$$\lim_{r \to 1-} \int_{-\pi}^{\pi} |f(re^{ix})|^p \, dx < \infty.$$

It is well known that for $f \in H_p$ the radial limit

$$\lim_{r \to 1-} f(re^{ix}) = f(e^{ix})$$

exists almost everywhere ([4], Chapter 2.2). By H_p^+ we denote the subset of H_p consisting of functions f satisfying the conditions $f(0) > 0$ and $f(z) \neq 0$ for $|z| < 1$.

In what follows δ_n will denote the Kronecker δ-function: $\delta_0 = 1$ and $\delta_n = 0$ for $n \neq 0$. A sequence $\mathbf{U} = \{U_n\}$ $(n \in \mathbb{Z})$ of random variables from M is called *orthonormal* if δ_n is its covariance function.

The main representation theorem says that each completely non-deterministic sequence $\mathbf{X}$ has a unique representation $\mathbf{X} = (F, \mathbf{U})$ where $F \in H_2^+$ and the orthonormal sequence $\mathbf{U}$ and the sequence $\mathbf{X}$ are retrospectively connected. This means that $\mathbf{X}$ is the moving average

$$X_n = \sum_{k=0}^{\infty} a_k U_{n-k} \qquad (n \in \mathbb{Z})$$

where $F(z) = \sum_{k=0}^{\infty} a_k z^k$ for $|z| < 1$ ([1], Chapter 12.4). Moreover, the spectral measure μ of $\mathbf{X}$ is of the form

$$\mu(dx) = \frac{1}{2\pi} |F(e^{-ix})|^2 dx.$$

2. Dual sequences. Let $\mathbf{X} = \{X_n\}$ $(n \in \mathbb{Z})$ be a stationary sequence. A sequence $\mathbf{X}^* = \{X_n^*\}$ $(n \in \mathbb{Z})$ of random variables from $[\mathbf{X}]$ is called the *dual* of $\mathbf{X}$ if

$$(X_n, X_m^*) = \delta_{n-m} \qquad (n, m \in \mathbb{Z}). \tag{2.1}$$

It is clear that the dual sequence is uniquely determined provided it exists. Thus, taking the unitary operator T induced by $\mathbf{X}$ on $[\mathbf{X}]$, we conclude that for every $r \in \mathbb{Z}$ the sequence $\{T^r X_{n-r}^*\}$ $(n \in \mathbb{Z})$ is also the dual of $\mathbf{X}$ and, consequently, $X_n^* = T^r X_{n-r}^*$ $(n, r \in \mathbb{Z})$. This shows that the sequence $\mathbf{X}^*$ is also stationary.

EXAMPLE 2.1. For orthonormal sequences $\mathbf{U}$ we have $\mathbf{U}^* = \mathbf{U}$.

EXAMPLE 2.2. Let $\mathbf{X}$ be a stationary Markov sequence with covariance function $R(n) = a^n R(0)$ where $n \geq 0$, $R(0) > 0$ and $|a| < 1$ ([1], p. 477). Then we have

$$X_n^* = (1 - |a|^2)^{-1}((1 + |a|^2)X_n - aX_{n-1} - \overline{a}X_{n+1}) \qquad (n \in \mathbb{Z}).$$

In what follows K will stand for the set of all stationary sequences admitting the dual sequence. Further, K_0 will denote the subset of K consisting of sequences $\mathbf{X}$ satisfying $[\mathbf{X}^*] = [\mathbf{X}]$. The following statement is evident.

PROPOSITION 2.1. *If* $\mathbf{X} \in K_0$, *then* $\mathbf{X}^* \in K_0$ *and* $(\mathbf{X}^*)^* = \mathbf{X}$.

PROPOSITION 2.2. $\mathbf{X} \in K$ *if and only if* $\mathbf{X}$ *is non-deterministic and* $\mathbf{X}' \in K$. *Then the formula* $(\mathbf{X}')^* = \mathbf{X}^*$ *is true.*

Proof. Let $\mathbf{X} \in K$. First we shall prove that the sequence $\mathbf{X}$ is non-deterministic. Suppose the contrary. Since $X_0^* \perp [\mathbf{X}, A_0]$ we have, by (1.1), $X_0^* = 0$, which contradicts the equality $(X_0, X_0^*) = 1$. Thus the sequence $\mathbf{X}$

is non-deterministic. Consider its Wold decomposition $\mathbf{X} = \mathbf{X}' + \mathbf{X}''$. Since $X_m^* \perp [\mathbf{X}, A_n]$ for $n < m$ we have, by (1.4),

$$X_m^* \perp \bigcap_{n\in\mathbb{Z}} [\mathbf{X}, A_n] = [\mathbf{X}''] \quad (m \in \mathbb{Z}).$$

Consequently, by (1.2), $X_m^* \in [\mathbf{X}']$ $(m \in \mathbb{Z})$ and $(X_n', X_m^*) = (X_n, X_m^*) = \delta_{n-m}$, which shows that $\mathbf{X}^*$ is the dual of $\mathbf{X}'$.

Conversely, suppose that $\mathbf{X}' \in K$. Then, by (1.3), $(X_m')^* \in [\mathbf{X}]$ and $(X_m')^* \perp [\mathbf{X}'']$. Consequently, $(X_n, (X_m')^*) = (X_n', (X_m')^*) = \delta_{n-m}$ $(n, m \in \mathbb{Z})$, which yields $\mathbf{X} \in K$. This completes the proof.

The next result is less trivial.

PROPOSITION 2.3. *Let X be a completely non-deterministic sequence with the representation $(F, \mathbf{U})$ such that $F^{-1} \in H_2$. Then $\mathbf{X} \in K_0$,*

$$X_n^* = \sum_{k=0}^{\infty} \overline{b}_k U_{k+n}, \tag{2.2}$$

with the coefficients b_k determined by the expansion $F^{-1}(z) = \sum_{k=0}^{\infty} b_k z^k$ for $|z| < 1$ and the sequences $\mathbf{X}^*$ and $\mathbf{U}$ are progressively connected.

Proof. Observe that $\sum_{k=0}^{\infty} |b_k|^2 < \infty$. This shows that the right-hand side of (2.2) is well defined. Denote it by Y_n. It is clear that

$$Y_n \in [\mathbf{U}, B_{n-1}] \subset [\mathbf{X}] \quad (n \in \mathbb{Z}),$$

which yields the relation $Y_n \perp [\mathbf{U}, A_n]$ $(n \in \mathbb{Z})$. Since the sequences $\mathbf{X}$ and $\mathbf{U}$ are retrospectively connected the last relation implies the equalities $(X_k, Y_n) = 0$ if $k < n$. Further, if $k = n + r$ and $r \geq 0$, then

$$(X_k, Y_n) = \sum_{s,j=0}^{\infty} a_j b_s (U_{n+r-j}, U_{n+s}) = \sum_{j=0}^{r} a_j b_{r-j} = \delta_r,$$

which shows that the sequence $\{Y_n\}$ $(n \in \mathbb{Z})$ is the dual of $\mathbf{X}$. Formula (2.2) and the relation $\mathbf{X} \in K$ are thus proved.

Now we shall prove that the sequences $\mathbf{X}^*$ and $\mathbf{U}$ are progressively connected. By formula (2.2) we have the inclusion

$$[\mathbf{X}^*, B_0] \subset [\mathbf{U}, B_0]. \tag{2.3}$$

To prove the reverse inclusion we suppose that a random variable Y satisfies

$$Y \perp [\mathbf{X}^*, B_0] \tag{2.4}$$

and belongs to $[\mathbf{U}, B_0]$. Consequently, it can be written in the form

$$Y = \sum_{k=1}^{\infty} c_k U_k \tag{2.5}$$

where $\sum_{k=1}^{\infty} |c_k|^2 < \infty$. From (2.4) we get the equalities

$$(2.6) \qquad (Y, X_n) = \sum_{s=0}^{\infty} \sum_{k=1}^{\infty} c_k b_s \delta_{k-s-n} = 0 \quad (n \geq 1).$$

Introduce the notation

$$(2.7) \quad f(z) = \sum_{k=0}^{\infty} \overline{a}_k z^k, \quad g(z) = \sum_{k=0}^{\infty} \overline{b}_k z^k, \quad h(z) = \sum_{k=1}^{\infty} c_k z^k \quad (|z| < 1).$$

It is clear that

$$(2.8) \qquad f, g, h \in H_2$$

and

$$(2.9) \qquad f(z)g(z) = 1 \quad (|z| < 1).$$

By Parseval's formula and (2.6) we get the equalities

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{ix}) \overline{g}(e^{ix}) e^{-inx} \, dx = \sum_{s=0}^{\infty} \sum_{k=1}^{\infty} c_k b_s \delta_{k-s-n} = 0 \quad (n \geq 1).$$

Consequently, the function $\overline{h}(e^{ix}) g(e^{ix})$ integrable on the interval $[-\pi, \pi]$ has the Fourier expansion of the form $\sum_{k=0}^{\infty} p_k e^{ikx}$. Setting $p(z) = \sum_{k=0}^{\infty} p_k z^k$ $(|z| < 1)$ we infer, by Theorem 6.1 in [1], Chapter 4, that

$$(2.10) \qquad p \in H_1$$

and

$$(2.11) \qquad p(e^{ix}) = \overline{h}(e^{ix}) g(e^{ix})$$

almost everywhere. Put $q(z) = p(z) f(z)$ $(|z| < 1)$. Taking into account (2.8), (2.10) and the inequality

$$|q(z)|^{1/2} \leq |p(z)| + |f(z)|$$

we conclude that $q \in H_{1/2}$ and, by (2.9) and (2.11),

$$(2.12) \qquad q(e^{ix}) = \overline{h}(e^{ix})$$

almost everywhere. Thus, by (2.8), the radial limit $q(e^{ix})$ is square integrable on the interval $[-\pi, \pi]$. Applying Smirnov's theorem ([4], p. 116) we have $q \in H_2$. Consequently, from (2.12) it follows that

$$h(e^{ix}) = \sum_{k=0}^{\infty} d_k e^{-ikx}$$

for some coefficients d_k with $\sum_{k=0}^{\infty} |d_k|^2 < \infty$. Comparing this with (2.7) we have $c_k = 0$ for $k \geq 1$, which, by (2.6), yields $Y = 0$. This completes the proof of the inclusion $[\mathbf{X}^*, B_0] \supset [\mathbf{U}, B_0]$, which together with (2.3) yields the equality $[\mathbf{X}^*, B_0] = [\mathbf{U}, B_0]$. Since

$$[\mathbf{X}^*, B_n] = T^n [\mathbf{X}^*, B_0], \quad [\mathbf{U}, B_n] = T^n [\mathbf{U}, B_0] \quad (n \in \mathbb{Z}),$$

where T is the unitary operator induced by the sequence $\mathbf{X}$ on $[\mathbf{X}]$, we have

$$[\mathbf{X}^*, B_n] = [\mathbf{U}, B_n] \quad (n \in \mathbb{Z}).$$

In other words, the sequences $\mathbf{X}^*$ and $\mathbf{U}$ are progressively connected. Hence in particular it follows that $[\mathbf{X}^*] = [\mathbf{U}]$. On the other hand, $[\mathbf{X}] = [\mathbf{U}]$ because the sequences X and U are retrospectively connected. Thus $[\mathbf{X}^*] = [\mathbf{X}]$ and, consequently, $\mathbf{X} \in K_0$, which completes the proof.

We are now in a position to prove a characterization of the class K. In what follows we shall use the notation $C_n = A_{-n} \cup B_n$ $(n \geq 0)$.

THEOREM 2.1. *The following conditions are equivalent*:

(i) $\mathbf{X} \in K$,
(ii) $[\mathbf{X}, C_0] \neq [\mathbf{X}]$,
(iii) $\mathbf{X}$ *is non-deterministic and* $\mathbf{X}' = (F, \mathbf{U})$ *with* $F^{-1} \in H_2$,
(iv) $\mathbf{X}$ *is non-deterministic and* $\mathbf{X}' \in K_0$.

Proof. (i)⇒(ii). Since $X_0^* \neq 0$ and $X_0^* \perp [\mathbf{X}, C_0]$ we have condition (ii).

(ii)⇒(iii). Condition (ii) and equalities (1.3) and (1.5) yield the condition $[\mathbf{X}', C_0] \neq [\mathbf{X}']$. Taking the representation $\mathbf{X}' = (F, \mathbf{U})$ we have, by Kolmogorov's Theorem ([5], Chapter 2, Theorem 10.2), $\int_{-\pi}^{\pi} |F(e^{-ix})|^{-2}\, dx < \infty$. Since $F(z) \neq 0$ for $|z| < 1$ we have $F^{-1} \in H_2$.

(iii)⇒(iv) and (iv)⇒(i) are immediate consequences of Propositions 2.3 and 2.2 respectively. The theorem is thus proved.

Given $Q \subset \mathbb{Z}$ we denote by Q^{c} the complement $\mathbb{Z} \backslash Q$. Let $\mathbf{X} \in K$. Since $(X_n, X_m^*) = 0$ for $n \in Q$ and $m \in Q^{\mathrm{c}}$, we have the inclusion

$$[\mathbf{X}, Q] \subset [\mathbf{X}^*, Q^{\mathrm{c}}]^{\perp} \tag{2.13}$$

where the orthogonal complement is taken in the space $[\mathbf{X}]$. We shall denote by $\Lambda(\mathbf{X})$ the family of all subsets Q of $\mathbb{Z}$ satisfying

$$[\mathbf{X}, Q] = [\mathbf{X}^*, Q^{\mathrm{c}}]^{\perp}.$$

Since $[\mathbf{X}, \mathbb{Z}] = [\mathbf{X}]$ and $[\mathbf{X}^*, \emptyset] = \{0\}$ we conclude that $\mathbb{Z} \in \Lambda(\mathbf{X})$ for every $\mathbf{X} \in K$.

PROPOSITION 2.4. *Let* $\mathbf{X} \in K$. *Suppose that* Q *and* S *are disjoint subsets of* $\mathbb{Z}$ *and the set* Q *is finite. Then* $S \cup Q \in \Lambda(\mathbf{X})$ *if and only if* $S \in \Lambda(\mathbf{X})$.

Proof. Suppose that

$$V \in [\mathbf{X}, Q] \cap [\mathbf{X}^*, S^{\mathrm{c}}]^{\perp}.$$

The random variable V can be written in the form $V = \sum_{n\in Q} c_n X_n$ where c_n $(n \in Q)$ are complex numbers. Since $Q \subset S^{\mathrm{c}}$ the random variable $V_0 = \sum_{n \in Q} c_n X_n^*$ belongs to $[\mathbf{X}^*, S^{\mathrm{c}}]$. Consequently, $0 = (V, V_0) = \sum_{n\in Q} |c_n|^2$, which yields the equality $V = 0$. Thus we have the formula

$$[\mathbf{X}, Q] \cap [\mathbf{X}^*, S^{\mathrm{c}}]^{\perp} = \{0\}. \tag{2.14}$$

Further, taking into account (2.13), we get $[\mathbf{X}, Q] \cap [\mathbf{X}, S] = \{0\}$. Since the subspace $[\mathbf{X}, Q]$ is finite-dimensional we conclude that the subspace $[\mathbf{X}, S \cup Q]$ can be represented as a direct sum

$$[\mathbf{X}, S \cup Q] = [\mathbf{X}, S] + [\mathbf{X}, Q]. \tag{2.15}$$

Using (2.13) we get the inclusion

$$[\mathbf{X}, Q] + [\mathbf{X}^*, S^{\mathrm{c}}]^{\perp} \subset [\mathbf{X}^*, S^{\mathrm{c}} \cap Q^{\mathrm{c}}]^{\perp}. \tag{2.16}$$

To prove the reverse inclusion we assume that

$$W \in [\mathbf{X}^*, S^{\mathrm{c}} \cap Q^{\mathrm{c}}]^{\perp}.$$

Setting

$$W_Q = \sum_{n \in Q} (W, X_n^*) X_n$$

we have the relations $W_Q \in [\mathbf{X}, Q]$, $W_Q \perp [\mathbf{X}^*, Q^{\mathrm{c}}]$ and $W - W_Q \perp [\mathbf{X}^*, Q]$. Moreover, by the formula $S^{\mathrm{c}} = Q \cup (S^{\mathrm{c}} \cap Q^{\mathrm{c}})$, we have $W - W_Q \perp [\mathbf{X}^*, S^{\mathrm{c}}]$. Consequently, $W \in [\mathbf{X}, Q] + [\mathbf{X}^*, S^{\mathrm{c}}]^{\perp}$, which, by (2.16), yields

$$[\mathbf{X}^*, S^{\mathrm{c}} \cap Q^{\mathrm{c}}]^{\perp} = [\mathbf{X}^*, S^{\mathrm{c}}]^{\perp} + [\mathbf{X}, Q].$$

Comparing this with (2.14) and (2.15) we conclude that $[\mathbf{X}, S] = [\mathbf{X}^*, S^{\mathrm{c}}]^{\perp}$ if and only if $[\mathbf{X}, S \cup Q] = [\mathbf{X}^*, S^{\mathrm{c}} \cap Q^{\mathrm{c}}]^{\perp}$, which completes the proof.

PROPOSITION 2.5. *If $\mathbf{X} \in K_0$ and $Q \in \Lambda(\mathbf{X}^*)$, then $Q^{\mathrm{c}} \in \Lambda(\mathbf{X}^*)$.*

Proof. First observe that the complementations in $[\mathbf{X}]$ and $[\mathbf{X}^*]$ coincide. Now our assertion is a consequence of Proposition 2.1 and the formula

$$[\mathbf{X}^*, Q^{\mathrm{c}}] = [\mathbf{X}, Q]^{\perp} = [(\mathbf{X}^*)^*, (Q^{\mathrm{c}})^{\mathrm{c}}]^{\perp}.$$

PROPOSITION 2.6. *If $\mathbf{X} \in K_0$, then $A_n, B_n \in \Lambda(\mathbf{X})$ for all $n \in \mathbb{Z}$.*

Proof. Let $\mathbf{X} \in K_0$. Then, by Proposition 2.2, the sequence $\mathbf{X}$ is competely non-deterministic and, consequently, has a representation $\mathbf{X} = (F, \mathbf{U})$, where the sequences $\mathbf{X}$ and $\mathbf{U}$ are retrospectively connected. Further, by Proposition 2.4, the sequences $\mathbf{X}^*$ and $\mathbf{U}$ are progressively connected. Hence we get the equalities

$$[\mathbf{X}, A_n] = [\mathbf{U}, A_n] = [\mathbf{U}, A_n^{\mathrm{c}}]^{\perp} = [\mathbf{X}^*, A_n^{\mathrm{c}}]^{\perp},$$

which yields $A_n \in \Lambda(\mathbf{X})$ for $\mathbf{X} \in K_0$ and $n \in \mathbb{Z}$. According to Proposition 2.1, $\mathbf{X}^* \in K_0$. Thus $A_n \in \Lambda(\mathbf{X}^*)$, which, by Proposition 2.5, implies $A_n^{\mathrm{c}} \in \Lambda(\mathbf{X})$. Since $A_{n+1}^{\mathrm{c}} = B_n$ we get the assertion.

PROPOSITION 2.7. *If Q is a finite subset of $\mathbb{Z}$ and $\mathbf{X} \in K_0$, then $Q, Q^{\mathrm{c}} \in \Lambda(\mathbf{X})$.*

Proof. By Proposition 2.1, $\mathbf{X}^* \in K_0$. Applying Proposition 2.5 to the evident relation $\mathbb{Z} \in \Lambda(\mathbf{X}^*)$ we get $\emptyset = \mathbb{Z}^{\mathrm{c}} \in \Lambda(\mathbf{X})$. Setting $S = \emptyset$ in Proposition 2.4 we conclude that every finite subset Q of $\mathbb{Z}$ belongs to $\Lambda(\mathbf{X})$. Now the remaining relation $Q^c \in \Lambda(\mathbf{X})$ is an immediate consequence of Proposition 2.5. This completes the proof.

3. Prediction problems. The linear least squares prediction problem for stationary sequences $\{X_n\}$ $(n \in \mathbb{Z})$ based on the observations X_n with $n \in Q$ consists in approximating X_r by linear combinations of X_n with $n \in Q$ minimizing the mean square error. The unique solution $\widehat{X}_r(Q)$ to this problem is the orthogonal projection of X_r on the subspace $[\mathbf{X}, Q]$.

Some special cases of this problem have drawn much attention and have a long history. The extrapolation problem based on the past $Q = A_n$ for some $n \in \mathbb{Z}$ and the interpolation problem corresponding to sets Q with finite complement Q^c were treated by A. N. Kolmogorov in [2] and [3] and N. Wiener in [6].

By the stationarity of the sequence in question the prediction problem can be reduced to the case $r = 0$. In what follows $\sigma(\mathbf{X}, Q)$ will denote the mean square error $\|X_0 - \widehat{X}_0(Q)\|$. It is clear that

$$\lim_{n\to\infty} \sigma(\mathbf{X}, Q_n) = \sigma(\mathbf{X}, Q) \tag{3.1}$$

whenever $Q_1 \subset Q_2 \subset \ldots$ and $Q = \bigcup_{n=1}^{\infty} Q_n$. For non-deterministic sequences $\mathbf{X}$ with the Wold decomposition $\mathbf{X} = \mathbf{X}' + \mathbf{X}''$ we have, by (1.6), the inequality

$$\sigma^2(\mathbf{X}, Q) \ge \sigma^2(\mathbf{X}', Q) + \sigma^2(\mathbf{X}'', Q) \tag{3.2}$$

for every subset Q of $\mathbb{Z}$. Moreover, by (1.5),

$$\sigma(\mathbf{X}, S) = \sigma(\mathbf{X}', S) \tag{3.3}$$

if $S \supset A_n$ for some $n \in \mathbb{Z}$.

For $Q \subset C_0$ we put $Q^* = C_0 \backslash Q$. Of course, $(Q^*)^* = Q$. The following statements can be regarded as a duality principle for stationary sequences and their duals.

THEOREM 3.1. *If* $\mathbf{X} \in K_0$, $Q \subset C_0$ *and* $Q \in \Lambda(\mathbf{X})$, *then*

$$\sigma(\mathbf{X}, Q)\sigma(\mathbf{X}^*, Q^*) = 1.$$

Proof. We note that, by Proposition 2.1, $\mathbf{X}^* \in K_0$. Since $Q \subset C_0$ we have, by Theorem 2.1 (part (ii)), $[\mathbf{X}^*, Q^*] \ne [\mathbf{X}^*]$, which yields the inequality $\sigma(\mathbf{X}^*, Q^*) > 0$. Put

$$Y = X_0 - \sigma^{-2}(\mathbf{X}^*, Q^*)(X_0^* - \widehat{X}_0^*(Q^*)).$$

It is clear that $\widehat{X}_0^*(Q^*) \in [\mathbf{X}^*, Q^*]$ and $X_0^* - \widehat{X}_0^*(Q^*) \perp [\mathbf{X}^*, Q^*]$. Consequently,

$$(X_0^* - \widehat{X}_0^*(Q^*), X_0^*) = \sigma^2(\mathbf{X}^*, Q^*),$$

which yields the relations $Y \perp [\mathbf{X}^*, Q^*]$ and $(Y, X_0^*) = 0$. As $Q^c = Q^* \cup \{0\}$ the last relations can be written in the form $Y \perp [\mathbf{X}^*, Q^c]$. From this and the assumption $Q \in \Lambda(\mathbf{X})$ we get $Y \in [X, Q]$. It is clear that $X_0^* - \widehat{X}_0^*(Q^*) \in [\mathbf{X}^*, Q^c]$. Consequently, $X_0 - Y \in [\mathbf{X}^*, Q^c]$. Since, by Proposition 2.5, $Q^c \in \Lambda(\mathbf{X}^*)$ the last relation can be written in the form $X_0 - Y \perp [\mathbf{X}, Q]$, which shows that the random variable Y is the orthogonal projection of X_0 on $[\mathbf{X}, Q]$. Thus $Y = \widehat{X}_0(Q)$ and, consequently,

$$\sigma(\mathbf{X}, Q) = \|X_0 - Y\| = \sigma^{-2}(\mathbf{X}^*, Q^*)\|X_0^* - \widehat{X}_0^*(Q^*)\| = \sigma^{-1}(\mathbf{X}^*, Q^*),$$

which completes the proof.

THEOREM 3.2. *Let $\mathbf{X} \in K$. Then for every $Q \subset C_0$,*

$$\sigma(\mathbf{X}, Q)\sigma(\mathbf{X}^*, Q^*) \geq 1.$$

Proof. Given $Q \subset C_0$ we define an auxiliary sequence $\{R_n\}$ $(n \geq 1)$ of subsets of C_0 by setting $R_n = Q^* \cap C_n^c$ $(n \geq 1)$. Of course, $R_1 \subset R_2 \subset \ldots$ and $Q^* = \bigcup_{n=1}^{\infty} R_n$, which, by formula (3.1), yields

$$\lim_{n\to\infty} \sigma(\mathbf{X}^*, R_n) = \sigma(\mathbf{X}^*, Q^*). \tag{3.4}$$

By Theorem 2.1 (part (iv)) and Propositions 2.1 and 2.2 we infer that $\mathbf{X}^* \in K_0$ and $(\mathbf{X}^*)^* = \mathbf{X}'$. Moreover, by Proposition 2.7, the finite sets R_n belong to $\Lambda(\mathbf{X}^*)$. Consequently, by Theorem 3.1, we have the equality

$$\sigma(\mathbf{X}^*, R_n)\sigma(\mathbf{X}', R_n^*) = 1 \qquad (n = 1, 2, \ldots). \tag{3.5}$$

Observe that $R_n^* = Q \cup C_n \supset Q$, which, by (3.2), yields $\sigma(\mathbf{X}', R_n^*) \leq \sigma(\mathbf{X}, Q)$. Thus, by (3.5),

$$\sigma(\mathbf{X}^*, R_n)\sigma(\mathbf{X}, Q) \geq 1 \qquad (n = 1, 2, \ldots)$$

and this, by (3.4), completes the proof.

Let $\mathbf{X}$ be a stationary sequence. A stationary sequence $\mathbf{Y}$ is said to be a *generalized dual* of $\mathbf{X}$ if for every subset Q of C_0 the inequality

$$\sigma(\mathbf{X}, Q)\sigma(\mathbf{Y}, Q^*) \geq 1$$

is true. In what follows $D(\mathbf{X})$ will denote the set of all generalized duals of X.

THEOREM 3.3. *$D(\mathbf{X}) \neq \emptyset$ if and only if $\mathbf{X} \in K$. Then $\mathbf{X}^* \in D(\mathbf{X})$ and*

$$\sigma(\mathbf{X}^*, Q) = \min\{\sigma(\mathbf{Y}, Q) : \mathbf{Y} \in D(\mathbf{X})\} \tag{3.6}$$

for every subset Q of C_0.

Proof. Suppose that $D(\mathbf{X}) \neq \emptyset$. Then $\sigma(\mathbf{X}, C_0) > 0$ and, consequently, $[\mathbf{X}, C_0] \neq [\mathbf{X}]$, which, by Theorem 2.1 (part (ii)), yields $\mathbf{X} \in K$. The reverse implication and the relation $\mathbf{X}^* \in D(\mathbf{X})$ are an immediate consequence of Theorem 3.2.

It remains to prove formula (3.6). Suppose that $\mathbf{X} \in K$ and $Q \subset C_0$. By Theorem 2.1 and Proposition 2.2 we conclude that $\mathbf{X}' \in K_0$ and $(\mathbf{X}')^* = \mathbf{X}^*$.

Put $S_n = Q^* \cup C_n$ $(n \geq 1)$. Since S_n^{c} are finite we deduce, by Proposition 2.7, that $S_n \in \Lambda(\mathbf{X}')$. Applying Theorem 3.1 we get the equalities

$$\sigma(\mathbf{X}', S_n)\sigma(\mathbf{X}^*, S_n^*) = 1 \quad (n = 1, 2, \ldots). \tag{3.7}$$

Observe that $S_n \supset A_{-n}$. Consequently, by (3.3), $\sigma(\mathbf{X}', S_n) = \sigma(\mathbf{X}, S_n)$, which, by (3.7), yields

$$\sigma(\mathbf{X}, S_n)\sigma(\mathbf{X}^*, S_n^*) = 1 \quad (n = 1, 2, \ldots).$$

Comparing this with the inequality

$$\sigma(\mathbf{X}, S_n)\sigma(\mathbf{Y}, S_n^*) \geq 1 \quad (n = 1, 2, \ldots)$$

for $\mathbf{Y} \in D(\mathbf{X})$ we get

$$\sigma(\mathbf{X}^*, S_n^*) \leq \sigma(\mathbf{Y}, S_n^*) \quad (n = 1, 2, \ldots). \tag{3.8}$$

From the formula $S_n^* = Q \cap C_n^{\mathrm{c}}$ it follows that $S_1^* \subset S_2^* \subset \ldots$ and $Q = \bigcup_{n=1}^{\infty} S_n^*$. Thus letting $n \to \infty$ in (3.8) we get, by (3.1), the inequality $\sigma(\mathbf{X}^*, Q) \leq \sigma(\mathbf{Y}, Q)$ for all $\mathbf{Y} \in D(\mathbf{X})$. This completes the proof of (3.6).

REFERENCES

[1] J. L. Doob, *Stochastic Processes*, Wiley, New York, 1953.

[2] A. N. Kolmogorov, *Sur l'interpolation et extrapolation des suites stationnaires*, C. R. Acad. Sci. Paris 208 (1939), 2043–2045.

[3] —, *Interpolation und Extrapolation von stationären zufälligen Folgen*, Bull. Acad. Sci. U.R.S.S. Sér. Math. 5 (1941), 3–14.

[4] I. I. Privalov, *Limit Properties of Analytic Functions*, G.I.T.-T.L., Moscow, 1950 (in Russian).

[5] Yu. A. Rozanov, *Stationary Random Processes*, G.I.F.-M.L., Moscow, 1963 (in Russian).

[6] N. Wiener, *Extrapolation, Interpolation and Smoothing of Stationary Time Series. With Engineering Applications*, The Technology Press of MIT, Cambridge, MA, 1949.

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