COLLOQUIUM MATHEMATICUM

VOL. 86 2000 NO. 1

SYMMETRIC PARTITIONS AND PAIRINGS

BY

FERENC ORAVECZ (BUDAPEST)

Abstract. The lattice of partitions and the sublattice of non-crossing partitions of a finite set are important objects in combinatorics. In this paper another sublattice of the partitions is investigated, which is formed by the symmetric partitions. The measure whose nth moment is given by the number of non-crossing symmetric partitions of n elements is determined explicitly to be the "symmetric" analogue of the free Poisson law.

1. Preliminaries. For a linearly ordered set S, $\pi = \{V_1, \ldots, V_p\}$ is a *partition* of S if the V_i are pairwise disjoint (non-empty) sets with union S. The sets V_i are called the *blocks* of the partition. We call the partition π *crossing* if in π there are two blocks V_i and V_j $(i \neq j)$ and elements $v_1, v_2 \in V_i$, $w_1, w_2 \in V_j$ such that $v_1 < w_1 < v_2 < w_2$. Otherwise the partition is called *non-crossing*. A partition π of S is called a *pair-partition* or *pairing* if every block of π contains exactly two elements of S. Of course the number of elements in S must be even if S admits such a partition.

It is well known (and can be verified easily) that the number of all partitions of a set of n elements is given by the nth Bell number B_n satisfying

$$B_0 = 1, \qquad B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k,$$

while the number of all pair-partitions of a set of $2n$ elements is $(2n)!/(2^n n!)$ (see [2] for example).

An interesting fact about the non-crossing partitions is that the number of non-crossing partitions of a set of n elements equals the number of non-crossing pair-partitions of a set of $2n$ elements, which is the nth Catalan number $c_k = \frac{1}{n+1}\binom{2n}{n}$ (see [4] and [6]).

One can define a partial order structure on the set of partitions (or non-crossing partitions) of an n-element set in the following way: let $\pi_1 \leq \pi_2$ if and only if each block of π_1 is contained in a block of π_2. With this order both the set of all partitions and the set of non-crossing partitions of an n-element set become lattices (see [4] and [2]).

2000 *Mathematics Subject Classification*: 60E10, 05A18.

Supported by OTKA T016924.

It is also interesting to see the connection between the number of certain partitions and certain probability distributions. The most important examples are the following. It is easy to check that the nth moment of the standard Gaussian law is given by the number of pair-partitions of a set of n elements. On the other hand, the number of non-crossing pair-partitions of a set of n elements gives the nth moment of the standard Wigner law (it is also called the semicircle law), which is the analogue of the Gaussian law in free probability theory (see e.g. [3]). The nth moment of a Poisson law of mean λ is a polynomial of λ in which the coefficient of λ^i is the number of all partitions of a set of n elements into i blocks. Again, taking the polynomial of λ in which the coefficient of λ^i is the number of non-crossing partitions of a set of n elements into i blocks, we get the nth moment of the Marchenko–Pastur distribution, which from many points of view is the free analogue of the Poisson law (see [5]).

2. Combinatorial results. In this paper we examine another special type of partition which is given by the following

DEFINITION 2.1. A partition π of a linearly ordered set $S = \{s_1, \dots, s_n\}$ is *symmetric* if whenever s_i and s_j are in the same block, then so are s_{n+1-i} and s_{n+1-j}.

The reason to call these partitions symmetric is that their linear representation (where instead of S we take the subset $\{1, \dots, n\}$ of the real line, and the blocks of π are represented by "bridges" connecting the elements of the same block) is symmetric with respect to its center, or (what means the same) inverting the order in S does not change the linear representation. In Fig. 1 one can see the linear representations of $\pi_1 = \{\{1,4\},\{2\},\{3,5,6\}\}$ and $\pi_2 = \{\{1,6\},\{2,3\},\{4,5\}\}$ of $\{1,2,3,4,5,6\}$, the latter being symmetric.

Fig. 1

Note that from the linear representation of a partition it can also be seen immediately whether it is crossing or non-crossing. If it is possible to draw the bridges corresponding to the blocks without crossing one another, then the corresponding partition is non-crossing (as for example π_2 of Fig. 1), while if it is impossible, the partition is crossing (as for example π_1 of Fig. 1). It is also easy to see from the linear representation whether the given partition is a pairing, as for example π_2 of Fig. 1.

Note that just as in the case of all partitions, both the set of all symmetric partitions and the set of non-crossing symmetric partitions of an n-element set form lattices with respect to the usual partial order $\leq$ of partitions.

PROPOSITION 2.1. *Both the number of non-crossing symmetric partitions of a set of n elements and the number of non-crossing symmetric pair-partitions of a set of $2n$ elements are $\binom{n}{[n/2]}$ (the numbers in the center of the Pascal triangle).*

Proof. Let A_n denote the number of non-crossing symmetric partitions of n elements. First we show the following recursion:

$$\begin{aligned} A_{2l+1} &= 2A_{2l} - c_l \quad && \text{for } l = 0, 1, 2, \ldots, \\ A_{2l+2} &= 2A_{2l+1} && \text{for } l = 0, 1, 2, \ldots, \end{aligned}$$

where c_l is the lth Catalan number (the number of all non-crossing partitions of l elements), with $A_0 = c_0 = 1$.

If π is any non-crossing symmetric partition of $S_0 = \{1, 2, \ldots, l, l+2, \ldots, 2l+1\}$, then $\widetilde{\pi} = \pi \cup \{\{l+1\}\}$ is a non-crossing symmetric partition of $S = \{1, \ldots, 2l+1\}$. Starting from different partitions of S_0 we clearly get different partitions of S. This way we get A_{2l} non-crossing symmetric partitions of S.

We get different—still symmetric and non-crossing—partitions of S if we let the inserted central element $l+1$ be the member of the block which contains the greatest element $i_0 \leq l$ together with its "symmetric counterpart" $2l+1-i_0$. The only case when such a block does not exist is when every element of any block is in the same half of the set, and as in this case the partition of the first l elements defines the partition of the last l elements (because of the requirement of symmetry), the number of these cases is c_l, the number of non-crossing partitions of l elements. This gives another $A_{2l} - c_l$ partitions, as starting from different partitions of S_0 we clearly get different partitions of S again. It is also clear that these $A_{2l} - c_l$ partitions differ from the A_{2l} partitions obtained before; to check this it is enough to take the block of the central element.

What is left is to show that any symmetric non-crossing partition ν of S can be obtained by one of those two methods. The element $l+1$ can either form a block by itself, or can be the central element of a block which contains both smaller and greater elements, in a symmetric arrangement. In the first case it is clear that the partition ν can be obtained by the first method from the partition $\nu_0 = \nu \setminus \{l+1\}$ of S_0. In the second case, as the partition is non-crossing, the block which contains the greatest element $i_0 \leq l$ together with its "symmetric counterpart" $2l+1-i_0$ is the block of the central element. So now the partition ν can be obtained by the second method from the partition of S_0 which arises by eliminating the central element from its block (and from the set itself).

Now starting from any non-crossing symmetric partition of $2l+1$ elements, we can obtain a non-crossing symmetric partition of $2l+2$ elements in two different ways again. Duplicating the central element (s_{l+1}), we can put the two elements in the same block or in different ones. (In the first case the number of blocks does not change, while in the second case it increases by one.) Starting from different partitions both procedures give different partitions of the resulting $2l+2$ elements, and of course the two methods can never give the same partition; to check this it is enough to take the block(s) of the two central elements.

We show that any symmetric non-crossing partition η of $2l+2$ elements can be obtained by one of those two methods. The elements s_{l+1} and s_{l+2} can be either in the same block or in different ones. In the first case it is clear that η can be obtained by the first method from the partition η_0 of $2l+1$ elements which arises by fusing the elements s_{l+1} and s_{l+2}. In the second case the partition η can be obtained from the same partition η_0 of $2l+1$ elements by the second method.

The fact that the sequence $\binom{n}{[n/2]}$ satisfies the same recursion with the same starting element can be verified directly.

For the second part of the statement we recall a well known natural bijection between the set of non-crossing partitions of n elements and the set of non-crossing pair-partitions of $2n$ elements. Let $\pi = \{V_1, \dots, V_k\}$ be a non-crossing partition of $S = \{s_1, \dots, s_n\}$. Let $\widehat{S}$ be the ordered set $\{s_1^-, s_1^+, \dots, s_n^-, s_n^+\}$, and let $\widehat{\pi}$ be the partition whose blocks are defined in the following way: whenever $V_i = \{s_{i_1}, \dots, s_{i_m}\} \in \pi$, let $\{s_{i_1}^-, s_{i_m}^+\}, \{s_{i_1}^+, s_{i_2}^-\}$, $\{s_{i_2}^+, s_{i_3}^-\}, \dots, \{s_{i_{m-1}}^+, s_{i_m}^-\} \in \widehat{\pi}$. (For an example see Fig. 2.) It is obvious that this map preserves symmetry, which completes the proof. ∎

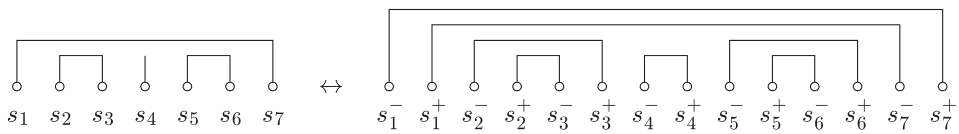

Fig. 2

We note that for the number of all symmetric partitions and all symmetric pair-partitions we could not find any useful expressions apart from obvious recurrences.

3. Solution of the moment problem in the case of symmetric partitions. In this part we consider the question whether the numbers of non-crossing symmetric partitions of n elements ($n = 0, 1, 2, \dots$) can form the moment sequence of some probability measure μ.

THEOREM 3.1. *The moment sequence of the probability distribution*

$$\mu = \frac{1}{2\pi}\sqrt{\frac{2+x}{2-x}}\chi(x)dx,$$

where χ stands for the characteristic function of the interval $[-2, 2]$, *is* $\{A_n\}_{n\in\mathbb{N}}$, *that is, the sequence of numbers of all non-crossing symmetric partitions of n elements* ($n = 0, 1, \dots$).

Proof. Denote the nth moment of μ by m_n. Integration by parts gives

$$m_{2l} = \frac{1}{2\pi}\int_{-2}^{2} x^{2l}\sqrt{\frac{2+x}{2-x}}\,dx = \frac{1}{2\pi}\Big[2\pi 4^l - 4l\int_{-2}^{2} x^{2l-1}\arcsin(x/2)\,dx\Big],$$

$$m_{2l+1} = \frac{1}{2\pi}\int_{-2}^{2} x^{2l+1}\sqrt{\frac{2+x}{2-x}}\,dx = \frac{1}{2\pi}(2l+1)\int_{-2}^{2} x^{2l}\sqrt{4-x^2}\,dx.$$

Another integration by parts (applied to the expression obtained for m_{2l}) gives $m_{2l} = 2m_{2l-1}$. (In the integral we recognize the above forms of m_{2l} and m_{2l-1}, leading to $m_{2l} = 4lm_{2l-1} - (2l-1)m_{2l}$, which yields the given result.) In a similar manner, integration by parts applied to the expression obtained for m_{2l+1} gives

$$m_{2l+1} = \frac{2l+1}{l+1}m_{2l}.$$

m_0 and m_1 can be computed directly; one gets $m_0 = 1$ and $m_1 = 1$. The fact that the sequence $\binom{n}{[n/2]}$ satisfies the same recursions with the same starting element can be verified directly, which completes the proof. ■

REMARK 3.2. A constructive way to obtain the measure μ is via its moment generating series. The moment generating series defined by the numbers A_n is

$$M(x) = \sum_{n=0}^{\infty} A_n x^n = \sum_{k=0}^{\infty} A_{2k}x^{2k} + \sum_{k=0}^{\infty} A_{2k+1}x^{2k+1}$$

Applying the recursions $A_{2k} = 2A_{2k-1}$ and $A_{2k+1} = 2A_{2k} - c_k$ one gets

$$M(x) = \frac{1 - xW(x)}{1-2x},$$

where $W(x)$ stands for the moment generating series $\sum_{n=0}^{\infty} c_n x^{2n}$ of the standard Wigner law:

$$W(x) = \frac{1-\sqrt{1-4x^2}}{2x^2}$$

(see e.g. [3]). Solving the above equation for $M(x)$ one gets

$$M(x) = \frac{2x + \sqrt{1-4x^2} - 1}{2x - 4x^2},$$

which gives the Cauchy transform of the corresponding probability measure:

$$G(z) = \frac{1}{z} M\left(\frac{1}{z}\right) = \frac{1}{2}\left[\sqrt{\frac{z+2}{z-2}} - 1\right].$$

According to the Stieltjes inversion formula this determines the corresponding measure:

$$\frac{d\mu}{dx} = \lim_{y \downarrow 0}\left[-\frac{1}{\pi}\operatorname{Im} G(x+iy)\right] = \frac{1}{2\pi}\sqrt{\frac{2+x}{2-x}}\chi(x).$$

From these results one can check that

$$G(z) = \frac{1}{z - 1 - G_w(z)},$$

where

$$G_w(z) = \frac{z - \sqrt{z^2-4}}{2}$$

is the Cauchy transform of the standard Wigner law, having the continued fraction form

$$G_w(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{1}{\ddots}}}$$

(see e.g. [3]), so we obtain

$$G(z) = \cfrac{1}{z - 1 - \cfrac{1}{z - \cfrac{1}{z - \cfrac{1}{\ddots}}}},$$

which in turn gives the recursion for the orthogonal polynomials for μ:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x - 1, \\ xP_n(x) &= P_{n+1}(x) + P_{n-1}(x) \qquad (n \geq 1). \end{aligned}$$

We note that there is no probability distribution with moments equal to the numbers of (non-crossing) symmetric pair-partitions of n elements ($n = 0, 1, 2, \ldots$). A necessary condition for a sequence $\{a_i\}_{i=0,1,\ldots}$ of numbers to be the moment sequence of some probability measure is that the determinant

$$\begin{vmatrix} a_0 & a_1 & \dots & a_k \\ a_1 & a_2 & \dots & a_{k-1} \\ . & . & . & . \\ a_k & a_{k+1} & \dots & a_{2k} \end{vmatrix}$$

should be non-negative for every $k = 0, 1, \dots$ (see e.g. [1]). Calculating the corresponding 4×4 determinants we get negative numbers in both (crossing and non-crossing) cases. (In the non-crossing case the same follows from the fact that for the probability measure μ of Theorem 3.1, $\operatorname{supp}\mu \nsubseteq [0,\infty)$.)

THEOREM 3.3. *The moment sequence of the probability distribution with parameter λ ($\lambda > 0$) defined by*

$$\mu_\lambda = \begin{cases} \dfrac{1}{2\pi}\sqrt{\dfrac{(1+\lambda)+x}{(1+\lambda)-x}(1-\dfrac{(1-\lambda)^2}{x^2})}\chi(x)dx & \text{if } \lambda \ge 1, \\ (1-\lambda)\delta_0 + \dfrac{1}{2\pi}\sqrt{\dfrac{(1+\lambda)+x}{(1+\lambda)-x}(1-\dfrac{(1-\lambda)^2}{x^2})}\widetilde{\chi}(x)dx & \text{if } \lambda < 1, \end{cases}$$

where χ is the characteristic function of $[-1-\lambda, 1-\lambda]\cup[-1+\lambda, 1+\lambda]$ and $\widetilde{\chi}$ is the characteristic function of $[-1-\lambda, -1+\lambda]\cup[1-\lambda, 1+\lambda]$, is given by

$$m_n^{(\lambda)} = \sum_{i=1}^{n} A_n^{(i)}\lambda^i,$$

where $A_n^{(i)}$ is the number of non-crossing symmetric partitions of a set of n elements into i blocks.

P r o o f. First one has to derive a recursive formula for $A_n^{(i)}$. This can be done following the ideas of the proof of Proposition 2.1, arriving at

$$\begin{aligned} A_0^{(0)} &= A_n^{(1)} = A_n^{(n)} = 1 && \text{if } n \ge 1, \\ A_n^{(k)} &= 0 && \text{if } 0 \le n < k \text{ or } k = 0 < n, \\ A_{2n+1}^{(2k)} &= A_{2n}^{(2k-1)} + A_{2n}^{(2k)} - Q_n^{(k)} && \text{if } 1 \le k \le n, \\ A_n^{(k)} &= A_{n-1}^{(k-1)} + A_{n-1}^{(k)} && \text{if } 1 \le k \le n \text{ and } (n \text{ is even or } k \text{ is odd}), \end{aligned}$$

where $Q_n^{(k)}$ means the number of non-crossing partitions of a set of n elements into k blocks.

From this one easily gets the following recursion for the moments:

$$\begin{aligned} m_{2n}^{(\lambda)} &= (1+\lambda)m_{2n-1}^{(\lambda)} && \text{for } n = 1, 2, \dots, \\ m_{2n+1}^{(\lambda)} &= (1+\lambda)m_{2n}^{(\lambda)} - \widetilde{m}_n^{(\lambda)} && \text{for } n = 0, 1, \dots, \\ m_0^{(\lambda)} &= 1, \end{aligned}$$

where $\widetilde{m}_n^{(\lambda)}$ is the nth moment of the free Poisson law with parameter λ^2 (see [5]), that is,

$$\widetilde{m}_n^{(\lambda)} = \sum_{i=1}^{n} Q_n^{(i)} \lambda^{2i}.$$

For the moment generating series defined by the numbers $m_n^{(\lambda)}$ one gets

$$M_\lambda(x) = \sum_{n=0}^{\infty} m_n^{(\lambda)} x^n = \sum_{k=0}^{\infty} m_{2k}^{(\lambda)} x^{2k} + \sum_{k=0}^{\infty} m_{2k+1}^{(\lambda)} x^{2k+1}.$$

(The moment generating series of the free Poisson law or the classical Poisson law with parameter λ clearly majorize this power series. As those series are absolutely convergent for sufficiently small values of x, the same holds for $M_\lambda(x)$.)

Applying the above recursions, passing to the Cauchy transform

$$G_\lambda(z) = \frac{1}{z} M_\lambda\left(\frac{1}{z}\right)$$

and using the known fact that the Cauchy transform of the free Poisson law with parameter λ^2 is

$$\widetilde{G}_{\lambda^2}(z) = \frac{z + (1-\lambda^2) - \sqrt{(z-1-\lambda^2)^2 - 4\lambda^2}}{2z}$$

(see e.g. [3] or [7]) one gets

$$G_\lambda(z) = \frac{1}{2}\left[\frac{1-\lambda}{z} - 1 + \sqrt{\frac{z+(1+\lambda)}{z-(1+\lambda)}(1 - \frac{(1-\lambda)^2}{z^2})}\right].$$

Applying the Stieltjes inversion formula, after some easy calculation one gets μ_λ as stated above. ∎

REMARK 3.4. Note that if $\lambda = 1$ then μ_λ coincides with the probability distribution μ given in Proposition 3.1.

Calculating the orthogonal polynomials for μ_λ one gets

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x - \lambda, \\ xP_n(x) &= P_{n+1}(x) + (-1)^n(\lambda - 1)P_n(x) + \lambda P_{n-1}(x) \qquad (n \geq 1). \end{aligned}$$

This can be shown by verifying the equality

$$G_\lambda(z) = \frac{1}{z - \lambda - F_\lambda(z)},$$

where $F_\lambda(z)$ is defined by the formula

$$F_\lambda(z) = \cfrac{\lambda}{z + (\lambda - 1) - \cfrac{\lambda}{z - (\lambda - 1) - F_\lambda(z)}}.$$

(Computation was carried out by MAPLE.)

The probability measure μ_λ given in Proposition 3.3 can be considered as the "symmetric analogue" of the free Poisson law of mean λ. We have seen that the "symmetric analogue" of the Wigner law and that of the Gaussian law do not exist (as probability distributions). Finally we have to mention that we could not derive a "symmetric analogue" of the Poisson law of mean λ (whose nth moment would be a polynomial of λ in which the coefficient of λ^i is the number of all symmetric partitions of a set of n elements into i blocks), neither could we prove that it does not exist.

REFERENCES

[1] N. I. Akhiezer, *Classical Moment Problem*, Gos. Izdat. Fiz.-Mat. Liter., Moscow, 1961 (in Russian).

[2] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, 1976.

[3] F. Hiai and D. Petz, *The semi-circle law, free random variables and entropy*, preprint.

[4] G. Kreweras, *Sur les partitions non-croisées d'un cycle*, Discrete Math. 1 (1972), 333–350.

[5] F. Oravecz and D. Petz, *On the eigenvalue distribution of some symmetric random matrices*, Acta Sci. Math. (Szeged) 63 (1997), 383–395.

[6] J. Touchard, *Sur un problème de configurations et sur les fractions continues*, Canad. J. Math. 4 (1952), 2–25.

[7] D. Voiculescu, K. Dykema and A. Nica, *Free Random Variables*, CRM Monogr. Ser. 1, Amer. Math. Soc., Providence, RI, 1992.

Rényi Alfréd Institute of Mathematics
Hungarian Academy of Sciences
Reáltanoda u. 13-15
H-1053 Budapest, Hungary
E-mail: oravecz@math-inst.hu

Received 13 October 1998;
revised version 12 September 1999 (3643)