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APPROXIMATING RADON MEASURES ON FIRST-COUNTABLE COMPACT SPACES

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Abstract. The assertion *every Radon measure defined on a first-countable compact space is uniformly regular* is shown to be relatively consistent. We prove an analogous result on the existence of uniformly distributed sequences in compact spaces of small character. We also present two related examples constructed under CH.

In this note we consider some properties of finite Radon measures defined on compact spaces. In Section 1 we recall basic definitions and mention some auxiliary results used later on. In Section 2 we give a brief account of known results and open problems concerning measures on first-countable spaces. In Section 3 we prove that the following statements are relatively consistent:

(1) *Every Radon measure on a first-countable compact space is uniformly regular.*

(2) *Every Radon measure on a compact space of character* $< \mathfrak{c}$ *has a uniformly distributed sequence.*

Assertion (1) is related to a problem posed by D. Fremlin [9], while (2) is a generalization of a result due to Mercourakis [16]. In Section 4 we present two examples constructed under CH. One of them describes a separable Radon measure on a first-countable separable compact space which has a uniformly distributed sequence but is not uniformly regular.

1. Preliminaries. All the measures considered in what follows are assumed to be finite. We say that a measure μ is *separable* if $L_1(\mu)$ is separable as a Banach space. In other words, μ is separable if its Maharam type is at most countable (see [6] for the terminology concerning measure algebras).

Recall that a Radon measure μ defined on a compact space K is called *uniformly regular* if there is a continuous surjection g from K onto a compact metric space such that $\mu(g^{-1}(g(F))) = \mu(F)$ for every compact $F \subseteq K$.

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Uniformly regular measures are also called strongly countably determined (see Pol [20]).

Given two families $\mathcal{A}$ and $\mathcal{B}$ of μ-measurable sets, we find it convenient to say that $\mathcal{A}$ *approximates* $\mathcal{B}$ *from below* if for every $\varepsilon > 0$ and every $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ such that $A \subseteq B$ and $\mu(B \setminus A) < \varepsilon$.

We recall a standard lemma concerning uniformly regular measures (see Babiker [1]).

Lemma 1.1. *The following are equivalent for a Radon measure μ defined on a compact space K:*

(i) *μ is uniformly regular;*

(ii) *there is a countable family of zero subsets of K approximating all open sets from below;*

(iii) *there is a countable family of cozero subsets of K approximating all open sets from below.*

Note that (ii) of Lemma 1.1 and outer regularity imply that for a uniformly regular Radon measure one can find a countable family approximating all measurable sets in the sense of symmetric difference. Thus uniform regularity is stronger than separability.

Given a probability measure μ on K, a sequence $(x_n) \subseteq K$ is said to be *uniformly distributed* (with respect to μ) if for every real-valued continuous function f defined on K one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \int_K f\, d\lambda.$$

The book by Kuipers and Niederreiter [12] surveys the classical theory of uniformly distributed sequences on metric spaces and topological groups. Some results concerning the existence of uniformly distributed sequences in nonmetrizable compact spaces may be found in Losert [15], Mercourakis [16] and Frankiewicz & Plebanek [3].

Let $\mathcal{J}$ be a proper ideal of subsets of a space X with $\bigcup \mathcal{J} = X$. Recall that the cardinal numbers $\operatorname{cov}(\mathcal{J})$ and $\operatorname{non}(\mathcal{J})$ of $\mathcal{J}$ are defined as

$$\operatorname{cov}(\mathcal{J}) = \min\Big\{|\mathcal{E}| : \mathcal{E} \subseteq \mathcal{J},\ \bigcup \mathcal{E} = X\Big\},$$
$$\operatorname{non}(\mathcal{J}) = \min\{|Y| : Y \notin \mathcal{J}\}.$$

Denote by $\mathbb{L}_\kappa$ the σ-ideal of subsets of the Cantor cube 2^κ which are null with respect to the usual product measure. Basic facts concerning ideals $\mathbb{L}_\kappa$ and their cardinal coefficients, as well as further references, may be found e.g. in [6], [21]. Kraszewski [11] offers a detailed discussion on cardinal invariants of a larger class of σ-ideals in Cantor cubes. Recall that $\operatorname{non}(\mathbb{L}_\omega) = \operatorname{non}(\mathbb{L}_{\omega_1})$ and $\operatorname{cov}(\mathbb{L}_\omega) \geq \operatorname{cov}(\mathbb{L}_{\omega_1})$; the relation $\operatorname{non}(\mathbb{L}_\omega) < \operatorname{cov}(\mathbb{L}_{\omega_1})$ is relatively consistent. It is perhaps worth recalling that the existence of an atomlessly measurable cardinal implies $\omega_1 = \mathrm{non}(\mathbb{L}) < \mathrm{cov}(\mathbb{L}_{\omega_1})$ (see [7], 6G and 6L).

Given a Radon measure μ, a cardinal number κ is a *caliber* of μ if for every family $(B_\xi)_{\xi<\kappa}$ of μ-measurable sets of positive measure we have $\bigcap_{\xi\in X} B_\xi \neq \emptyset$ for some $X \subseteq \kappa$ of cardinality κ (this is equivalent to saying that κ is a precaliber of the measure algebra of μ; see [7], A2T, and [19], Section 2).

The following lemma links the notion of caliber with the covering number (see [7], A2U).

LEMMA 1.2. *Let μ be a finite Radon measure, Σ be the family of μ-measurable sets, and $\mathcal{N} = \{E \in \Sigma : \mu(E) = 0\}$. Given a cardinal κ of uncountable cofinality, if κ is not a caliber of μ then there is a family $(E_\xi)_{\xi<\kappa} \subseteq \mathcal{N}$ such that $\bigcup_{\xi<\kappa} E_\xi \in \Sigma \setminus \mathcal{N}$. If, moreover, κ is regular then E_ξ's may be chosen increasing.*

Using Lemma 1.2 and the Maharam theorem one can check that ω_1 is a caliber of Radon measures if and only if $\mathrm{cov}(\mathbb{L}_{\omega_1}) > \omega_1$ (a remark due to D. Fremlin). We note the following simple fact (see [18], Lemma 1).

LEMMA 1.3. *Let μ be a finite Radon measure defined on a compact space K. Suppose that K is written as an increasing union of its arbitrary subsets: $K = \bigcup_{\alpha<\kappa} Y_\alpha$, where κ is an uncountable regular cardinal. If κ is a caliber of μ then there is an $\alpha < \kappa$ such that $\mu^*(Y_\alpha) = \mu(K)$.*

2. Some results and problems. Let μ be a uniformly regular probability Radon measure on a compact space K. It is clear that μ is separable and that there is a separable subspace $S \subseteq K$ with $\mu(S) = 1$. Moreover, μ has a uniformly distributed sequence (see Mercourakis [16]).

Clearly, every Radon measure on a metrizable K is uniformly regular. On the other hand, it is not difficult to give an example of a separable Radon measure on a compact space of topological weight $\mathfrak{c}$ which is not uniformly regular. Consider, for instance, the Lebesgue measure λ on $[0,1]$, and let K be the Stone space of its measure algebra. Then λ is separable but not uniformly regular, when considered as a measure on K, and there is no uniformly distributed sequence in K.

Note that if we want to discuss compact spaces K on which every Radon measure is uniformly regular then K has to be first-countable, since the Dirac measure δ_x, $x \in K$, is uniformly regular if and only if x is a point of countable character. This cannot be reversed: The results of Haydon [10] and Kunen [13] revealed that under CH there is a compact first-countable space K supporting a nonseparable Radon measure μ which vanishes on all separable subspaces of K (so, in particular, μ is not uniformly regular and has no uniformly distributed sequences).

The assertion *every Radon measure on a first-countable space is separable* is equivalent to $\operatorname{cov}(\mathbb{L}_{\omega_1}) > \omega_1$ (see [18] and [14]). Assuming MA + ¬CH, Fremlin [8] showed that for an arbitrary compact space K, every Radon measure on K is separable if and only if there is no continuous surjection from K onto $[0,1]^{\omega_1}$.

Let μ be a probability Radon measure on a first-countable compact space K. Assuming MA + ¬CH, one can pose the following questions:

(2.1) Is μ uniformly regular?

(2.2) Does μ admit a uniformly distributed sequence?

(2.3) Is μ uniformly regular provided it has a uniformly distributed sequence?

Problem (2.1) has been posed by Fremlin (see [9], Problem DH) and seems to be quite difficult. I am writing down the other questions with the hope they are easier to answer (or in case the answer to (2.1) is negative).

3. Assuming non < cov**.** We now show that (2.1) is answered in the affirmative if we impose some conditions on the values of $\operatorname{non}(\mathbb{L}_{\omega_1})$ and $\operatorname{cov}(\mathbb{L}_{\omega_1})$ (necessarily contradiction of Martin's axiom).

THEOREM 3.1. *Assume that* $\omega_1 = \operatorname{non}(\mathbb{L}) < \operatorname{cov}(\mathbb{L}_{\omega_1})$. *Then every Radon measure on a first-countable compact space is uniformly regular.*

Proof. (1) Consider a Radon probability measure μ on a first-countable space K. Clearly, we can assume that μ vanishes on all singletons. For every $x \in K$ we fix a countable local base $(V_n(x))_{n\in\omega}$ at x consisting of cozero sets. Write $\mathcal{N}$ for the σ-ideal of μ-null subsets of K.

Since $\operatorname{cov}(\mathbb{L}_{\omega_1}) > \omega_1$, it follows from [18], Theorem 3, that μ is a separable measure (using the remark from Section 1). Then $\operatorname{non}(\mathcal{N}) \le \operatorname{non}(\mathbb{L}) = \omega_1$.

(2) Given a set $X \subseteq K$, let $\mathcal{A}(X)$ be the family of all finite unions of sets from the family $\{V_n(x) : x \in X,\ n \in \omega\}$. Note that the family $\mathcal{A}(X)$ approximates all open sets from below whenever $\mu^*(X) = 1$. Indeed, if U is open then there is a subfamily $\mathcal{A}'$ of $\mathcal{A}(X)$ such that putting $G = \bigcup \mathcal{A}'$ we have $U \cap X \subseteq G \subseteq U$, which implies $\mu(U \setminus G) = 0$. Now it is clear that finite unions of elements of $\mathcal{A}'$ approximate U from below.

(3) By (1) we can find and fix a set $X \subseteq K$ of cardinality ω_1 such that $\mu^*(X) = 1$. Then the family $\mathcal{A}(X)$ has cardinality ω_1 and we can write $\mathcal{A}(X) = \bigcup_{\alpha<\omega_1} \mathcal{C}_\alpha$, where $(\mathcal{C}_\alpha)_{\alpha<\omega_1}$ is an increasing sequence of countable families, and every $\mathcal{C}_\alpha$ is closed under taking finite unions. For every $\alpha < \omega_1$ denote by Y_α the set of those points x in K for which the family $\mathcal{C}_\alpha$ approximates $(V_n(x))_{n\in\omega}$ from below.

By (2), $\mathcal{A}(X)$ approximates all open sets from below, so for any $x \in K$ there is an $\alpha < \omega_1$ such that $x \in Y_\alpha$. Thus we have $K = \bigcup_{\alpha<\omega_1} Y_\alpha$, and $Y_\beta \subseteq Y_\alpha$ for every $\beta < \alpha < \omega_1$.

(4) Using the assumption $\mathrm{cov}(\mathbb{L}_{\omega_1}) > \omega_1$ again we infer that $\mu^*(Y_\alpha) = 1$ for some $\alpha < \omega_1$ (see Lemma 1.3). It follows from (2) that $\mathcal{A}(Y_\alpha)$ approximates all open sets from below, and thus $\mathcal{C}_\alpha$ approximates all open sets from below. Hence μ is uniformly regular by Lemma 1.1, and the proof is complete.

We remark that the above result may be proved under the weaker assumption $\mathrm{non}(\mathbb{L}) < \min(\mathrm{cov}(\mathbb{L}_{\omega_1}), \omega_\omega)$ (a similar argument works).

Pol [20] proved that if every Radon measure on a compact space K is uniformly regular then the space $P(K)$ of all Radon probability measures on K is first-countable in its weak* topology. In particular, the space $P(K)$ has countable tightness, which implies that the Banach space $C(K)$ has property (C) of Corson (see [20]). Thus the theorem above implies that $C(K)$ has property (C) for first-countable K provided $\omega_1 = \mathrm{non}(\mathbb{L}) < \mathrm{cov}(\mathbb{L}_{\omega_1})$. I do not know if this is also true under MA + ¬CH (see [4] for related results).

Mercourakis [16], Proposition 2.21, showed that under MA + ¬CH every Radon probability measure on a space of weight $< \mathfrak{c}$ has a uniformly distributed sequence. As our next result shows, that assertion may hold in a larger class of compact spaces.

THEOREM 3.2. *It is relatively consistent that every Radon probability measure defined on a compact space of character* $< \mathfrak{c}$ *has a uniformly distributed sequence.*

Proof. (1) Assume that

$$\mathrm{non}(\mathbb{L}_{\mathfrak{c}}) = \omega_1 < \mathrm{cov}(\mathbb{L}_{\mathfrak{c}}) = \omega_2 = \mathfrak{c}.$$

Consider a Radon probability nonatomic measure μ on a compact space K of character $\le \omega_1$. Note that it follows from Lemma 1.2 and our assumption that $\omega_2 = \mathfrak{c}$ is a caliber of Radon measures. Then the measure μ is of Maharam type at most ω_1 (see [18], the remark on page 160).

(2) Write $\mathcal{N}$ for the σ-ideal of μ-null sets, and $\mathcal{N}_\omega$ for the σ-ideal of subsets of K^ω which are null with respect to the product measure μ^ω. By Lemma 6.4 from [6], μ^ω is of Maharam type at most ω_1. It follows that $\mathrm{non}(\mathcal{N}) = \omega_1$ and $\mathrm{cov}(\mathcal{N}_\omega) = \omega_2$.

(3) We choose a set $X \subseteq K$ of cardinality ω_1 and such that $\mu^*(X) = 1$, and for every $x \in X$ choose a local base $(V_\alpha(x))_{\alpha<\omega_1}$ at x. Consider the family $\mathcal{A}$ which is the closure of $\{V_\alpha(x) : x \in X,\ \alpha < \omega_1\}$ under taking finite unions and intersections. Again $\mathcal{A}$ is a family of cardinality at most ω_1, and it approximates all open sets from below.

(4) Now we can follow the proof of a classical result on uniform distribution of sequences; see [12], page 183, or [16], Theorem 2.5 for details. For

$A \in \mathcal{A}$ denote by $T(A)$ the set of all sequences (x_n) from K^ω satisfying

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_A(x_i) = \mu(A).$$

Then $\mu^\omega(T(A)) = 1$, so using (2) and (3) the set $\bigcap_{A\in\mathcal{A}} T(A)$ is nonempty. But $\mathcal{A}$ is a lattice approximating the family of all open sets from below, so $\mathcal{A}$ is the so-called convergence determining class (see [16], Lemma 2.7). This means that any sequence $(x_n) \in \bigcap_{A\in\mathcal{A}} T(A)$ is uniformly distributed with respect to μ, and the proof is complete.

4. Under CH. Let us add the following results to the long list of constructions of "pathological" first-countable spaces carrying Radon measures; see Haydon [10], Kunen [13], Džamonja and Kunen [2], Kunen and van Mill [14]. We follow here some ideas from Plebanek [19].

THEOREM 4.1. *Assume CH. There exist a compact space K and a Radon probability measure on K with the following properties*:

(i) *K is separable and first-countable*;
(ii) *μ is separable and has a uniformly distributed sequence*;
(iii) *μ is not uniformly regular.*

We need some preparations for the proof of this theorem. Recall that the *asymptotic density* of a set $A \subseteq \omega$, denoted here by $d(A)$, is defined as

$$d(A) = \lim_{n\to\infty} \frac{|A \cap n|}{n},$$

provided the limit exists. Let $\mathcal{D}$ denote the family of sets for which d is defined.

LEMMA 4.2. *Let $\mathcal{A}$ be a countable algebra contained in $\mathcal{D}$, and let $(A_k)_k$ be an increasing sequence in $\mathcal{A}$. Then there is a set $X \subseteq \omega$ such that $A_k \subseteq^* X$ for every k, the algebra generated by $\mathcal{A} \cup \{X\}$ is contained in $\mathcal{D}$, and $d(X) = \lim_{k\to\infty} d(A_k)$.*

Proof. We can write $\mathcal{A} = \bigcup_{k=1}^{\infty} \mathcal{A}_k$, where every $\mathcal{A}_k$ is a finite algebra containing A_k. For every k we choose a number m_k such that

$$\left| \frac{|B \cap n|}{n} - d(B) \right| < \frac{1}{k}$$

whenever $B \in \mathcal{A}_k$ and $n \geq m_k$. Now one can check that the set

$$X = \bigcup_{k=1}^{\infty} [m_k, m_{k+1}) \cap A_k$$

has the required properties.

Proof of Theorem 4.1. (1) Let $\mathcal{A}_0$ be a countable algebra in $\mathcal{D}$ containing all finite subsets of ω and such that d is nonatomic on $\mathcal{A}_0$. Using CH we fix an enumeration $(s_\alpha)_{\alpha<\omega_1}$ of all decreasing sequences from $\mathcal{A}_0$ on which d tends to 0 (so for every $\alpha < \omega_1$ we have $s_\alpha(0) \supseteq s_\alpha(1) \supseteq \ldots$, and $\lim_{k\to\infty} d(s_\alpha(k)) = 0$).

(2) We construct an increasing sequence $(\mathcal{A}_\alpha)_{\alpha<\omega_1}$ of countable algebras contained in $\mathcal{D}$ in the following way. At stage α may find an increasing sequence $(A_k)_k$ in $\mathcal{A}_0$ such that $d(A_k) \le 1/2$, and whenever $\beta < \alpha$ there are natural numbers i and k with $s_\beta(i) \subseteq A_k$. Let $X_\alpha = X$ be a set as in Lemma 4.2. We let $\mathcal{A}_\alpha$ be the algebra generated by $\bigcup_{\beta<\alpha} \mathcal{A}_\beta \cup \{X_\alpha\}$.

(3) Now we write $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$, and denote by K the Stone space of all ultrafilters in the algebra $\mathcal{A}$. Since d is finitely additive on $\mathcal{A}$, it defines a probability Radon measure μ on K.

The inclusion $\mathcal{A} \subseteq \mathcal{D}$ means that the sequence of natural numbers is uniformly distributed with respect to μ. The set ω may be seen as a dense subset of K (of isolated points in K). For every $x \in K \setminus \omega$ there is $\alpha < \omega_1$ such that $s_\alpha(i) \in x$ for every i, and it follows from the construction that there is a local base at x contained in $\mathcal{A}_\alpha$. The algebra $\mathcal{A}_0$ witnesses that μ is a separable measure.

(4) Note that there is no separable subset of $K \setminus \omega$ of measure one. Indeed, if $(x_k)_k$ is a sequence (of ultrafilters) in $K \setminus \omega$ then for every k there is an $\alpha_k < \omega_1$ with $s_{\alpha_k}(i) \in x_k$ for every i. Hence, taking $\alpha > \alpha_k$, we see that all x_k are elements of the clopen set corresponding to X_α (of measure at most 1/2).

By (4), μ is not uniformly regular when considered on the space $K \setminus \omega$ (since uniform regularity implies the existence of a separable subspace of full measure), and hence is not uniformly regular on K.

THEOREM 4.3. *Assume CH. There exist a compact space S and a Radon probability measure ν on S with the following properties*:

(i) *S is separable and first-countable*;
(ii) *ν is separable*;
(iii) *ν has no uniformly distributed sequences.*

Proof (sketch). We first slightly modify the construction from the proof of Theorem 4.1. Keeping the notation from (1), we change (2) so that at step α we add a sequence $(X^n_\alpha)_n$ of sets to the algebra, where $d(X^n_\alpha) \le 1/(n+1)$, and every X^n_α is as in Lemma 4.2. We define K and μ as in (3).

Now μ has the additional property that it is zero on all separable subspaces of $K \setminus \omega$ (see (4)). This implies that if $(x_n)_n \subseteq K$ is uniformly distributed then so is the sequence $(x_n)_{x_n\in\omega}$.

Using CH find an enumeration $(\varphi_\alpha)_{\alpha<\omega_1}$ of all functions $\varphi \in \omega^\omega$ for which the sequence $(\varphi_\alpha(n))_n$ is uniformly distributed with respect to μ. For every $\alpha < \omega_1$ we can find a set $Y_\alpha \subseteq \omega \setminus X^1_\alpha$ for which the limit

$$\lim_{n\to\infty} \frac{|\{i \le n : \varphi_\alpha(i) \in Y_\alpha\}|}{n}$$

does not exist.

Let $\mathcal{C}$ be the algebra generated by $\mathcal{A}$ and $(Y_\alpha)_{\alpha<\omega_1}$, and let S denote the Stone space of $\mathcal{C}$. Then, again, S is a first-countable compact space. It should be clear from the construction that if ν is a Radon measure on S (in other words, a finitely additive measure on $\mathcal{C}$) such that ν restricted to $\mathcal{A}$ equals d, then ν has no uniformly distributed sequence. Now it suffices to show that there is such a measure ν which is separable.

This follows from the following general fact. If μ is a finitely additive measure on an algebra $\mathcal{A}$ and $\mathcal{A}$ is contained in some algebra $\mathcal{C}$ then the set of all extensions of μ to $\mathcal{C}$ is convex and nonempty. Moreover, every extreme point ν of this set has the property that $\mathcal{A}$ is dense in $\mathcal{C}$ with respect to symmetric difference (see Plachky [17]). In particular, if μ is separable then it has a separable extension to $\mathcal{C}$.

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