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H^1-BMO DUALITY ON GRAPHS

BY

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Abstract. On graphs satisfying the doubling property and the Poincaré inequality, we prove that the space $H^1_{\max}$ is equal to H^1_{at}, and therefore that its dual is BMO. We also prove the atomic decomposition for $H^p_{\max}$ for $p \leq 1$ close enough to 1.

I. INTRODUCTION

In [RUS], Theorem 1, it is proved that, on any complete Riemannian manifold M satisfying the doubling property and the Poincaré inequality, the spaces $H^1_{\max}(M)$ and $H^1_{\mathrm{at}}(M)$ are equal and their dual is $\mathrm{BMO}(M)$. Moreover, in Theorem 2 of the same paper, the atomic decomposition for $H^p_{\max}$ is obtained for all p sufficiently close to 1. Those results are closely related to the existence of estimates for the heat kernel h_t (or the Poisson kernel p_t), and essentially an upper estimate for the oscillation of h_t (or p_t).

In the present paper, we give the analogous results in the discrete setting of graphs. We rely on estimates recently obtained for Markov chains on graphs. In [DEL2], T. Delmotte shows that, if Γ is a graph satisfying the doubling property and the Poincaré inequality, and p is a Markov kernel with suitable assumptions, then a Gaussian upper bound and a Gaussian lower bound hold for the nth iteration $p_n(x,y)$ of p (the same result is obtained in a different way by P. Auscher and T. Coulhon in [AC]). Moreover, $p_n(x,y)$ is Hölderian with respect to the first variable. Those results allow us to prove that, as in the case of manifolds, $H^1_{\max}(\Gamma) = H^1_{\mathrm{at}}(\Gamma)$ and their dual is $\mathrm{BMO}(\Gamma)$. We are also able to get the atomic decomposition for $H^p_{\max}$ when p is sufficiently close to 1.

We state precisely our main result. The following presentation is borrowed from [DEL2]. Let Γ be an infinite connected graph, endowed with its natural metric and a symmetric weight $\mu_{xy} = \mu_{yx}$ on $\Gamma \times \Gamma$. Assume that x and y are neighbors if and only if $\mu_{xy} \neq 0$. Define, for every $x \in \Gamma$,

$$m(x) = \sum_{y \sim x} \mu_{xy}.$$

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For every real $r \geq 0$, the ball $B(x,r)$ is defined as follows:

$$B(x,r) = \{y \in \Gamma : d(y,x) \leq r\},$$

and, if A is a subset of Γ, its volume is

$$V(A) = \sum_{x \in A} m(x).$$

When A is a ball $B(x,r)$, $V(A)$ will be denoted by $V(x,r)$.

For any $p > 0$, we denote by $L^p(\Gamma)$ the set of all complex-valued functions f defined on Γ and satisfying

$$\sum_x |f(x)|^p m(x) < +\infty.$$

When $f \in L^p(\Gamma)$, set

$$\|f\|_p = \Big[\sum_x |f(x)|^p m(x)\Big]^{1/p}.$$

The graph Γ is said to satisfy the *doubling property* if there exists a constant $C > 0$ such that, for every $x \in \Gamma$ and $r > 0$,

$$V(x,2r) \leq CV(x,r). \tag{1}$$

The graph Γ is said to satisfy the *Poincaré inequality* if there exists a constant $C > 0$ such that, for every function f from Γ to $\mathbb{R}$, every $x_0 \in \Gamma$ and $r > 0$, one has

$$\sum_{x \in B(x_0,r)} m(x)|f(x) - f_B|^2 \leq Cr^2 \sum_{x,y \in B(x_0,2r)} \mu_{xy}|f(x) - f(y)|^2, \tag{2}$$

where

$$f_B = \frac{1}{V(x_0,r)} \sum_{x \in B(x_0,r)} m(x)f(x).$$

Finally, one says that Γ satisfies $\Delta^*(\alpha)$ for $\alpha > 0$ if

$$x \sim y \Rightarrow \mu_{xy} \geq \alpha m(x). \tag{3}$$

One may then consider on Γ a discrete-time Markov kernel. Set

$$p(x,y) = \mu_{xy}/m(x)$$

and define the iterated kernel p_n as follows:

$$p_0(x,y) = \delta(x,y), \qquad p_n(x,y) = \sum_z p(x,z)p_{n-1}(z,y).$$

We define $H^p_{\max}(\Gamma)$ when $0 < p \leq 1$. If $f \in L^1_{\mathrm{loc}}(\Gamma)$, define

$$P^n f(x) = \sum_\Gamma p_n(x,y)f(y)$$

for any $n \in \mathbb{N}$, $x \in \Gamma$ and $f^+(x) = \sup_{n \in \mathbb{N}} |P^n f(x)|$.

If $f \in L^1(\Gamma)$ satisfies $\sum f(x)m(x) = 0$, we say that $f \in H^1_{\max}(\Gamma)$ if $f^+ \in L^1(\Gamma)$, and we define

$$\|f\|_{H^1_{\max}} = \|f^+\|_1.$$

Notice that $\|\cdot\|_{H^1_{\max}}$ is a norm on $H^1_{\max}$, and $H^1_{\max}$ is a Banach space.

When $p < 1$, $H^p_{\max}(\Gamma)$ is defined as a space of distributions. For any $\alpha > 0$, define the Hölder space $\mathcal{L}_\alpha$ as being the space of all functions f on Γ such that there exists $C > 0$ satisfying, for any x and $y \in \Gamma$ and any ball B containing both x and y,

$$|f(x) - f(y)| \leq C[V(B)]^\alpha. \tag{4}$$

When (4) holds, define $\|f\|^{(\alpha)}$ to be the infimum of all constants $C > 0$ satisfying (4).

If $p \in]0, 1[$ is sufficiently close to 1 and f is a continuous linear form on $\mathcal{L}_{1/p-1}$, one may define, for any $n \in \mathbb{N}$ and any $x \in \Gamma$,

$$P^n f(x) = \langle f, p_n(x, \cdot)\rangle \quad \text{and} \quad f^+(x) = \sup_{n\in\mathbb{N}} |P^n f(x)|.$$

Indeed, the estimates for p_n which will be given later on (Lemma 5) ensure that, under the assumptions (1)–(3) and an extra assumption (see Theorem 1 below), for each fixed $x \in \Gamma$, $p_n(x, \cdot) \in \mathcal{L}_{1/p-1}$ for p close enough to 1. When $f \in \mathcal{L}^*_{1/p-1}$, we say that $f \in H^p_{\max}(\Gamma)$ if $f^+ \in L^p$, and we set

$$\|f\|_{H^p_{\max}} = \|f^+\|_p.$$

This is not a norm, but the function

$$d(f, g) = \|f - g\|^p_{H^p_{\max}}$$

is a distance on $H^p_{\max}$. Moreover, the metric space $(H^p_{\max}, d)$ is complete.

We now give the definition of H^p_{at} for $p \leq 1$. Whenever $p \leq 1$, a function a is said to be a *p-atom* if $\sum a(x)m(x) = 0$, a is supported in a ball $B = B(x_0, r_0)$ and $\|a\|_\infty \leq (1/V(B))^{1/p}$.

A function $f \in L^1(\Gamma)$ is said to be in H^1_{at} if there exist a sequence $(\lambda_n)_{n\geq1} \in l^1$ of numbers and a sequence $(a_n)_{n\geq1}$ of atoms such that

$$f = \sum_{n=1}^{+\infty} \lambda_n a_n,$$

where the convergence is to be understood in the sense of L^1; we let

$$\|f\|_{H^1_{\mathrm{at}}} = \inf \sum_{n=1}^{+\infty} |\lambda_n|$$

where the infimum is taken over all such decompositions.

The definition of H^p_{at} for $p < 1$ is a bit more complicated. When $p < 1$, a p-atom a defines a bounded linear form on $\mathcal{L}_{1/p-1}(\Gamma)$ with norm ≤ 1. A linear form f on $\mathcal{L}_{1/p-1}(\Gamma)$ is said to be in H^p_{at} if there exist a sequence $(\lambda_n)_{n\geq 1}\in l^p$ of numbers and a sequence $(a_n)_{n\geq 1}$ of atoms such that

$$f=\sum_{n=1}^{+\infty}\lambda_n a_n,$$

with convergence in the sense of $(\mathcal{L}_{1/p-1})^*$; we let

$$\|f\|_{H^p_{\mathrm{at}}}=\inf\Big(\sum_{n=1}^{+\infty}|\lambda_n|^p\Big)$$

where the infimum is taken over all such decompositions. This is not a norm, but the function

$$d_p(f,g)=\|f-g\|_p$$

is a distance on H^p_{at}.

When $f\in L^1_{\mathrm{loc}}(\Gamma)$, we say that $f\in\mathrm{BMO}$ if

$$\sup_B\frac{1}{V(B)}\sum_B|f(x)-f_B|m(x)<+\infty,$$

the supremum being taken over all balls of Γ; we let

$$\|f\|_{\mathrm{BMO}}=\sup_B\frac{1}{V(B)}\sum_B|f(x)-f_B|m(x).$$

Whenever f belongs to BMO, one has $\|f\|_{\mathrm{BMO}}=0$ if and only if f is constant almost everywhere. For $f,g\in\mathrm{BMO}$, we say that $f\sim g$ if $f-g$ is constant almost everywhere. Thus, one obtains a set of equivalence classes, which is again denoted by BMO. The norm of an equivalence class is defined as $\|f\|_{\mathrm{BMO}}$ where f is any of its members, and BMO, equipped with that norm, is a Banach space.

Here is the main result of this paper:

THEOREM 1. *Let Γ be an infinite graph endowed with a symmetric weight, and consider the corresponding Markov kernel $p_n(x,y)$. Assume that Γ satisfies the doubling property, the Poincaré inequality and $\Delta^*(\alpha)$ for a certain $\alpha>0$. Assume also that there exists $\varrho\in\mathbb{N}$ such that, for all $x\in\Gamma$, there exists a path of length $2\varrho+1$ which starts from x and returns to x, i.e. a finite sequence of vertices $(x_i)_{0\leq i\leq 2\varrho+1}\subset\Gamma$ such that for all i, $x_i\sim x_{i+1}$ and $x_0=x_{2\varrho+1}=x$. Then there exists $p_0\in\,]0,1[$ such that, for any $p\in\,]p_0,1]$, $H^p_{\max}=H^p_{\mathrm{at}}$. As a consequence, the dual of $H^1_{\max}(\Gamma)$ is equal to* $\mathrm{BMO}(\Gamma)$, *and, for any $p\in\,]p_0,1[$, the dual of $H^p_{\max}(\Gamma)$ is $\mathcal{L}_{1/p-1}$.*

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II. THE ATOMIC DECOMPOSITION FOR $H^p_{\max}$

In the present section, we prove Theorem 1, which is the discrete analogue of Theorems 1 and 2 of [RUS]. Adapting continuous methods to this discrete setting creates technical difficulties (especially with the area integral). That is the reason why the strategy of the proof will be completely different. The inclusion $H^1_{\mathrm{at}} \subset H^1_{\max}$ is easily obtained as in Theorem 1 of [RUS]. The converse inclusion and the atomic decomposition for H^p with $p < 1$ will be obtained by methods which are very much inspired by [UCH].

More precisely, the most natural way to obtain an atomic decomposition for $H^p_{\max}$ is to use a variant of the Calderón–Zygmund decomposition: see, for instance, [ST] in the Euclidean setting. Roughly speaking, when $f \in H^p_{\max}$, one writes, for any $j \in \mathbb{Z}$, the Calderón–Zygmund decomposition at level 2^j: $f = g^j + b^j$. Then one easily sees that $g^j \to f$ in $H^p_{\max}$ as $j \to +\infty$, whereas $g^j \to 0$ uniformly as $j \to -\infty$. Therefore, $f = \sum_j [g^{j+1} - g^j]$, and, after some manipulations using the properties of the Calderón–Zygmund decomposition, one finally obtains an atomic decomposition for f. Such ideas are used by Coifman of [COI] to get the atomic decomposition for $H^p(\mathbb{R})$, by Latter of [LAT] for the analogous result in $\mathbb{R}^n$, and by Macías and Segovia of [MS2] in the general context of a normal space (which is, approximately, a space with linear volume growth).

The point is that, in those papers, the $H^p_{\max}$ spaces are defined by means of suitable test functions. Namely, in [MS2], for any $f \in L^1(X)$ where X is a normal space, the authors define, for any $x \in X$,

$$f^*(x) = \sup \Big\{ \Big| \int f(y)\phi(y)\, d\mu(y) \Big| \Big\},$$

where the supremum is taken over all functions $\phi : X \to \mathbb{R}_+$ satisfying an appropriate Hölder regularity condition. Then they say that $f \in H^p_{\max}$ if $f^* \in L^p(X)$. When one considers a maximal H^p space defined by means of a kernel, as we do in the present paper, the methods developed by Coifman, Latter, Macías and Segovia cannot be applied directly. That is why, for the proof of the inclusion $H^p_{\max} \subset H^p_{\mathrm{at}}$, we consider (following Uchiyama [UCH]) another maximal H^p space, defined in [MS2]. Then we prove that our $H^p_{\max}$ space (defined by means of the Markov kernel) is included in that maximal H^p space, for which it is well known (see [MS2], Theorem 4.13) that one has an atomic decomposition. Thus, the inclusion $H^p_{\max} \subset H^p_{\mathrm{at}}$ is proved.

Once this is done, one finds that $H^p_{\mathrm{at}}(\Gamma) = H^p_{\max}(\Gamma)$. Since Γ is a space of homogeneous type, Theorem B of [CW] ensures that the dual of $H^1_{\mathrm{at}}(\Gamma)$ is $\mathrm{BMO}(\Gamma)$ and the dual of $H^p_{\mathrm{at}}(\Gamma)$ is $\mathcal{L}_{1/p-1}(\Gamma)$, which implies that the dual of $H^1_{\max}(\Gamma)$ is $\mathrm{BMO}(\Gamma)$ and the dual of $H^p_{\max}(\Gamma)$ is $\mathcal{L}_{1/p-1}(\Gamma)$. Therefore, the proof of Theorem 1 will be complete provided that one shows $H^p_{\mathrm{at}} \subset H^p_{\max}$ and $H^p_{\max} \subset H^p_{\mathrm{at}}$ for $p \in \,]p_0, 1]$.

It is important to observe that the techniques developed for the proof of Theorem 1 do not depend on the graph structure: they only use the doubling property (1) and the estimates for the kernel. In other words, the results of the present paper hold in the general context of a space of homogeneous type endowed with a kernel satisfying a Gaussian upper bound, an on-diagonal lower bound and a Hölder regularity assumption. Consequently, they give an alternative proof of the equality $H^1_{\max,H} = H^1_{\mathrm{at}}$ and of the atomic decomposition for $H^p_{\max,H}$ on a Riemannian manifold satisfying suitable assumptions, which is part of Theorems 1 and 2 of [RUS]. However, the estimates for the Poisson kernel do not allow one to apply such techniques, and that is the reason why, in [RUS], one uses other techniques to get results about $H^p_{\max,P}$.

Let us point out that, to prove that $H^1_{\mathrm{at}} = H^1_{\max}$, it is possible to prove a representation theorem for BMO functions, using Carleson's [CAR] ideas, as done by Y. Meyer [MEY]. Precisely, the following result is true:

THEOREM 2. *Let Γ be a graph satisfying the assumptions of Theorem* 1. *Then there exists $C_1 > 0$ such that, for all measurable functions $k : \Gamma \to \mathbb{N}^*$ and all functions $b_1, b_2 \in L^\infty$, the function*

$$f(x) = b_1(x) + \sum_{y\in\Gamma} p_{k(y)}(x,y)b_2(y)$$

belongs to BMO *and satisfies $\|f\|_{\mathrm{BMO}} \le C_1[\|b_1\|_\infty + \|b_2\|_\infty]$. Conversely, there exists $C_2 > 0$ such that every $f \in$ BMO has a representation as above with $\|b_1\|_\infty + \|b_2\|_\infty \le C_2\|f\|_{\mathrm{BMO}}$.*

This theorem is interesting in its own right, and provides a proof of the equality $H^1_{\max} = H^1_{\mathrm{at}}$. However, it does not give any result about H^p for $p < 1$.

The paper is organized as follows. We first give estimates for p_n, particularly a Hölder regularity result, from which we deduce that $H^p_{\mathrm{at}} \subset H^p_{\max}$. Then we prove a theorem which allows us to compare $\|f^+\|_p$ with $\|f^*\|_p$ (where f^* is the maximal function considered by Macías and Segovia [MS2]). This last theorem gives the atomic decomposition for $H^p_{\max}$, $p \le 1$, thanks to Macías and Segovia's work [MS2].

1. Kernel estimates. In the present section, we will need the following definition. We say that Γ satisfies $\Delta(\alpha)$ if

$$x \sim y \Rightarrow \mu_{xy} \ge \alpha m(x), \quad \forall x \in \Gamma, \quad \mu_{xx} \ge \alpha m(x). \tag{5}$$

In other words, Γ satisfies $\Delta^*(\alpha)$ and, for all $x \in \Gamma$, $x \sim x$.

Recall that, when (1), (2) and (5) hold, one has the following estimate for p_k:

THEOREM 3. *Let Γ satisfy the doubling property, the Poincaré inequality and $\Delta(\alpha)$ for $\alpha > 0$. Then there exist $c_1, C_1, c_2, C_2 > 0$ such that*

$$d(x,y) \leq k \Rightarrow \frac{c_1 m(y)}{V(x,\sqrt{k})} e^{-C_1 d(x,y)^2/k} \leq p_k(x,y) \leq \frac{C_2 m(y)}{V(x,\sqrt{k})} e^{-c_2 d(x,y)^2/k}.$$

This theorem is shown by T. Delmotte in [DEL2], Theorem 1.7 (see also [AC], Theorem 2.5).

As a consequence of Theorem 3 and of Proposition 4.1 in [DEL2], one gets, under the same assumptions, the following estimate:

LEMMA 4. *Let Γ satisfy the doubling property, the Poincaré inequality and $\Delta(\alpha)$ for $\alpha > 0$. Then there exist $C_3, c_3 > 0$ and $h \in]0,1[$ such that, for any $k \in \mathbb{N}$ and $x, y, y_0 \in \Gamma$ such that $d(y_0, y) \leq \sqrt{k}$,*

$$|p_k(y,x) - p_k(y_0,x)| \leq C_3 \left[\frac{d(y,y_0)}{\sqrt{k}}\right]^h \frac{m(x)}{V(x,\sqrt{k})} e^{-c_3 d^2(x,y_0)/k}.$$

Proof. Assume first that $d(y,y_0) \leq \frac{1}{2}\sqrt{k}$ and $x \in \Gamma$. Proposition 4.1 of [DEL2] may be applied to $u(k,z) = p_k(z,x)$, with $R \sim \frac{1}{2}\sqrt{k}$ and $n_0 \sim \frac{5}{4}k$. Since $y \in B(y_0, R)$ and $k \in \mathbb{Z} \cap [n_0 - R^2, n_0]$, one gets

$$|p_k(y,x) - p_k(y_0,x)| \leq C \left[\frac{d(y,y_0)}{R}\right]^h \sup_Q p_l(z,x)$$

where $Q = (\mathbb{Z} \cap [n_0 - 2R^2, n_0]) \times B(y_0, 2R)$. But, thanks to Theorem 3, when $n_0 - 2R^2 \leq l \leq n_0$ and $z \in B(y_0, 2R)$,

$$p_l(z,x) \leq \frac{C_2 m(x)}{V(x,\sqrt{l})} e^{-c_2 d(x,z)^2/l} \leq \frac{C_2 m(x)}{V(x,\sqrt{n_0 - 2R^2})} e^{-c_2 d(x,z)^2/n_0}.$$

One has

$$-\frac{d(x,z)^2}{n_0} \leq -\frac{d(x,y_0)^2}{2n_0} + \frac{d(y_0,z)^2}{n_0} \leq -\frac{d(x,y_0)^2}{2n_0} + 4,$$

so that

$$p_l(z,x) \leq \frac{C_3 m(x)}{V(x,\sqrt{k})} e^{-c_2 d^2(y_0,x)/k}.$$

It follows that

$$|p_k(y,x) - p_k(y_0,x)| \leq C_4 \left[\frac{d(y,y_0)}{\sqrt{k}}\right]^h \frac{m(x)}{V(x,\sqrt{k})} e^{-c_2 d^2(y_0,x)/k}. \tag{6}$$

Finally, if $d(y,y_0) \leq \sqrt{k}$, consider a point y_1 such that $d(y,y_1) \leq \frac{1}{2}\sqrt{k}$ and $d(y_1,y_2) \leq \frac{1}{2}\sqrt{k}$, and apply (6) to $|p_k(y,x) - p_k(y_1,x)|$ and to $|p_k(y_1,x) - p_k(y_0,x)|$. ■

We now deduce from Lemma 4 some estimates for $p_k(x,y)$ when Γ satisfies the assumptions of Theorem 1, which are weaker than the ones of Theorem 3.

LEMMA 5. *Assume that Γ satisfies the doubling property, the Poincaré inequality and $\Delta^*(\alpha)$ for a certain $\alpha>0$. Assume also that there exists $\varrho\in\mathbb{N}$ such that, for all $x\in\Gamma$, there exists a path of length $2\varrho+1$ which starts from x and goes back to x, i.e. a finite sequence of vertices $(x_i)_{0\le i\le 2\varrho+1}\subset\Gamma$ such that for all i, $x_i\sim x_{i+1}$ and $x_0=x_{2\varrho+1}=x$. Then there exist $c_1',C_1',c_2',C_2'>0$ such that, for all $x,y\in\Gamma$ and $k\in\mathbb{N}$,*

$$p_k(x,y)\le\frac{C_2'm(y)}{V(x,\sqrt{k})}e^{-c_2'd(x,y)^2/k},$$

and, when $d(x,y)\le 2k$,

$$\frac{C_1'm(y)}{V(x,\sqrt{k})}e^{-c_1'd(x,y)^2/k}\le p_{2k}(x,y).$$

Moreover, there exist $C_3',c_3'>0$ and $h\in\,]0,1[$ such that, for any $k\in\mathbb{N}$ and $x,y,y_0\in\Gamma$ such that $d(y_0,y)\le\sqrt{k}$,

$$|p_k(y,x)-p_k(y_0,x)|\le C_3'\left[\frac{d(y,y_0)}{\sqrt{k}}\right]^h\frac{m(x)}{V(x,\sqrt{k})}e^{-c_3'd(x,y_0)^2/k}.$$

Proof. Following [DEL1], p. 122, one considers the iterated graph $(\Gamma,\mu^{(2)})$ where

$$\mu_{xy}^{(2)}=\sum_{z\in\Gamma}\frac{\mu_{xz}\mu_{zy}}{m(z)}.$$

The corresponding kernel $p_k^{(2)}(x,y)$ satisfies $p_k^{(2)}(x,y)=p_{2k}(x,y)$, whereas the corresponding weights $m^{(2)}(x)$ satisfy $m^{(2)}(x)=m(x)$. It is easy to check that, under the assumptions of Lemma 5, the induced distance $d^{(2)}$ satisfies

$$d(x,y)/2\le d^{(2)}(x,y)\le(2\varrho+2)d(x,y). \tag{7}$$

As a consequence, under the assumptions of Lemma 5, $(\Gamma,\mu^{(2)})$ satisfies the doubling property (see [DEL1], p. 123, Proposition 7.5), the Poincaré inequality (see [DEL1], p. 123, Proposition 7.6) and the condition $\Delta(\alpha^2)$ (see [DEL1], p. 125). Applying Theorem 3, Lemma 4 and (7), one gets the first two assertions of Lemma 5 (see also [DEL1], p. 124, Théorème 7.7) and the third one when k is even. When $k=2l+1$ and $d(y_0,y)\le\sqrt{k}$, one has

$$\begin{aligned}|p_k(y,x)-p_k(y_0,x)|&\le\sum_{u\sim x}|p_{2l}(y,u)-p_{2l}(y_0,u)|p(u,x)\\&\le\sum_{u\sim x}C_3\left[\frac{d(y,y_0)}{\sqrt{2l}}\right]^h\frac{m(u)}{V(u,\sqrt{2l})}e^{-c_3d(u,y_0)^2/(2l)}p(u,x)\end{aligned}$$

$$\leq C_3'\left[\frac{d(y,y_0)}{\sqrt{k}}\right]^h \frac{m(x)}{V(x,\sqrt{k})}e^{-c_3'd(x,y_0)^2/k}.$$

In the last line, one uses the doubling property and the fact that $d(u,x) \leq 1$. ∎

2. Reduction to the case of a normal space. The proof of the inclusion $H^p_{\mathrm{at}} \subset H^p_{\max}$ is exactly analogous to the proof of $H^p_{\mathrm{at}} \subset H^p_{\max,H}$ in [RUS], II, 2, and we do not repeat it. Note that it uses the Hölder regularity of p_n (Lemma 5), which is due to the Poincaré inequality.

To prove the converse inclusion, as well as $H^p_{\max} \subset H^p_{\mathrm{at}}$ for $p < 1$ sufficiently close to 1, we need a theorem which is very close to Uchiyama's result in [UCH] (Theorem 1′). That theorem deals with a particular class of spaces of homogeneous type, namely the normal spaces. Via an appropriate reduction, the inclusion $H^p_{\max} \subset H^p_{\mathrm{at}}$ for $p \in \,]p_0, 1]$ is an easy consequence of that result, which we are going to state now.

Let X be a set, equipped with a non-negative quasi-distance $d : X\times X \to \mathbb{R}_+$. Precisely, d is symmetric, $d(x, y) = 0 \Leftrightarrow x = y$, and there exists $A > 0$ such that, for any $x, y, z \in X$,

$$d(x,y) \leq A[d(x,z) + d(z,y)]. \tag{8}$$

Let μ be a σ-finite measure on X such that $\mu(X) = +\infty$. Assume that (X, d, μ) is a *normal space*, which means that there exists $\kappa > 0$ such that, for any $x \in X$ and any $r > 0$,

$$\begin{aligned} &V(x,r) \geq A^{-1}r,\\ &V(x,r) \leq Ar \quad \text{if } r \geq \kappa\mu(\{x\}),\\ &B(x,r) = \{x\} \quad \text{if } r < \kappa\mu(\{x\}). \end{aligned} \tag{9}$$

Assume that $K : \mathbb{R}_+^* \times X \times X \to \mathbb{R}_+$ is a symmetric measurable function and that, for any $\beta > 0$, there exists C_β such that, for $t > 0$ and $x, y \in X$,

$$\begin{aligned} &K(t,x,y) \leq \frac{C_\beta}{V(x,t)}\left[1+\frac{d(x,y)}{t}\right]^{-1-\beta},\\ &K(t,x,x) \geq \frac{A^{-1}}{V(x,t)}. \end{aligned} \tag{10}$$

Assume also that there exist $\gamma > 0$ and $C > 0$ such that, for all $t > 0$ and $x, y, z \in X$ satisfying $d(y, z) \leq (t + d(x, y))/(4A)$,

$$|K(t,x,y) - K(t,x,z)| \leq \frac{C}{V(x,t)}\left[\frac{d(y,z)}{t}\right]^\gamma\left[1+\frac{d(x,y)}{t}\right]^{-1-2\gamma}. \tag{11}$$

Notice that, under the assumptions (10) and (11), there exist $C_1, C_2 > 0$

such that, for any $t>0$ and $x,y\in X$ satisfying $d(x,y)\le C_2t$,

$$K(t,x,y)\ge \frac{1}{C_1V(x,t)}.$$

For any $x\in X$, we say that a function $\phi : X\to \mathbb{R}_+$ belongs to the class $T_\gamma(x)$ if it is supported in a ball $B(x,r)$ with $r\ge \kappa\mu(\{x\})$, $\|\phi\|_\infty\le 1/r$ and, for any $y,z\in X$,

$$|\phi(y)-\phi(z)|\le [d(y,z)/r]^\gamma.$$

If $f\in \mathcal{L}^*_\gamma(X)$, set, for $r>0$ and $x\in X$,

$$Kf(r,x)=\langle f,K_r(x,\cdot)\rangle,\qquad f^+(x)=\sup_{r>0}|Kf(r,x)|.$$

Define also

$$f^*(x)=\sup\{|\langle f,\phi\rangle| : \phi\in T_\gamma(x)\}.$$

Theorem 6. *Let* (X,d,μ) *be a normal space equipped with a kernel* K *satisfying* (10) *and* (11). *Then there exists* $p_0\in\,]0,1[$ *such that*, *for any* $p\in\,]p_0,1]$, *there exists* C_p *such that*, *for any* $f\in\mathcal{L}^*_\gamma$,

$$\|f^*\|_p\le C_p\|f^+\|_p.$$

Before proving that result, we explain how to use it to prove that, under the assumptions of Theorem 1, one has $H^p_{\max}(\Gamma)\subset H^p_{\rm at}(\Gamma)$ for $p\in\,]p_0,1]$.

The results of [MS1] show that there exists a quasidistance δ on Γ such that (Γ,δ,μ) satisfies the assumptions of Theorem 6. Moreover, for any $p\in\,]0,1]$, the p-atoms for d and for δ coincide, and therefore, the same is true for the $H^p_{\rm at}$ spaces defined with respect to d and to δ.

If one defines, for all $t>0$ and $x,y\in\Gamma$,

$$K(t,x,y)=p_{2n^2}(x,y)/m(y),$$

where $n=\inf\{p\in\mathbb{N} : V(x,p)\ge t\}$, then it is easy to check that K also satisfies the assumptions of Theorem 6. (See an analogous reduction in [SC], Section 4, p. 322, in the setting of Lie groups.)

Apply Theorem 6 with (X,δ,μ,K): there exists $p_0\in\,]0,1[$ such that the conclusion of Theorem 6 holds. Let $p\in\,]p_0,1]$, and consider $f\in H^p_{\max}(\Gamma)$ defined with respect to the kernel p. The definitions of K and of $H^p_{\max}$ show that f also belongs to $H^p_{\max}$ defined with respect to K. Theorem 6 proves that $f^*\in L^p(\Gamma)$, and Theorem 4.13 in [MS2] ensures that $f\in H^p_{\rm at}$ defined with respect to δ, which coincides with $H^p_{\rm at}$ defined with respect to d. Thus, $f\in H^p_{\rm at}(\Gamma)$ and the inclusion $H^p_{\max}\subset H^p_{\rm at}$ is proved. As was explained at the beginning of the present part of the paper, the duality of $H^1_{\max}(\Gamma)$ and $\mathrm{BMO}(\Gamma)$ and the one of $H^p_{\max}(\Gamma)$ and $\mathcal{L}_{1/p-1}(\Gamma)$ follows, and the proof of Theorem 1 is complete. ∎

3. Proof of the fundamental theorem. The rest of this paper is devoted to a self-contained proof of Theorem 6. Note first that it is a bit more general than Theorem 1′ in [UCH], p. 586. Indeed, the assumptions on the volume in Theorem 1′ of [UCH] say that $V(x,r) \le Ar$ for any $x \in X$ and $r > 0$, which, in particular, implies that $\mu(\{x\}) = 0$ for all $x \in X$ and excludes precisely the case of graphs.

We follow Uchiyama's methods in [UCH], modifying some points, in particular the statement and proof of Lemma 1 of [UCH].

We claim that it is enough to prove that there exist $p_0 \in]0,1[$ and $C > 0$ such that, for any $f \in H^p_{\max}$ and $x_0 \in X$,

$$f^*(x_0) \le CM[(f^+)^{p_0}]^{1/p_0}(x_0), \tag{12}$$

where M denotes the Hardy–Littlewood maximal function. Indeed, assume that (12) is proved. Then, for any $p > p_0$, writing $g = (f^+)^{p_0}$, we get

$$\|f^*\|_p \le C\|Mg\|_{p/p_0}^{1/p_0} \le C'\|g\|_{p/p_0}^{1/p_0} = C'\|f^+\|_p,$$

which is the conclusion of Theorem 6 (the second inequality holds because $p/p_0 > 1$). Therefore, we turn to the proof of (12).

One has to show that, for any function ϕ supported in $B(x_0, r_0)$ with $r_0 \ge \kappa\mu(\{x_0\})$, such that, for any $x, y \in X$,

$$|\phi(x) - \phi(y)| \le 1/r_0^\gamma$$

and $\|\phi\|_\infty \le 1$, one has

$$\Big|\int f\phi\, d\mu\Big|/r_0 \le CM[(f^+)^{p_0}]^{1/p_0}(x_0), \tag{13}$$

where $C > 0$ is independent of f, ϕ, x_0, r_0.

It is sufficient to show (13) when $r_0 = 1$. Indeed, if it is proved in that case, consider the quasidistance $d' = d/r_0$, the measure $\mu' = \mu/r_0$ and the kernel $K' = r_0K(t/r_0, x, y)$. Then (X, d', μ', K') satisfies the same assumptions as (X, d, μ, K) with the same constants, and an elementary computation proves that (13) holds.

From now on, assume that $r_0 = 1$. For any $x \in X$, set

$$d(x) = 1 + d(x_0, x).$$

We will make use of the following lemma:

Lemma 7. *Let g be a non-negative function on X and $t < (2A)^{-5}$. Then there exists a sequence $(x_j)_{j\in\mathbb{N}} \subset X$ and constants $C_i > 0$ for $i = 3,4,5$ such that*:

- $X = \bigcup_j B(x_j, C_2td(x_j))$,
- *each point of X belongs at most to C_3 balls $B(x_j, C_2td(x_j))$*,
- $g(x_j) \le C_4Kg(td(x_j), x_j)$.

Moreover, there exists $C > 0$ *only depending on* X *such that, for any* $k \in \mathbb{N}$, $r > 0$ *and* $x \in X$,

$$\sum_{2^{k-1} \leq d(x_j) < 2^k,\, x_j \in B(x,r),\, t2^{k-1} \leq r} V(x_j, td(x_j)) \leq CV(x,r). \tag{14}$$

Finally, if $0 \leq a$, $a + \gamma/2 \leq b \leq 2\gamma$, $M \geq 0$, *and*

$$u_j(x) = d(x_j)^{-1-a} \left[1 + \frac{d(x_j,x)}{td(x_j)}\right]^{-1-b} \mathbf{1}_{[M,+\infty]}\left[\frac{d(x,x_j)}{td(x_j)}\right],$$

then, for all $x \in X$,

$$\sum_j u_j(x) \leq C_5 d(x)^{-1-a} \max(t^b, (1+M)^{-b}).$$

We postpone the technical proof of Lemma 7 to the appendix.

Another lemma, which is necessary to the proof of Theorem 6 (and which should be compared with Lemma 1 of [UCH]), deals with any measure ν over $X \times \mathbb{R}_+$ supported in $B(x_0, R) \times [0, R]$, where $x_0 \in X$ and $r > 0$, and satisfying, for any $r > 0$ and any $x \in X$,

$$\nu(\{B(x,r) \times [0,r]\}) \leq V(x,r)^{1+\delta}. \tag{15}$$

LEMMA 8. *Let* $p \in \,]1, +\infty[$ *and* $\delta \geq 0$. *Then, for each* $\beta > 0$, *there exists* $C_{p,\delta,\beta} > 0$ *such that, for any* $x_0 \in X$, $R > 0$, $k \in \mathbb{N}$, *any positive measure* ν *over* $X \times \mathbb{R}_+$ *supported in* $B(x_0, R) \times [0, R]$ *and satisfying* (15) *and any function* $f \in L^p(X, \mu)$ *supported in* $B(x_0, 2^{k+1}R) \setminus B(x_0, 2^k R)$,

$$\|Kf\|_{L^{p(1+\delta)}(\nu)} \leq C_{p,\delta,\beta} 2^{-k\beta} \|f\|_{L^p(\mu)}.$$

Proof. The idea of the proof is borrowed from [HOR]. Notice first that (15) implies that there exists $C > 0$ such that, for any $x \in X$ and $r > 0$,

$$\nu(\{(y,s) : B(y,s) \subset B(x,r)\}) \leq CV(x,r)^{1+\delta}. \tag{16}$$

Indeed, assume first that $r \geq \kappa\mu(\{x\})$. Then, if $B(y,s) \subset B(x,r)$, then

$$A^{-1}s \leq V(y,s) \leq V(x,r) \leq r.$$

Moreover, y then belongs to $B(x,r)$. Consequently,

$$\begin{aligned}\nu(\{(y,s) : B(y,s) \subset B(x,r)\}) &\leq \nu(B(x,r) \times [0, Ar]) \\ &\leq CV(x, Ar)^{1+\delta} \leq CV(x,r)^{1+\delta}.\end{aligned}$$

If $r \leq \kappa\mu(\{x\})$, then $B(x,r) = \{x\}$. If $B(y,s) \subset B(x,r)$, then $y = x$ and $A^{-1}s \leq V(y,s) \leq \mu(\{x\})$. Therefore,

$$\begin{aligned}\nu(\{(y,s) : B(y,s) \subset B(x,r)\}) &\leq \nu(\{x\} \times [0, A\mu(\{x\})]) \\ &\leq V(x, A\mu(\{x\}))^{1+\delta} \\ &\leq CV\left(x, \frac{\kappa}{2}\mu(\{x\})\right)^{1+\delta} = CV(x,r)^{1+\delta}.\end{aligned}$$

Thus, (16) is shown.

We recall that K is the linear operator which, to a locally integrable function f defined on X, associates the function Kf defined on $X \times \mathbb{R}_+$ by

$$Kf(x,r) = \int K(r,x,y) f(y)\, d\mu(y).$$

For any $f \in L^1(\mu)$, consider the "maximal" function

$$Mf(x,r) = \sup \left\{ \frac{1}{V(y,s)} \int_{B(y,s)} |f(z)|\, d\mu(z) : B(y,s) \supset B(x,r) \right\}.$$

We claim that, for any $p > 1$, M is bounded from $L^p(X,\mu)$ into $L^{p(1+\delta)}(X \times \mathbb{R}_+^*, \nu)$. Since it is clear that it maps continuously $L^\infty(X,\mu)$ into $L^\infty(X \times \mathbb{R}_+^*, \nu)$, thanks to the Marcinkiewicz interpolation theorem, we just have to show that M maps continuously $L^1(X,\mu)$ into $L^{(1+\delta),\infty}(X \times \mathbb{R}, \nu)$, which means that there exists a constant $C > 0$ such that, for any $\lambda > 0$ and $f \in L^1(\mu)$,

$$\nu(\{(x,r) : |Mf(x,r)| > \lambda\}) \le \frac{C}{\lambda^{1+\delta}} \|f\|_{L^1(\mu)}^{1+\delta}. \tag{17}$$

The argument is very much inspired by [HOR]. First, we prove the following proposition (see [HOR], Lemma 2.2, p. 67):

PROPOSITION 9. *Assume that $E \subset X \times \mathbb{R}_+$ and there exists $R > 0$ such that, for any $(x,r) \in E$, $r \le R$. Assume also that there exists no infinite sequence of points (x_i, r_i) in E such that the balls $B(x_i, r_i)$ are pairwise disjoint. Then there exist finitely many points (x_i, r_i) in E such that the balls $B(x_i, r_i)$ are pairwise disjoint and*

$$E \subset \{(x,r) : \exists i,\ B(x,r) \subset B(x_i, 5A^2 r_i)\}.$$

Proof. Set $R_1 = \sup\{r > 0 : (x,r) \in E\}$ and choose $(x_1, r_1) \in E$ such that $r_1 \ge R_1/2$. Assume that $N \ge 2$ and that (x_i, r_i) have been constructed for $i \le N-1$. Define

$$R_N = \sup\{r > 0 : (x,r) \in E \text{ and } \forall i \le N-1,\ B(x_i, r_i) \cap B(x,r) = \emptyset\}$$

if this set is not empty, and choose $(x_N, r_N) \in E$ such that $r_N \ge R_N/2$ and the ball $B(x_N, r_N)$ is disjoint from $B(x_i, r_i)$ for each $i \le N-1$. The assumption about E implies that this construction must stop after a finite number of steps. The balls $B(x_i, r_i)$ which have been constructed are pairwise disjoint. It remains to prove that, for any $(x,r) \in E$, there exists i such that $B(x,r) \subset B(x_i, 5A^2 r_i)$. Take $(x,r) \in E$ and define i to be the smallest integer such that $B(x,r) \cap B(x_i, r_i) \ne \emptyset$. Then $r \le R_i$. Let $y \in B(x,r) \cap B(x_i, r_i)$. Then $B(x,r) \subset B(x_i, 5r_i)$. Indeed, let $u \in B(x,r)$. One has

$$\begin{aligned} d(u, x_i) &\le A d(u,x) + A d(x, x_i) \le Ar + A^2 d(x,y) + A^2 d(y, x_i) \\ &\le Ar + A^2 r + A^2 r_i \le 5A^2 r_i. \ \blacksquare \end{aligned}$$

We are now ready to prove (17). Assume that $f \in L^1(X,\mu)$. For any $\lambda > 0$, set

$$E_\lambda = \{(x,r) : Mf(x,r) > \lambda\}.$$

For any $\varepsilon > 0$, define

$$E_\lambda^\varepsilon = \Big\{(x,r) : \int_{B(x,r)} |f(y)|\, d\mu(y) > \lambda(\varepsilon + V(x,r))\Big\},$$

$$E_\lambda'^\varepsilon = \{(x,r) : \exists (y,s) \in E_\lambda^\varepsilon,\ B(x,r) \subset B(y,s)\}.$$

Observe that there exists no infinite sequence $(x_i, r_i) \in E_\lambda^\varepsilon$ such that the balls $B(x_i,r_i)$ are pairwise disjoint. Indeed, if $(x_i,r_i) \in E_\lambda^\varepsilon$ and the $B(x_i,r_i)$ are pairwise disjoint, then, for each i,

$$\lambda(\varepsilon + V(x_j,r_j)) < \int_{B(x_j,r_j)} |f(y)|\, d\mu(y),$$

so that

$$\sum_j \lambda(\varepsilon + V(x_j,r_j)) < \int |f(y)|\, d\mu(y) < +\infty,$$

which implies that the sequence (x_i,r_i) is finite. Applying Proposition 9, one gets a finite sequence (x_i,r_i) in E_λ^ε such that the balls $B(x_i,r_i)$ are pairwise disjoint and

$$E_\lambda^\varepsilon \subset \bigcup_i \{(x,r) : B(x,r) \subset B(x_i, 5A^2 r_i)\}. \tag{18}$$

Hence

$$E_\lambda'^\varepsilon \subset \bigcup_i \{(x,r) : B(x,r) \subset B(x_i, 5A^2 r_i)\}.$$

Therefore, using (16), one gets

$$\begin{aligned}
\nu(E_\lambda'^\varepsilon) &\le \sum_i \nu(\{(x,r) : B(x,r) \subset B(x_i, 5A^2 r_i)\}) \\
&\le C \sum_i V(x_i, 5A^2 r_i)^{1+\delta} \le C \sum_i V(x_i,r_i)^{1+\delta} \\
&\le \frac{C}{\lambda^{1+\delta}} \sum_i \Big(\int_{B(x_i,r_i)} |f| \Big)^{1+\delta} \le \frac{C}{\lambda^{1+\delta}} \Big(\sum_i \int_{B(x_i,r_i)} |f| \Big)^{1+\delta} \\
&\le \frac{C}{\lambda^{1+\delta}} \|f\|_{L^1(\mu)}^{1+\delta}.
\end{aligned}$$

The second inequality follows from (16). The fourth one holds because $(x_i,r_i) \in E_\lambda^\varepsilon$, and the sixth one is true because the balls $B(x_i,r_i)$ are pairwise disjoint. Therefore, letting $\varepsilon \to 0$ yields (17). Thus, for any $p > 1$, M maps continuously $L^p(X,\mu)$ into $L^{p(1+\delta)}(X \times \mathbb{R}_+, \nu)$.

We are now able to conclude the proof of Lemma 8. Consider $r \leq R$, $x \in B(x_0, R)$ and $f \in L^1(\mu)$ supported in $B(x_0, 2^{k+1}R) \setminus B(x_0, 2^k R)$. Then, whenever $y \in B(x_0, 2^{k+1}R) \setminus (x_0, 2^k R)$, one has $d(x,y) \geq c2^{k-1}r$ for a constant $c > 0$ (use (8)), so that

$$\begin{aligned}
&|Kf(x,r)| \\
&= \Big| \sum_{i=k-1}^{+\infty} \int_{B(x,c2^{i+1}r)\setminus B(x,c2^i r)} K(r,x,y) f(y)\, d\mu(y) \Big| \\
&\leq C_\beta \sum_{i=k-1}^{+\infty} \frac{1}{V(x,r)} [1 + c2^i]^{-1-\beta} \int_{B(x,2^{i+1}r)} |f(y)|\, d\mu(y) \\
&\leq C \sum_{i=k-1}^{+\infty} \frac{V(x,2^{i+1}r)}{V(x,r)} [1 + c2^i]^{-1-\beta} \frac{1}{V(x,2^{i+1}r)} \int_{B(x,2^{i+1}r)} |f(y)|\, d\mu(y) \\
&\leq C' \sum_{i=k-1}^{+\infty} 2^{i+1} [1 + 2^i]^{-1-\beta} Mf(x,r) \leq C'' 2^{-k\beta} Mf(x,r).
\end{aligned}$$

Since M maps continuously $L^p(X,\mu)$ into $L^{p(1+\delta)}(X \times \mathbb{R}_+, \nu)$, it follows that

$$\|Kf\|_{L^{p(1+\delta)}(X\times\mathbb{R}_+,\nu)} \leq C2^{-k\beta} \|Mf\|_{L^{p(1+\delta)}(X\times\mathbb{R}_+,\nu)} \leq C2^{-k\beta} \|f\|_{L^p(X,\mu)}.$$

Lemma 8 is proved. ∎

We now use Lemmas 7 and 8 to prove (13) when $r_0 = 1$. Define $C_6 = 2(2A)^{1+\gamma/2} C_1$, $C_7 = 4C_5 C_6 C_\beta$ and $\varepsilon = \inf(1/C_7, (2A)^{-1-\gamma/2})$. Let $\eta > 0$ be sufficiently small, to be chosen at the end of the proof.

STEP 1: *Representation of ϕ by means of the kernel K.* We build a sequence $(x_j^{(n)})_{j\in\mathbb{N}} \subset X$ and a sequence $(\varepsilon_j^{(n)})_{j\in\mathbb{N}} \subset \{-1, 0, 1\}$ such that, for any $n \in \mathbb{N}$, the points $(x_j^{(n)})$ satisfy all the requirements of Lemma 7 for $t = \eta^n$ and $g = \sqrt{f^+}$, and

$$\begin{aligned}
\phi(x) = \sum_n \sum_j & C_6 \varepsilon (1-\varepsilon)^{n-1} \varepsilon_j^{(n)} d(x_j^{(n)})^{-1-\gamma/2} \\
& \times V(x_j^{(n)}, \eta^n d(x_j^{(n)})) K(\eta^n d(x_j^{(n)}), x_j^{(n)}, x),
\end{aligned}$$

the convergence being uniform on X.

For that purpose, for each $n \in \mathbb{N}$, we are going to build a sequence $(x_j^{(n)})_{j\in\mathbb{N}} \subset X$ and a sequence $(\varepsilon_j^{(n)})_{j\in\mathbb{N}} \subset \{-1, 0, 1\}$ such that, for any $n \in \mathbb{N}$, the points $(x_j^{(n)})$ satisfy all the requirements of Lemma 7 for $t = \eta^s$

and $g = \sqrt{f^+}$, and

$$|\phi_n(x)| \leq (1-\varepsilon)^n d(x)^{-1-\gamma/2}, \tag{19}$$

where

$$\begin{aligned}\phi_n(x) = \phi(x) - \sum_{i=1}^{n} C_6 \varepsilon (1-\varepsilon)^{i-1} \\ \times \sum_j \varepsilon_j^{(i)} d(x_j^{(i)})^{-1-\gamma/2} V(x_j^{(i)}, \eta^i d(x_j^{(i)})) K(\eta^i d(x_j^{(i)}), x_j^{(i)}, x).\end{aligned}$$

Set $\phi_0 = \phi$. Assume that the construction is done up to $n-1$ and let $(x_j^{(n)})$ be given by Lemma 7 applied with $t = \eta^n$ and $g = \sqrt{f^+}$. Define also

$$\varepsilon_j^{(n)} = \operatorname{sgn} \phi_{n-1}(x_j^{(n)}).$$

We claim that (19) holds. Indeed, if

$$\begin{aligned}\psi_n(x) = C_6 C_\beta \varepsilon (1-\varepsilon)^{n-1} \\ \times \sum_j \varepsilon_j^{(n)} d(x_j^{(n)})^{-1-\gamma/2} V(x_j^{(n)}, \eta^n d(x_j^{(n)})) K(\eta^n d(x_j^{(n)}), x_j^{(n)}, x),\end{aligned}$$

then Lemma 7 proves that, for any $x \in X$,

$$\begin{aligned}|\psi_n(x)| &\leq C_6 \varepsilon (1-\varepsilon)^{n-1} \sum_j d(x_j^{(n)})^{-1-\gamma/2} \left(1 + \frac{d(x_j^{(n)}, x)}{\eta^n d(x_j^{(n)})}\right)^{-1-\gamma} \\ &\leq \tfrac{1}{4} (1-\varepsilon)^{n-1} d(x)^{-1-\gamma/2}.\end{aligned} \tag{20}$$

Set $C_8 = (\varepsilon (2A)^{-1-3\gamma/2}/2)^{1/\gamma}$. Let $x, y \in X$ satisfy $d(x,y) \leq C_8 \eta^{n-1} d(y)$. We now prove that

$$|\phi_{n-1}(x) - \phi_{n-1}(y)| \leq \varepsilon (1-\varepsilon)^{n-1} d(y)^{-1-\gamma/2}. \tag{21}$$

One has $d(x,y) \leq \eta^{n-1} d(x)/(4A)^2$. Consequently, for any $i \leq n-1$,

$$d(x,y) \leq \frac{\eta^{n-1} d(x_j^{(i)}) + d(x_j^{(i)}, x)}{4A}.$$

Therefore,

$$\begin{aligned}|\phi_{n-1}(x) - \phi_{n-1}(y)| &\leq |\phi(x) - \phi(y)| \\ &\quad + \sum_{i=1}^{n-1} C_6 \varepsilon (1-\varepsilon)^{i-1} \sum_j d(x_j^{(i)})^{-1-\gamma/2} V(x_j^{(i)}, \eta^i d(x_j^{(i)})) \\ &\quad \times |K(\eta^i d(x_j^{(i)}), x_j^{(i)}, x) - K(\eta^i d(x_j^{(i)}), x_j^{(i)}, y)| \\ &= |\phi(x) - \phi(y)| + S_{n-1},\end{aligned} \tag{22}$$

where

$$
\begin{aligned}
S_{n-1} &\le 2\sum_{i=1}^{n-1} C_6\varepsilon(1-\varepsilon)^i \\
&\quad \times \sum_j d(x_j^{(i)})^{-1-\gamma/2}\left(\frac{d(x,y)}{\eta^i d(x_j^{(i)})}\right)^{\gamma}\left(1+\frac{d(x_j^{(i)},x)}{\eta^i d(x_j^{(i)})}\right)^{-1-2\gamma} \\
&\le 2d(x,y)^{\gamma} C_6\varepsilon\sum_{i=1}^{n-1}(1-\varepsilon)^i\eta^{-i\gamma} \\
&\quad \times \sum_j d(x_j^{(i)})^{-1-3\gamma/2}\left(1+\frac{d(x_j^{(i)},x)}{\eta^i d(x_j^{(i)})}\right)^{-1-2\gamma} \\
&\le 2d(x,y)^{\gamma} C_6\varepsilon\sum_{i=1}^{n-1}\left(\frac{1-\varepsilon}{\eta^{\gamma}}\right)^i C_5 d(x)^{-1-3\gamma/2} \\
&\le d(x,y)^{\gamma}\left(\frac{1-\varepsilon}{\eta^{\gamma}}\right)^{n-1} d(x)^{-1-3\gamma/2}.
\end{aligned}
\tag{23}
$$

Since $d(x,y) \le C_8\eta^{n-1}d(y)$, one has

$$\frac{d(y)}{2A} \le d(x) \le 2Ad(y),$$

which implies that

$$|\phi(x)-\phi(y)| \le \frac{\varepsilon}{2}(1-\varepsilon)^{n-1}d(y)^{-1-\gamma/2}. \tag{24}$$

Indeed, one may assume that either x or y belongs to $B(x_0,1)$. Therefore, $1/(2A) \le d(y) \le 4A$, and

$$
\begin{aligned}
|\phi(x)-\phi(y)| \le d(x,y)^{\gamma} &\le \frac{\varepsilon}{2}\eta^{\gamma(n-1)}d(y)^{\gamma} \\
&\le \frac{\varepsilon}{2}(1-\varepsilon)^{n-1}d(y)^{-1-\gamma/2}.
\end{aligned}
$$

It follows from (22)–(24) that (21) holds.

Consequently, if $\phi_{n-1}(y) \le 0$ and $d(x,y) \le C_8\eta^{n-1}d(y)$, then

$$\phi_{n-1}(x) \le \varepsilon(1-\varepsilon)^{n-1}d(y)^{-1-\gamma/2} \le (2A)^{1+\gamma/2}\varepsilon(1-\varepsilon)^{n-1}d(x)^{-1-\gamma/2}.$$

We have proved that, if $\phi_{n-1}(y) \le 0$, then for any $x \in B(y, C_8\eta^{n-1}d(y))$,

$$\phi_{n-1}(x) \le \tfrac{1}{2}(1-\varepsilon)^{n-1}d(x)^{-1-\gamma/2}.$$

Now, it is clear that (19) holds when $\phi_{n-1}(x) > \frac{1}{2}(1-\varepsilon)^{n-1}d(x)^{-1-\gamma/2}$.

Indeed,

$$\psi_n(x) = C_6\varepsilon(1-\varepsilon)^{n-1} \times \sum_{j;\phi_{n-1}(x_j^{(n)})>0} d(x_j^{(n)})^{-1-\gamma/2} V(x_j^{(n)}, \eta^n d(x_j^{(n)})) K(\eta^n d(x_j^{(n)}), x_j^{(n)}, x)$$
$$- C_6\varepsilon(1-\varepsilon)^{n-1} \times \sum_{j;\phi_{n-1}(x_j^{(n)})<0} d(x_j^{(n)})^{-1-\gamma/2} V(x_j^{(n)}, \eta^n d(x_j^{(n)})) K(\eta^n d(x_j^{(n)}), x_j^{(n)}, x),$$

so that

$$\begin{aligned}\psi_n(x) &\geq 2\varepsilon(1-\varepsilon)^{n-1} d(x)^{-1-\gamma/2} \\ &\quad - C_6\varepsilon(1-\varepsilon)^{n-1} \\ &\quad \times \sum_j d(x_j^{(n)})^{-1-\gamma/2} V(x_j^{(n)}, \eta^n d(x_j^{(n)})) |K(\eta^n d(x_j^{(n)}), x_j^{(n)}, x)| \\ &\quad \times \mathbf{1}_{[C_8\eta^{-1}, +\infty[}\left(\frac{d(x, x_j^{(n)})}{\eta^n d(x_j^{(n)})}\right) \\ &\geq 2\varepsilon(1-\varepsilon)^{n-1} d(x)^{-1-\gamma/2} - C_6 C_\beta \varepsilon(1-\varepsilon)^{n-1} C_5 d(x)^{-1-\gamma/2} (C_8\eta^{-1})^{-\gamma} \\ &\geq \varepsilon(1-\varepsilon)^{n-1} d(x)^{-1-\gamma/2}\end{aligned}$$

provided that η is small enough. Thus, one has

$$\psi_{n-1}(x) \geq \varepsilon(1-\varepsilon)^{n-1} d(x)^{-1-\gamma/2}, \tag{25}$$

and (19) holds. In the same way, one may prove that, whenever $\phi_{n-1}(x) < -\frac{1}{2}(1-\varepsilon)^{n-1} d(x)^{-1-\gamma/2}$, then

$$\psi_{n-1}(x) \leq -\varepsilon(1-\varepsilon)^{n-1} d(x)^{-1-\gamma/2}, \tag{26}$$

which proves that, in that case, (19) holds. Finally, if $-\frac{1}{2}(1-\varepsilon)^{n-1} d(x)^{-1-\gamma/2} \leq \psi_{n-1}(x) \leq \frac{1}{2}(1-\varepsilon)^{n-1} d(x)^{-1-\gamma/2}$, then (20) shows that (19) holds again.

As claimed, we have shown that, for any $x \in X$,

$$\phi(x) = \sum_n \sum_j C_6\varepsilon(1-\varepsilon)^{n-1} \varepsilon_j^{(n)} d(x_j^{(n)})^{-1-\gamma/2} \times V(x_j^{(n)}, \eta^n d(x_j^{(n)})) K(\eta^n d(x_j^{(n)}), x_j^{(n)}, x), \tag{27}$$

the convergence being uniform on X.

STEP 2: *Proof of (13).* It follows from (27) that

$$|\langle f, \phi\rangle| \leq C_6\varepsilon \sum_n \sum_j (1-\varepsilon)^{n-1} d(x_j^{(n)})^{-1-\gamma/2} V(x_j^{(n)}, \eta^n d(x_j^{(n)})) f^+(x_j^{(n)})$$

$$\leq C \int_{X\times\mathbb{R}_+} (K\sqrt{f^+}(x,r))^2\, d\nu(x,r),$$

where the measure ν is defined by

$$\begin{aligned}
\nu &= \sum_n \sum_j \varepsilon(1-\varepsilon)^n d(x_j^{(n)})^{-1-\gamma/2} V(x_j^{(n)}, \eta^n d(x_j^{(n)}))\delta_{(x_j^{(n)},\eta^n d(x_j^{(n)}))} \\
&\leq C\varepsilon \sum_k 2^{-k\gamma/2} \\
&\quad\times \sum_{n,j;2^{k-1}\leq d(x_j^{(n)})<2^k} (1-\varepsilon)^n d(x_j^{(n)})^{-1} V(x_j^{(n)}, \eta^n d(x_j^{(n)}))\delta_{(x_j^{(n)},\eta^n d(x_j^{(n)}))} \\
&= C\varepsilon \sum_k 2^{-k\gamma/2}\nu_k.
\end{aligned}$$

We are going to apply Lemma 8 with ν_k. For that purpose, fix $x \in X$ and $r > 0$. Then

$$\begin{aligned}
&\nu_k(B(x,r)\times[0,r]) \\
&\quad= \sum_n (1-\varepsilon)^n \sum_{\substack{j;\, x_j^{(n)}\in B(x,r),\\ \eta^n d(x_j^{(n)})\leq r,\, 2^{k-1}\leq d(x_j^{(n)})<2^k}} d(x_j^{(n)})^{-1} V(x_j^{(n)}, \eta^n d(x_j^{(n)})) \\
&\quad\leq 2^{1-k} \sum_{n;\, \eta^n 2^{k-1}\leq r} (1-\varepsilon)^n \\
&\qquad\times \sum_{j;\, x_j^{(n)}\in B(x,r),\, \eta^n d(x_j^{(n)})\leq r,\, 2^{k-1}\leq d(x_j^{(n)})<2^k} V(x_j^{(n)}, \eta^n d(x_j^{(n)})) \\
&\quad\leq 2^{1-k} V(x,r) \sum_{n;\, \eta^n 2^{k-1}\leq r} (1-\varepsilon)^n \\
&\quad\leq C2^{-k}V(x,r)(2^{-k}r)^{\log(1-\varepsilon)/\log\eta} \leq C(2^{-k}V(x,r))^{1+\delta},
\end{aligned}$$

where $\delta = \log(1-\varepsilon)/\log\eta$. The third inequality holds thanks to Lemma 7.

Notice that ν_k is supported in $B(x_0, C2^k)\times[0, C2^k]$. Set $g = \sqrt{f^+}$ and

$$g_i = g[\mathbf{1}_{B(x_0,C2^{k+i})} - \mathbf{1}_{B(x_0,C2^{k+i-1})}] \quad \text{for } i \in \mathbb{N}^*, \qquad g_0 = g\mathbf{1}_{B(x_0,C2^k)}.$$

Applying Lemma 8 with $2^{k(1+\delta)}\nu_k$, $p = 2/(1+\delta)$, $R = C2^k$, one gets

$$\|Kg\|_{L^2(2^{k(1+\delta)}\nu_k)} \leq \sum_i \|Kg_i\|_{L^2(2^{k(1+\delta)}\nu_k)} \leq C\sum_i 2^{-i\gamma}\|g_i\|_{L^p(\mu)}.$$

As a consequence,

$$\begin{aligned}\|Kg\|_{L^2(\nu_k)} &\le C\sum_i 2^{-i\gamma}2^{-k(1+\delta)/2}\|g_i\|_{L^p(\mu)}\\ &\le C\sum_i 2^{-i\gamma}2^{-k(1+\delta)/2}\Big[\int\limits_{B(x_0,C2^{k+i})}|g|^p\Big]^{1/p}\\ &\le C\sum_i 2^{-i\gamma}2^{-k(1+\delta)/2}V(x_0,C2^{k+i})^{1/p}[M|g|^p]^{1/p}(x_0)\\ &\le C[M|g|^p]^{1/p}(x_0),\end{aligned}$$

provided that γ is chosen $> (1+\delta)/2$. Finally,

$$\begin{aligned}\int\limits_{X\times\mathbb{R}_+}(K\sqrt{f^+}(x,r))^2\,d\nu_k(x,r) &= \|K\sqrt{f^+}\|^2_{L^{p(1+\delta)}(\nu_k)}\\ &\le CM[(f^+)^{1/(1+\delta)}]^{1+\delta}(x_0).\end{aligned}$$

Thus, (13) and, consequently, (12) are proved, which concludes the proof of Theorem 6. ∎

4. Appendix: proof of the constructive lemma. We give the proof of Lemma 7.

STEP 1: *Definition of the x_j's.* Consider a sequence $(y_i^{(0)}) \subset B(x_0,2)$ satisfying

$$d(y_i^{(0)},y_j^{(0)}) \ge \tfrac{1}{8}(2A)^{-2}C_2t \quad \forall i\ne j,$$

and maximal for this property. For any $k\in\mathbb{N}$, consider a sequence $(y_i^{(k)}) \subset B(x_0,2^{k+1})\setminus B(x_0,2^k)$ satisfying

$$d(y_i^{(k)},y_j^{(k)}) \ge 2^{k-3}(2A)^{-2}C_2t \quad \forall i\ne j,$$

and maximal for this property.

It is important to notice that the balls $B(y_i^{(k)},2^{k-3}(2A)^{-3}C_2t)$ are pairwise disjoint for any $k\in\mathbb{N}$. Moreover,

$$X=\bigcup_k\bigcup_i B(y_i^{(k)},2^{k-3}(2A)^{-2}C_2t).$$

In every ball $B(y_i^{(k)},2^{k-3}(2A)^{-2}C_2t)$, consider a point $x_i^{(k)}$ such that

$$g(x_i^{(k)}) \le \frac{2}{V(y_i^{(k)},2^{k-3}(2A)^{-2}C_2t)}\int\limits_{B(y_i^{(k)},2^{k-3}(2A)^{-2}C_2t)} g(y)\,d\mu(y).$$

Note that such a point always exists. Thus, we obtain a sequence x_j. The triangle inequality (8) shows that

$$[1+(2A)^{-1}2^k] \le d(x_j^{(k)}) \le A2^{k+2}.$$

STEP 2: *Properties of the x_j's.* Before checking that the conclusions of Lemma 7 hold, we give some properties of the points x_j.

PROPOSITION 10. *For all $l \in \mathbb{N}$ and $x \in X$, define*

$$I_l(x) = \{j : x \in B(x_j^{(l)}, C_2 t d(x_j^{(l)}))\}.$$

Then $|I_l(x)| \leq C$.

For all $k \in \mathbb{N}$, define

$$J_k = \{x_j : 2^{k-1} \leq d(x_j) < 2^k\}.$$

Then $|J_k| \leq C/t$.

For all $x \in X$ and all integers $k, i \in \mathbb{N}$, set

$$L_{k,i}(x) = \{x_j : 2^{k-1} \leq d(x_j) < 2^k \text{ and } d(x_j, x) < 2^{i+k-1}t\}.$$

Then $|L_{k,i}(x)| \leq C2^i$.

In all those assertions, $C > 0$ only depends on X.

Proof. We begin with the first assertion. Let $j \in I_l(x)$. Then

$$\begin{aligned} d(x, y_j^{(l)}) &\leq A[d(x, x_j^{(l)}) + d(x_j^{(l)}, y_j^{(l)})] \leq A[C_2 t d(x_j^{(l)}) + 2^{l-3}(2A)^{-2}C_2 t] \\ &\leq A^2 2^{l+3} C_2 t. \end{aligned}$$

It follows that, for any $j \in I_l(x)$, $B(y_j^{(l)}, 2^{l-3}(2A)^{-3}C_2 t) \subset B(x, A^3 2^{l+4} C_2 t)$, and

$$\begin{aligned} V(x, A^3 2^{l+4} C_2 t) &\geq \sum_{j \in I_l(x)} V(y_j^{(l)}, 2^{l-3}(2A)^{-3}C_2 t) \\ &\geq C \sum_{j \in I_l(x)} V(y_j^{(l)}, A^4 2^{l+5} C_2 t) \\ &\geq C|I_l(x)| V(x, A^3 2^{l+4} C_2 t). \end{aligned}$$

In the first line, we used the fact that the balls $B(y_j^{(l)}, 2^{l-3}(2A)^{-3}C_2 t)$ are pairwise disjoint. The second one holds thanks to the doubling property. The claim about $I_l(x)$ is proved.

We now turn to the assertion about J_k. Define first, for $k \in \mathbb{N}$, $J'_k = \{y_j^{(k)}\}$. Then $|J'_k| \leq C/t$. Indeed,

$$\begin{aligned} V(x_0, A2^{k+2}) &\geq \sum_j V(y_j^{(k)}, 2^{k-3}(2A)^{-3}C_2 t) \geq Ct \sum_j V(y_j^{(k)}, A^2 2^{k+3}) \\ &\geq Ct|J'_k| V(x_0, A2^{k+2}). \end{aligned}$$

From that, one deduces that $|\{x_j^{(k)}\}| \leq C/t$.

We are now able to prove the assertion about J_k. Indeed, if $x_j^{(l)} \in J_k$, then since $[1+(2A)^{-1}2^l] \leq d(x_j^{(l)}) < A2^{l+2}$, one has

$$2^{k-1} < A2^{l+2}, \quad 1+(2A)^{-1}2^l < 2^k,$$

which shows that the number of l's concerned is in $[k-k_0, k+k_0]$ where $k_0 \in \mathbb{N}$ only depends on A. Thus, the assertion about J_k is proved.

Finally, we prove the result about $L_{k,i}(x)$. We have just seen that if $x_j^{(l)} \in L_{k,i}$, then $k-k_0 \leq l \leq k+k_0$. Therefore, if, for each $l \in [k-k_0, k+k_0]$, $L^l_{k,i}(x) = \{j : x_j^{(l)} \in L_{k,i}(x)\}$, then

$$\begin{aligned} V(x, C2^{i+l}t) &\geq \sum_{j \in L^{(l)}_{k,i}(x)} V(y_j^{(l)}, 2^{l-3}(2A)^{-3}C_2 t) \\ &\geq c2^{-i} \sum_{j \in L^{(l)}_{k,i}(x)} V(y_j^{(l)}, C2^{i+l}t) \\ &\geq c2^{-i}|L^{(l)}_{k,i}(x)| V(x, C2^{i+l}t). \end{aligned}$$

The assertion about $L_{k,i}(x)$ is thus proved. ■

STEP 3: *End of Proof of Lemma 7.* To begin with, one has

$$X = \bigcup_j B(x_j, C_2 t d(x_j)).$$

Indeed, let $x \in X$. Then $x \in B(y_j^{(k)}, 2^{k-3}(2A)^{-2}C_2 t)$ for some $k \in \mathbb{N}$ and $j \in \mathbb{N}$. Thus,

$$\begin{aligned} d(x, x_j^{(k)}) &\leq A[d(x, y_j^{(k)}) + d(y_j^{(k)}, x_j^{(k)})] \\ &\leq A[2^{k-3}(2A)^{-2}C_2 t + 2^{k-3}(2A)^{-2}C_2 t] \\ &\leq 2^{k-3}(2A)^{-1}C_2 t \leq C_2 t d(x_j^{(k)}). \end{aligned}$$

Now, we prove that each point of X is an element of at most C_3 balls $B(x_j, C_2 t d(x_j))$. Take, for instance, $x \in B(x_0, 2^{k+1}) \setminus B(x_0, 2^k)$. Assume that $x \in B(x_j^{(l)}, C_2 t d(x_j^{(l)}))$. We claim that $k-3 \leq l \leq k+2$. Indeed, on the one hand,

$$2^k \leq d(x, x_0) \leq A[d(x, x_j^{(l)}) + d(x_j^{(l)}, x_0)] \leq A[1 + C_2 t] d(x_j^{(l)}) \leq A^2 2^{l+3}.$$

On the other hand,

$$(2A)^{-1}2^l \leq d(x_0, x_j^{(l)}) \leq A[d(x_0, x) + d(x, x_j^{(l)})] \leq A[2^{k+1} + A2^{l+2}C_2 t],$$

hence $l \leq k+3$, since $t < (2A)^{-5}$. Since the cardinality of each $I_l(x)$ is controlled by a constant (Proposition 10), x belongs to at most C_3 balls $B(x_j, C_2 t d(x_j))$.

As for the assertion about g, for any (j,k), one has

$$\begin{aligned} Kg(td(x_j^{(k)}), x_j^{(k)}) &= \int K(td(x_j^{(k)}), x_j^{(k)}, x) g(x)\, d\mu(x) \\ &\geq \int_{B(y_j^{(k)}, 2^{k-3}(2A)^{-2}C_2 t)} K(td(x_j^{(k)}), x_j^{(k)}, x) g(x)\, d\mu(x) \\ &\geq \frac{C}{V(y_j^{(k)}, 2^{k-3}(2A)^{-2}C_2 t)} \int_{B(y_j^{(k)}, 2^{k-3}(2A)^{-2}C_2 t)} g(x)\, d\mu(x) \\ &\geq Cg(x_j^{(k)}). \end{aligned}$$

We turn to the proof of (14). Consider $k \in \mathbb{N}$, $x \in X$ and $r > 0$. We want to estimate

$$\sum_{2^{k-1} \leq d(x_j) < 2^k,\, x_j \in B(x,r),\, 2^{k-1}t \leq r} V(x_j, td(x_j)). \tag{28}$$

The x_j's involved are of the form $x_j^{(l)}$ with $k - k_0 \leq l \leq k + k_0$. Thus, the sum is equal to

$$\sum_{l=k-k_0}^{k+k_0} \sum_{2^{k-1} \leq d(x_j^{(l)}) < 2^k,\, x_j^{(l)} \in B(x,r),\, 2^{k-1}t \leq r} V(x_j^{(l)}, td(x_j^{(l)})) = \sum_{l=k-k_0}^{k+k_0} S_l.$$

For any fixed integer l in $[k - k_0, k + k_0]$, one has

$$\begin{aligned} S_l &\leq \sum_{2^{k-1} \leq d(x_j^{(l)}) < 2^k,\, x_j^{(l)} \in B(x,r),\, 2^{k-1}t \leq r} V(x_j^{(l)}, 2^k t) \\ &\leq \sum_{2^{k-1} \leq d(x_j^{(l)}) < 2^k,\, x_j^{(l)} \in B(x,r),\, 2^{k-1}t \leq r} V(y_j^{(l)}, A2^{k+1} t) \\ &\leq C \sum_{2^{k-1} \leq d(x_j^{(l)}) < 2^k,\, x_j^{(l)} \in B(x,r),\, 2^{k-1}t \leq r} V(y_j^{(l)}, 2^{k-3}(2A)^{-3}C_2 t) \\ &\leq CV(x, Kr) \leq CV(x,r). \end{aligned}$$

In those computations, the third inequality is a consequence of the doubling property, and the fourth holds because the balls $B(y_j^{(l)}, 2^{k-3}(2A)^{-3}C_2 t)$ are pairwise disjoint. Thus, (14) is proved.

Finally, we show the assertion about u_j. Let n be the integer such that $2^n \leq M < 2^{n+1}$. For each $k \in \mathbb{N}$, write

$$\sum_{2^{k-1}\le d(x_j)<2^k} u_j(x)$$
$$\le 2^{-(k-1)(1+a)} \sum_{2^{k-1}\le d(x_j)<2^k,\, d(x_j,x)>Mt2^{k-1}} \left[1+\frac{d(x_j,x)}{t2^k}\right]^{-1-b}$$
$$\le 2^{-(k-1)(1+a)} \sum_{i\ge n}\left[\sum_{2^{k-1}\le d(x_j)<2^k,\, 2^{k+i}t>d(x_j,x)>2^{k-1+i}t} \left[1+\frac{d(x_j,x)}{t2^k}\right]^{-1-b}\right]$$
$$\le 2^{-(k-1)(1+a)} C\sum_{i\ge n} 2^i[1+2^{i-1}]^{-1-b} \le C2^{-(k-1)(1+a)}(1+M)^{-b}.$$

Moreover, if $2^{k-1} > 2Ad(x)$, then $d(x_j,x) > cd(x_j)$, and it follows that

$$\sum_{2^{k-1}\le d(x_j)<2^k} u_j(x) \le 2^{-(k-1)(1+a)}t^{1+b} \sum_{2^{k-1}\le d(x_j)<2^k} 1 \le C2^{-(k-1)(1+a)}t^b.$$

Finally, if $2^k < d(x)/(2A)$, then $d(x_j,x) > cd(x)$, and

$$\sum_{2^{k-1}\le d(x_j)<2^k} u_j(x) \le 2^{-(k-1)(1+a)}\left[1+\frac{d(x)}{t2^k}\right]^{-1-b} \sum_{2^{k-1}\le d(x_j)<2^k} 1$$
$$\le C2^{-(k-1)(1+a)}d(x)^{-1-b}t^b2^{k(1+b)}.$$

Lemma 7 is proved. ■

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