A DUALITY PRINCIPLE FOR STATIONARY RANDOM SEQUENCES

BY

K. URBANIK (WROCLAW)

Abstract. The paper is devoted to the study of stationary random sequences. A concept of dual sequences is discussed. The main aim of the paper is to establish a relationship between the errors of linear least squares predictions for sequences and their duals.

1. Preliminaries and notation. This paper is organized as follows. Section 1 collects together some basic facts and notation concerning stationary random sequences needed in what follows. In Section 2 a concept of dual sequences is discussed. In the last section a relationship between the errors of linear least squares predictions for sequences and their duals is established.

We suppose, as usual, that there is a probability measure defined on a \( \sigma \)-algebra of sets of some space \( \Omega \). Let \( M \) be the set of all complex-valued random variables whose squares are integrable. The set \( M \) is a Hilbert space under the inner product \( (X, Y) = EXY \) where \( E \) stands for the expectation of random variables. Throughout this paper \( \mathbb{Z} \) will denote the set of all integers.

A sequence \( X = \{X_n\} \ (n \in \mathbb{Z}) \) of random variables from \( M \) is said to be stationary if the inner product \( (X_n + m, X_m) \) does not depend on \( m \). The function \( R(n) = (X_{n+m}, X_m) \ (n \in \mathbb{Z}) \) is called the covariance function of the sequence in question. The Herglotz Theorem describes the covariance function as a Fourier transform

\[ R(n) = \int_{-\pi}^{\pi} e^{inx} \mu(dx) \]

where the measure \( \mu \) is concentrated on the interval \([-\pi, \pi]\). Of course, the correspondence \( R \leftrightarrow \mu \) is one-to-one ([1], Chapter 10.3). The measure \( \mu \) is called the spectral measure of the sequence \( X \).

For the empty set \( \emptyset \) we put \( [X, \emptyset] = \{0\} \). For a non-empty subset \( Q \) of \( \mathbb{Z} \), we denote by \([X, Q]\) the closed linear manifold of \( M \) generated by the random variables \( X_n \) with \( n \in Q \). For the sake of brevity we put \([X, \mathbb{Z}] = [X]\).
Each stationary sequence $\mathbf{X} = \{X_n\} \ (n \in \mathbb{Z})$ induces a unitary operator $T$ on $[\mathbf{X}]$ satisfying the condition $TX_n = X_{n+1} \ (n \in \mathbb{Z})$ ([1], Chapter 10.1).

In what follows we shall use the notation

$$A_n = \{k : k < n\} \quad \text{and} \quad B_n = \{k : k > n\} \quad (n \in \mathbb{Z}).$$

Two stationary sequences $\mathbf{X}$ and $\mathbf{Y}$ are said to be retrospectively or progressively connected if $[\mathbf{X}, A_n] = [\mathbf{Y}, A_n]$ for all $n \in \mathbb{Z}$ or $[\mathbf{X}, B_n] = [\mathbf{Y}, B_n]$ for all $n \in \mathbb{Z}$ respectively.

A stationary sequence $\mathbf{X}$ is called deterministic if

$$[\mathbf{X}, A_n] = [\mathbf{X}] \quad (n \in \mathbb{Z}).$$

A stationary sequence $\mathbf{X}$ is called completely non-deterministic if

$$[\mathbf{X}] \neq \{0\} \quad \text{and} \quad \bigcap_{n \in \mathbb{Z}} [\mathbf{X}, A_n] = \{0\}.$$

Each non-deterministic stationary sequence $\mathbf{X}$ has a unique Wold decomposition $\mathbf{X} = \mathbf{X}' + \mathbf{X}''$ into two stationary sequences $\mathbf{X}'$ and $\mathbf{X}''$ where $\mathbf{X}'$ is completely non-deterministic, $\mathbf{X}''$ is deterministic,

$$[\mathbf{X}] = [\mathbf{X}'] \oplus [\mathbf{X}']$$

and $[\mathbf{X}]$ is the orthogonal sum of $[\mathbf{X}']$ and $[\mathbf{X}''],$

$$[\mathbf{X}, A_n] = [\mathbf{X}', A_n] \oplus [\mathbf{X}''] \quad (n \in \mathbb{Z}),$$

which yields the equality

$$[\mathbf{X}, S] = [\mathbf{X}', S] \oplus [\mathbf{X}''].$$

The Hardy class $H_p \ (p > 0)$ consists of functions $f$ analytic on $|z| < 1$ and satisfying the condition

$$\lim_{r \to 1^-} \int_{-\pi}^{\pi} |f(re^{ix})|^p \, dx < \infty.$$

It is well known that for $f \in H_p$ the radial limit

$$\lim_{r \to 1^-} f(re^{ix}) = f(e^{ix})$$

exists almost everywhere ([4], Chapter 2.2). By $H_p^+$ we denote the subset of $H_p$ consisting of functions $f$ satisfying the conditions $f(0) > 0$ and $f(z) \neq 0$ for $|z| < 1.$
In what follows $\delta_n$ will denote the Kronecker $\delta$-function: $\delta_0 = 1$ and $\delta_n = 0$ for $n \neq 0$. A sequence $U = \{U_n\}$ ($n \in \mathbb{Z}$) of random variables from $M$ is called orthonormal if $\delta_n$ is its covariance function.

The main representation theorem says that each completely non-deterministic sequence $X$ has a unique representation $X = (F, U)$ where $F \in H^+_2$ and the orthonormal sequence $U$ and the sequence $X$ are retrospectively connected. This means that $X$ is the moving average

$$X_n = \sum_{k=0}^{\infty} a_k U_{n-k} \quad (n \in \mathbb{Z})$$

where $F(z) = \sum_{k=0}^{\infty} a_k z^k$ for $|z| < 1$ ([1], Chapter 12.4). Moreover, the spectral measure $\mu$ of $X$ is of the form

$$\mu(dx) = \frac{1}{2\pi} |F(e^{-ix})|^2 dx.$$  

2. Dual sequences. Let $X = \{X_n\}$ ($n \in \mathbb{Z}$) be a stationary sequence. A sequence $X^* = \{X^*_n\}$ ($n \in \mathbb{Z}$) of random variables from $[X]$ is called the dual of $X$ if

$$(X_n, X^*_m) = \delta_{n-m} \quad (n, m \in \mathbb{Z}).$$

It is clear that the dual sequence is uniquely determined provided it exists. Thus, taking the unitary operator $T$ induced by $X$ on $[X]$, we conclude that for every $r \in \mathbb{Z}$ the sequence $\{T^rX^*_n\}$ ($n \in \mathbb{Z}$) is also the dual of $X$ and, consequently, $X^*_n = T^rX^*_{n-r}$ ($n, r \in \mathbb{Z}$). This shows that the sequence $X^*$ is also stationary.

Example 2.1. For orthonormal sequences $U$ we have $U^* = U$.

Example 2.2. Let $X$ be a stationary Markov sequence with covariance function $R(n) = a^n R(0)$ where $n \geq 0$, $R(0) > 0$ and $|a| < 1$ ([1], p. 477). Then we have

$$X^*_n = (1 - |a|^2)^{-1}((1 + |a|^2)X_n - aX_{n-1} - \pi X_{n+1}) \quad (n \in \mathbb{Z}).$$

In what follows $K$ will stand for the set of all stationary sequences admitting the dual sequence. Further, $K_0$ will denote the subset of $K$ consisting of sequences $X$ satisfying $[X^*] = [X]$. The following statement is evident.

Proposition 2.1. If $X \in K_0$, then $X^* \in K_0$ and $(X^*)^* = X$.

Proposition 2.2. $X \in K$ if and only if $X$ is non-deterministic and $X' \in K$. Then the formula $(X')^* = X^*$ is true.

Proof. Let $X \in K$. First we shall prove that the sequence $X$ is non-deterministic. Suppose the contrary. Since $X_0^* \perp [X, A_0]$ we have, by (1.1), $X_0^* = 0$, which contradicts the equality $(X_0, X_0^*) = 1$. Thus the sequence $X$
is non-deterministic. Consider its Wold decomposition $X = X' + X''$. Since $X^*_m \perp [X, A_n]$ for $n < m$ we have, by (1.4),

$$X^*_m \perp \bigcap_{n \in \mathbb{Z}} [X, A_n] = [X''] \quad (m \in \mathbb{Z}).$$

Consequently, by (1.2), $X^*_m \in [X'] (m \in \mathbb{Z})$ and $(X^*_n, X^*_m) = (X_n, X^*_m) = \delta_{n-m}$, which shows that $X^*$ is the dual of $X'$.

Conversely, suppose that $X' \in K$. Then, by (1.3), $(X'_m)^* \in [X]$ and $(X'_m)^* \perp [X'']$. Consequently, $(X_n, (X'_m)^*) = (X'_n, (X'_m)^*) = \delta_{n-m} (n, m \in \mathbb{Z})$, which yields $X \in K$. This completes the proof.

The next result is less trivial.

**Proposition 2.3.** Let $X$ be a completely non-deterministic sequence with the representation $(F, U)$ such that $F^{-1} \in H_2$. Then $X \in K_0$.

(2.2)

$$X^*_n = \sum_{k=0}^{\infty} b_k U_{k+n},$$

with the coefficients $b_k$ determined by the expansion $F^{-1}(z) = \sum_{k=0}^{\infty} b_k z^k$ for $|z| < 1$ and the sequences $X^*$ and $U$ are progressively connected.

**Proof.** Observe that $\sum_{k=0}^{\infty} |b_k|^2 < \infty$. This shows that the right-hand side of (2.2) is well defined. Denote it by $Y_n$. It is clear that

$$Y_n \in [U, B_{n-1}] \subset [X] \quad (n \in \mathbb{Z}),$$

which yields the relation $Y_n \perp [U, A_n] (n \in \mathbb{Z})$. Since the sequences $X$ and $U$ are retrospectively connected the last relation implies the equalities $(X_k, Y_n) = 0$ if $k < n$. Further, if $k = n + r$ and $r \geq 0$, then

$$(X_k, Y_n) = \sum_{s, j=0}^{\infty} a_j b_s (U_{n+r-j}, U_{n+s}) = \sum_{j=0}^{r} a_j b_{r-j} = \delta_r,$$

which shows that the sequence $\{Y_n\} (n \in \mathbb{Z})$ is the dual of $X$. Formula (2.2) and the relation $X \in K$ are thus proved.

Now we shall prove that the sequences $X^*$ and $U$ are progressively connected. By formula (2.2) we have the inclusion

(2.3)

$$[X^*, B_0] \subset [U, B_0].$$

To prove the reverse inclusion we suppose that a random variable $Y$ satisfies

(2.4)

$$Y \perp [X^*, B_0]$$

and belongs to $[U, B_0]$. Consequently, it can be written in the form

(2.5)

$$Y = \sum_{k=1}^{\infty} c_k U_k$$
where \( \sum_{k=1}^{\infty} |c_k|^2 < \infty \). From (2.4) we get the equalities

\[
(2.6) \quad (Y, X_n) = \sum_{s=0}^{\infty} \sum_{k=1}^{\infty} c_k b_s \delta_{k-s-n} = 0 \quad (n \geq 1).
\]

Introduce the notation

\[
(2.7) \quad f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{k=0}^{\infty} b_k z^k, \quad h(z) = \sum_{k=1}^{\infty} c_k z^k \quad (|z| < 1).
\]

It is clear that

\[
(2.8) \quad f, g, h \in H_2
\]

and

\[
(2.9) \quad f(z)g(z) = 1 \quad (|z| < 1).
\]

By Parseval’s formula and (2.6) we get the equalities

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} h(e^{ix}) \overline{g(e^{ix})} e^{-inx} \, dx = \sum_{s=0}^{\infty} \sum_{k=1}^{\infty} c_k b_s \delta_{k-s-n} = 0 \quad (n \geq 1).
\]

Consequently, the function \( \overline{h}(e^{ix})g(e^{ix}) \) integrable on the interval \([-\pi, \pi]\) has the Fourier expansion of the form \( \sum_{k=0}^{\infty} p_k e^{ikx} \). Setting \( p(z) = \sum_{k=0}^{\infty} p_k z^k \) \((|z| < 1)\) we infer, by Theorem 6.1 in [1], Chapter 4, that

\[
(2.10) \quad p \in H_1
\]

and

\[
(2.11) \quad p(e^{ix}) = \overline{h}(e^{ix})g(e^{ix})
\]

almost everywhere. Put \( q(z) = p(z)f(z) \) \((|z| < 1)\). Taking into account (2.8), (2.10) and the inequality

\[
|q(z)|^{1/2} \leq |p(z)| + |f(z)|
\]

we conclude that \( q \in H_{1/2} \) and, by (2.9) and (2.11),

\[
(2.12) \quad q(e^{ix}) = \overline{h}(e^{ix})
\]

almost everywhere. Thus, by (2.8), the radial limit \( q(e^{ix}) \) is square integrable on the interval \([-\pi, \pi]\). Applying Smirnov’s theorem ([4], p. 116) we have \( q \in H_2 \). Consequently, from (2.12) it follows that

\[
h(e^{ix}) = \sum_{k=0}^{\infty} d_k e^{-ikx}
\]

for some coefficients \( d_k \) with \( \sum_{k=0}^{\infty} |d_k|^2 < \infty \). Comparing this with (2.7) we have \( c_k = 0 \) for \( k \geq 1 \), which, by (2.6), yields \( Y = 0 \). This completes the proof of the inclusion \([X^*, B_0] \supset [U, B_0] \), which together with (2.3) yields the equality \([X^*, B_0] = [U, B_0] \). Since

\[
[X^*, B_n] = T^n[X^*, B_0], \quad [U, B_n] = T^n[U, B_0] \quad (n \in \mathbb{Z}),
\]
where $T$ is the unitary operator induced by the sequence $X$ on $[X]$, we have

$$[X^*, B_n] = [U, B_n] \quad (n \in \mathbb{Z}).$$

In other words, the sequences $X^*$ and $U$ are progressively connected. Hence in particular it follows that $[X^*] = [U]$. On the other hand, $[X] = [U]$ because the sequences $X$ and $U$ are retrospectively connected. Thus $[X^*] = [X]$ and, consequently, $X \in K_0$, which completes the proof.

We are now in a position to prove a characterization of the class $K$. In what follows we shall use the notation $C_n = A_{-n} \cup B_n \ (n \geq 0)$.

**Theorem 2.1.** The following conditions are equivalent:

(i) $X \in K$,

(ii) $[X, C_0] \neq [X]$,

(iii) $X$ is non-deterministic and $X' = (F, U)$ with $F^{-1} \in H_2$,

(iv) $X$ is non-deterministic and $X' \in K_0$.

**Proof.** (i)$\Rightarrow$(ii). Since $X_0^* \neq 0$ and $X^*_0 \perp [X, C_0]$ we have condition (ii).

(ii)$\Rightarrow$(iii). Condition (ii) and equalities (1.3) and (1.5) yield the condition $[X', C_0] \neq [X']$. Taking the representation $X' = (F, U)$ we have, by Kolmogorov’s Theorem ([5], Chapter 2, Theorem 10.2), $\int_{-\infty}^{\infty} |F(e^{-ix})|^{-2} \, dx < \infty$. Since $F(z) \neq 0$ for $|z| < 1$ we have $F^{-1} \in H_2$.

(iii)$\Rightarrow$(iv) and (iv)$\Rightarrow$(i) are immediate consequences of Propositions 2.3 and 2.2 respectively. The theorem is thus proved.

Given $Q \subset Z$ we denote by $Q^c$ the complement $Z \setminus Q$. Let $X \in K$. Since $(X_n, X_n^*) = 0$ for $n \in Q$ and $m \in Q^c$, we have the inclusion

$$[X, Q] \subset [X^*, Q^c]^\perp$$

where the orthogonal complement is taken in the space $[X]$. We shall denote by $A(X)$ the family of all subsets $Q$ of $Z$ satisfying

$$[X, Q] = [X^*, Q^c]^\perp.$$ 

Since $[X, Z] = [X]$ and $[X^*, \emptyset] = \{0\}$ we conclude that $Z \in A(X)$ for every $X \in K$.

**Proposition 2.4.** Let $X \in K$. Suppose that $Q$ and $S$ are disjoint subsets of $Z$ and the set $Q$ is finite. Then $S \cup Q \in A(X)$ if and only if $S \in A(X)$.

**Proof.** Suppose that $V \in [X, Q] \cap [X^*, S^c]^\perp$.

The random variable $V$ can be written in the form $V = \sum_{n \in Q} c_n X_n$ where $c_n \ (n \in Q)$ are complex numbers. Since $Q \subset S^c$ the random variable $V_0 = \sum_{n \in Q} c_n X_n^*$ belongs to $[X^*, S^c]$. Consequently, $0 = (V, V_0) = \sum_{n \in Q} |c_n|^2$, which yields the equality $V = 0$. Thus we have the formula

$$[X, Q] \cap [X^*, S^c]^\perp = \{0\}.$$
Further, taking into account (2.13), we get $[X, Q] \cap [X, S] = \{0\}$. Since the subspace $[X, Q]$ is finite-dimensional we conclude that the subspace $[X, S \cup Q]$ can be represented as a direct sum

$$[X, S \cup Q] = [X, S] + [X, Q].$$

Using (2.13) we get the inclusion

$$[X, Q] + [X^*, S^c] \subseteq [X^*, S^c \cap Q^c] \subseteq [X^*, S^c].$$

To prove the reverse inclusion we assume that $W \in [X^*, S^c \cap Q^c]$. Setting $W_Q = \sum_{n \in Q} (W, X_n^*) X_n$ we have the relations $W_Q \in [X, Q]$, $W_Q \perp [X^*, Q^c]$ and $W - W_Q \perp [X^*, Q]$. Moreover, by the formula $S^c = Q \cup (S^c \cap Q^c)$, we have $W - W_Q \perp [X^*, S^c]$. Consequently, $W \in [X, Q] + [X^*, S^c] \perp$, which, by (2.16), yields

$$[X^*, S^c \cap Q^c] = [X^*, S^c] \subseteq [X^*, S^c \cap Q^c] \subseteq [X^*, S^c].$$

Comparing this with (2.14) and (2.15) we conclude that $[X, S] = [X^*, S^c] \subseteq$ if and only if $[X, S \cup Q] = [X^*, S^c \cap Q^c] \subseteq$, which completes the proof.

**Proposition 2.5.** If $X \in K_0$ and $Q \in \Lambda(X^*)$, then $Q^c \in \Lambda(X^*)$.

**Proof.** First observe that the complementations in $[X]$ and $[X^*]$ coincide. Now our assertion is a consequence of Proposition 2.1 and the formula

$$[X^*, Q^c] = [X, Q] \subseteq [(X^*)^*, (Q^c)^c] \subseteq [X^*, S^c \cap Q^c].$$

**Proposition 2.6.** If $X \in K_0$, then $A_n, B_n \in \Lambda(X)$ for all $n \in \mathbb{Z}$.

**Proof.** Let $X \in K_0$. Then, by Proposition 2.2, the sequence $X$ is completely non-deterministic and, consequently, has a representation $X = (F, U)$, where the sequences $X$ and $U$ are retrospectively connected. Further, by Proposition 2.4, the sequences $X^*$ and $U$ are progressively connected. Hence we get the equalities

$$[X, A_n] = [U, A_n] = [U, A_n^c] \subseteq [X^*, A_n^c],$$

which yields $A_n \in \Lambda(X)$ for $X \in K_0$ and $n \in \mathbb{Z}$. According to Proposition 2.1, $X^* \in K_0$. Thus $A_n \in \Lambda(X^*)$, which, by Proposition 2.5, implies $A_n^c \in \Lambda(X)$. Since $A_{n+1} = B_n$ we get the assertion.

**Proposition 2.7.** If $Q$ is a finite subset of $\mathbb{Z}$ and $X \in K_0$, then $Q, Q^c \in \Lambda(X)$.

**Proof.** By Proposition 2.1, $X^* \in K_0$. Applying Proposition 2.5 to the evident relation $Z \in \Lambda(X^*)$ we get $\emptyset = Z^c \in \Lambda(X)$. Setting $S = \emptyset$ in Proposition 2.4 we conclude that every finite subset $Q$ of $\mathbb{Z}$ belongs to $\Lambda(X)$. Now
the remaining relation $Q^c \in A(X)$ is an immediate consequence of Proposition 2.5. This completes the proof.

3. Prediction problems. The linear least squares prediction problem for stationary sequences $\{X_n\}$ ($n \in \mathbb{Z}$) based on the observations $X_n$ with $n \in Q$ consists in approximating $X_r$ by linear combinations of $X_n$ with $n \in Q$ minimizing the mean square error. The unique solution $\hat{X}_r(Q)$ to this problem is the orthogonal projection of $X_r$ on the subspace $[X, Q]$.

Some special cases of this problem have drawn much attention and have a long history. The extrapolation problem based on the past $Q = A_n$ for some $n \in \mathbb{Z}$ and the interpolation problem corresponding to sets $Q$ with finite complement $Q^c$ were treated by A. N. Kolmogorov in [2] and [3] and N. Wiener in [6].

By the stationarity of the sequence in question the prediction problem can be reduced to the case $r = 0$. In what follows $\sigma(X, Q)$ will denote the mean square error $\|X_0 - \hat{X}_0(Q)\|$. It is clear that

$$\sigma^2(X, Q) = \sigma(X, Q^c)$$

whenever $Q_1 \subset Q_2 \subset \ldots$ and $Q = \bigcup_{n=1}^{\infty} Q_n$. For non-deterministic sequences $X$ with the Wold decomposition $X = X' + X''$ we have, by (1.6), the inequality

$$\sigma^2(X, Q) \geq \sigma^2(X', Q) + \sigma^2(X'', Q)$$

for every subset $Q$ of $\mathbb{Z}$. Moreover, by (1.5),

$$\sigma(X, S) = \sigma(X', S)$$

if $S \supset A_n$ for some $n \in \mathbb{Z}$.

For $Q \subset C_0$ we put $Q^* = C_0 \setminus Q$. Of course, $(Q^*)^* = Q$. The following statements can be regarded as a duality principle for stationary sequences and their duals.

**Theorem 3.1.** If $X \in K_0$, $Q \subset C_0$ and $Q \in A(X)$, then

$$\sigma(X, Q)\sigma(X^*, Q^*) = 1.$$ 

**Proof.** We note that, by Proposition 2.1, $X^* \in K_0$. Since $Q \subset C_0$ we have, by Theorem 2.1 (part (ii)), $[X^*, Q^*] \neq [X^*]$, which yields the inequality $\sigma(X^*, Q^*) > 0$. Put

$$Y = X_0 - \sigma^{-2}(X^*, Q^*)(X_0^* - \hat{X}_0^*(Q^*))$$

It is clear that $\hat{X}_0^*(Q^*) \in [X^*, Q^*]$ and $X_0^* - \hat{X}_0^*(Q^*) \perp [X^*, Q^*]$. Consequently,

$$\langle X_0^* - \hat{X}_0^*(Q^*), X_0^* \rangle = \sigma^2(X^*, Q^*),$$

which yields the relations $Y \perp [X^*, Q^*]$ and $\langle Y, X_0^* \rangle = 0$. As $Q^c = Q^* \cup \{0\}$ the last relations can be written in the form $Y \perp [X^*, Q^c]$. From this and
the assumption $Q \in \Lambda(X)$ we get $Y \in [X, Q]$. It is clear that $X_0^* - \hat{X}_0^*(Q^*) \in [X^*, Q^*]$. Consequently, $X_0 - Y \in [X^*, Q^*]$. Since, by Proposition 2.5, $Q^* \in \Lambda(X^*)$ the last relation can be written in the form $X_0 - Y \perp [X, Q]$, which shows that the random variable $Y$ is the orthogonal projection of $X_0$ on $[X, Q]$. Thus $Y = \hat{X}_0(Q)$ and, consequently,

$$\sigma(X, Q) = \|X_0 - Y\| = \sigma^{-2}(X^*, Q^*)\|X_0^* - \hat{X}_0^*(Q^*)\| = \sigma^{-1}(X^*, Q^*),$$

which completes the proof.

**Theorem 3.2.** Let $X \in K$. Then for every $Q \subset C_0$,

$$\sigma(X, Q)\sigma(X^*, Q^*) \geq 1.$$

**Proof.** Given $Q \subset C_0$ we define an auxiliary sequence $\{R_n\}$ ($n \geq 1$) of subsets of $C_0$ by setting $R_n = Q^* \cap C_n^c$ ($n \geq 1$). Of course, $R_1 \subset R_2 \subset \ldots$ and $Q^* = \bigcup_{n=1}^{\infty} R_n$, which, by formula (3.1), yields

$$\lim_{n \to \infty} \sigma(X^*, R_n) = \sigma(X^*, Q^*).$$

By Theorem 2.1 (part (iv)) and Propositions 2.1 and 2.2 we infer that $X^* \in K_0$ and $(X^*)^* = X'$. Moreover, by Proposition 2.7, the finite sets $R_n$ belong to $\Lambda(X^*)$. Consequently, by Theorem 3.1, we have the equality

$$\sigma(X^*, R_n)\sigma(X', R_n^*) = 1 \quad (n = 1, 2, \ldots).$$

Observe that $R_n^* = Q \cup C_n \supset Q$, which, by (3.2), yields $\sigma(X', R_n^*) \leq \sigma(X, Q)$. Thus, by (3.5),

$$\sigma(X^*, R_n)\sigma(X, Q) \geq 1 \quad (n = 1, 2, \ldots)$$

and this, by (3.4), completes the proof.

Let $X$ be a stationary sequence. A stationary sequence $Y$ is said to be a generalized dual of $X$ if for every subset $Q$ of $C_0$ the inequality

$$\sigma(X, Q)\sigma(Y, Q^*) \geq 1$$

is true. In what follows $D(X)$ will denote the set of all generalized duals of $X$.

**Theorem 3.3.** $D(X) \neq \emptyset$ if and only if $X \in K$. Then $X^* \in D(X)$ and

$$\sigma(X^*, Q) = \min\{\sigma(Y, Q) : Y \in D(X)\}$$

for every subset $Q$ of $C_0$.

**Proof.** Suppose that $D(X) \neq \emptyset$. Then $\sigma(X, C_0) > 0$ and, consequently, $[X, C_0] \neq [X]$, which, by Theorem 2.1 (part (ii)), yields $X \in K$. The reverse implication and the relation $X^* \in D(X)$ are an immediate consequence of Theorem 3.2.

It remains to prove formula (3.6). Suppose that $X \in K$ and $Q \subset C_0$. By Theorem 2.1 and Proposition 2.2 we conclude that $X' \in K_0$ and $(X')^* = X^*$. 


Put $S_n = Q^* \cup C_n$ ($n \geq 1$). Since $S_n^*$ are finite we deduce, by Proposition 2.7, that $S_n \in \Lambda(X')$. Applying Theorem 3.1 we get the equalities

$$
\sigma(X', S_n) \sigma(X^*, S_n^*) = 1 \quad (n = 1, 2, \ldots).
$$

(3.7)

Observe that $S_n \supset A_{-n}$. Consequently, by (3.3), $\sigma(X', S_n) = \sigma(X, S_n)$, which, by (3.7), yields

$$
\sigma(X, S_n) \sigma(X^*, S_n^*) = 1 \quad (n = 1, 2, \ldots).
$$

Comparing this with the inequality

$$
\sigma(X, S_n) \sigma(Y, S_n^*) \geq 1 \quad (n = 1, 2, \ldots)
$$

for $Y \in D(X)$ we get

$$
\sigma(X^*, S_n^*) \leq \sigma(Y, S_n^*) \quad (n = 1, 2, \ldots).
$$

(3.8)

From the formula $S_n^* = Q \cap C_n^*$ it follows that $S_n^* \subset S_n^* \subset \ldots$ and $Q = \bigcup_{n=1}^{\infty} S_n^*$. Thus letting $n \to \infty$ in (3.8) we get, by (3.1), the inequality $\sigma(X^*, Q) \leq \sigma(Y, Q)$ for all $Y \in D(X)$. This completes the proof of (3.6).

REFERENCES


