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SINGULAR INTEGRALS WITH HIGHLY OSCILLATING KERNELS ON PRODUCT SPACES

BY

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Abstract. We prove the $L^2(\mathbb{T}^2)$ boundedness of the oscillatory singular integrals

$$\mathcal{P}_0 f(x,y)=\int\limits_{D_x}\frac{e^{i(M_2(x)y'+M_1(x)x')}}{x'y'}f(x-x',y-y')\,dx'\,dy'$$

for arbitrary real-valued L^∞ functions $M_1(x),M_2(x)$ and for rather general domains $D_x\subseteq\mathbb{T}^2$ whose dependence upon x satisfies no regularity assumptions.

Introduction. Convergence properties of Fourier series are linked to singular integrals. In one dimension, the convergence relates to the Hilbert transform [1], [3], [4]. In several dimensions the theory of singular integrals on product spaces was developed in [6]–[8] and [11] having in mind some open problems of convergence a.e. of multiple Fourier series [9], [10].

In particular in [6] we proved the L^p boundedness, $1<p<\infty$, of the following operator defined as a principal value:

$$\sum_{h=0}^{\infty}\psi_h(y')\sum_{2^{-k}\le r(h,x)}\psi_k(x')*f(x,y)$$

for any measurable $0<r(h,x)\le 1$, where $\psi_k(x')=2^k\psi(2^kx')$, $\psi_h(y')=2^h\psi(2^hy')$ with $\psi(x')$ a C^∞ function supported on $\{|x'|\le 2\pi\}$ such that $1/x'=\sum_{k=0}^{\infty}\psi_k(x')$, for $|x'|\le\pi$. Here $(x,y)\in\mathbb{T}^2$ and $\mathbb{T}=[0,2\pi]$. The operator norm turned out to be independent of the choice of $r(h,x)$.

Moreover in [8] we proved the L^2 boundedness of the operator

$$\sum_{h=0}^{\infty}e^{iM(x)y'}\psi_h(y')\sum_{2^{-k}\le r(h,x)}\psi_k(x')*f(x,y) \tag{1}$$

for any bounded real-valued $M(x)$. The operator norm is independent of the choice of $r(h,x)$ and furthermore of the choice of $M(x)$ and its L^∞ norm.

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The purpose of this paper is twofold. First we point out that the L^2 boundedness holds for operators more general than (1). Indeed we shall prove the following

THEOREM. *Let $M_1(x)$ and $M_2(x)$ be bounded real-valued functions. Then the operator*

$$\mathcal{P}_0 f(x,y) = \sum_{h=0}^{\infty} e^{iM_2(x)y'} \psi_h(y') \sum_{2^{-k} \le r(h,x)} e^{iM_1(x)x'} \psi_k(x') * f(x,y)$$

is bounded from $L^2(\mathbb{T}^2)$ to itself with norm independent of any measurable $0 < r(h,x) \le 1$ and of $M_1(x), M_2(x)$ and their L^∞ norms. Moreover the maximal operator $\widetilde{\mathcal{P}}_0$ satisfies the following pointwise inequality:

$$\begin{aligned}\widetilde{\mathcal{P}}_0 f(x,y) &= \sup_{h_0} \Big| \sum_{h \le h_0} e^{iM_2(x)y'} \psi_h(y') \sum_{2^{-k} \le r(h,x)} e^{iM_1(x)x'} \psi_k(x') * f(x,y) \Big| \\ &\le c\{M_{y'} \widetilde{C}_{x'} f(x,y) + M_{y'} \mathcal{P}_0 f(x,y)\}.\end{aligned}$$

Here $M_{y'}$ denotes the Hardy–Littlewood maximal function acting on the y' variable and $\widetilde{C}_{x'}$ denotes the Carleson maximal operator

$$\widetilde{C}_{x'} g(x) = \sup_{k_0} \Big| \sum_{k \le k_0} e^{iM(x)x'} \psi_k(x') * g(x) \Big|.$$

Equivalently $\widetilde{C}$ can be defined as the linear operator

$$\widetilde{C}_{x'} g(x) = \sum_{k \le k_0(x)} e^{iM(x)x'} \psi_k(x') * g(x)$$

with $k_0(x)$ arbitrarily depending upon x. It is known that

$$\widetilde{C}_{x'} g(x) \le Mg(x) + MCg(x)$$

where C denotes the Carleson operator [5]. See also Lemma 1 of [6] and [7].

Second we wish to point out that the L^p boundedness, $1 < p < \infty$, of the operator (1) is still an open problem since the proof in [2] is inconclusive.

The paper is organized as follows: in Section 1 we prove the Theorem; in Section 2, concerning the L^p theory, we summarize the known results and point out the unproven claims of [2].

1. The proof of the theorem is not significantly different from the one for the operator (1) in [8]. We sketch it below. The goal is to break up the binding between the integration in dx' and dy' given by $r(h,x)$ and to neutralize $M_2(x)$. This is obtained by an application of the Plancherel theorem on the y variable.

Proof of the Theorem. It is easy to check that $\sum_{h=0}^{\infty}|\widehat{\psi}_h(\eta)|\le c$. Then for almost every x fixed we have

$$\begin{aligned}\int|\mathcal{P}_0 f(x,y)|^2\,dy &= \int|\widehat{\mathcal{P}}_0 f(x,\eta)|^2\,d\eta\\ &= \int\Big|\sum_h \widehat{\psi}_h(\eta - M_2(x))\Big(\sum_{2^{-k}\le r(h,x)}\psi_k(x')*f(x,\cdot)\Big)^{\wedge}(\eta)\Big|^2 d\eta\\ &\le \int\sum_h|\widehat{\psi}_h(\eta-M_2(x))|\sup_{k_0}\Big|\sum_{k\ge k_0}\psi_k(x')*\widehat{f}(x,\eta)\Big|^2 d\eta\\ &\le c\int|\widetilde{C}_{x'}\widehat{f}(x,\eta)|^2\,d\eta.\end{aligned}$$

Now by switching the order of integration we obtain

$$\begin{aligned}\iint|\mathcal{P}_0 f(x,y)|^2\,dy\,dx &\le c\iint|\widetilde{C}_{x'}\widehat{f}(x,\eta)|^2\,dx\,d\eta\\ &\le c\iint|\widehat{f}(x',\eta)|^2\,dx'\,d\eta \le c\iint|f(x',y')|^2\,dx'\,dy'.\end{aligned}$$

The role played by $\widetilde{H}_{x'}$ in [8] is now taken by $\widetilde{C}_{x'}$.

Similarly the proof of the pointwise inequality follows the same steps of the proof of Theorem 2 in [6].

2. The L^p boundedness, $p\neq 2$, of $\mathcal{P}_0$ is known under additional assumptions. We list three cases.

Case 1: The domain of integration is a rectangle. In this case $\mathcal{P}_0$ equals

$$\mathcal{P}_1 f(x,y) = e^{iM_2(x)y}H_{y'}(e^{-iM_2(x)y'}C_{x'}f(x,y'))(y).$$

The domain of integration might even depend arbitrarily upon x, that is, $|x'|\le A(x)$ and $|y'|\le B(x)$. Then $\mathcal{P}_0$ is equal to

$$\mathcal{P}_2 f(x,y) = e^{iM_2(x)y}H_{y'}(e^{-iM_2(x)y'}\widetilde{C}_{x'}f(x,y'))(y)$$

with $H_{y'}$ denoting a fixed truncation (depending upon x) of the Hilbert transform.

Remark. We point out that already in the case of $\mathcal{P}_1$ the order of integration is crucial. Let us write $\mathcal{P}_1$ explicitly with the "wrong" order of integration:

$$\int_{-\pi}^{\pi}\frac{e^{iM_1(x)x'}}{x'}\Big(\int_{-\pi}^{\pi}\frac{e^{iM_2(x)y'}}{y'}f(x-x',y-y')\,dy'\Big)\,dx'. \tag{2}$$

Only if $M_2(x)$ is a constant the operator is equal to$C_{x'}H_{y'}f(x,y)$. Otherwise it is not immediately decodable.

CASE 2: $M_2(x)$ is a constant. The proof of the boundedness of the corresponding $\mathcal{P}_0$ can be found in [7], p. 278, or else one can follow the steps of the proof of Theorem 3 of [6], replacing $\widetilde{H}_{x'}$ by $\widetilde{C}_{x'}$.

CASE 3: If $M_2(x) = M_1(x)$ and $r(h,x) = 2^{-h}$ or $M_2(x) = M_1^2(x)$ and $r(h,x) = 2^{-h}M_1(x)$ then $\mathcal{P}_0$ is bounded from $L^r(\mathbb{T}^2)$ to $L^p(\mathbb{T}^2)$, $1 < p < r \le 2$ (see [11]).

Now we come to [2]. The proof is subdivided in three cases $p = 2$, $p > 2$, $p < 2$. In all of them the natural order of integration in (1), i.e. dx' first and dy' second, is reversed. The next step, in [2], is more easily seen by considering (2) with $M_1(x) = 0$ for all x and $M_2(x) = M(x)$, that is,

$$\int_{-\pi}^{\pi} \frac{1}{x'} \bigg(\int_{-\pi}^{\pi} \frac{e^{iM(x)y'}}{y'} f(x - x', y - y')\, dy' \bigg)\, dx'.$$

To evaluate the $L^p(dx)$ norm of the above operator, the boundedness of the Hilbert transform, acting on x', cannot be used (the inner core—the operator acting on y'—varies arbitrarily with x, due to the phase $M(x)$, and the variable x is saturated in the integration). In [2] (line 7 from below on p. 299, line 4 from below on p. 300 and (6)), apparently, to dominate from above the $L^p(dx)$ norm of the operator (1), the phase $M(x)$ is replaced by $M(x')$ to be integrated together with $f(x', \cdot)$ in dx' (which is equivalent to keeping $M(x)$, replacing $f(x', \cdot)$ by $f(x, \cdot)$ and integrating in dx, precisely as in [2]).

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