COLLOQUIUM MATHEMATICUM

VOL. 86 2000 NO. 1

RANDOM WEIGHTED SIDON SETS

BY

KATHRYN E. HARE (WATERLOO, ON)

Abstract. We investigate random Sidon-type sets in which the degrees of the representations are weighted. These variants of Sidon sets are of interest as there are compact non-abelian groups which admit no infinite Sidon sets. In this note we determine the largest weight function such that infinite random weighted Sidon sets exist in all infinite compact groups.

1. Introduction. In 1927 Sidon ([13]) proved the delightful result that if E was a lacunary set of positive integers and f was a continuous function on the circle whose Fourier transform was non-zero only on E, then f had summable Fourier coefficients. Generalizations of this result have been studied by many authors, and today any subset of the dual of a compact group which has the property that every (randomly) continuous function whose Fourier transform is supported on the subset has summable Fourier coefficients, is known as a (random) Sidon set. Sidon sets have proven to be quite useful, as well as ubiquitous in duals of abelian groups (see [7], [8] and the references cited therein). Indeed, every infinite subset of the dual of a compact abelian group contains an infinite Sidon set.

In contrast, it is known that there are non-abelian compact groups whose duals admit no infinite Sidon sets ([4], [11]). Since this is due, in part, to the existence of representations of unbounded degrees, Sidon-type sets, but with the degrees of the representations weighted in different ways, have been considered by a number of authors (cf. [3], [6]).

Motivated by these generalizations, a study of random weighted Sidon sets was initiated in [2]. In this note we determine the largest weight function such that random Sidon-type sets with respect to this weight exist in all compact groups. Furthermore, we show that such sets are plentiful: any set of representations with degrees growing sufficiently rapidly is a set of this type.

2. Definitions and notation. Let G be a compact group and $\widehat{G}$ denote its dual object, a maximal set of irreducible, pairwise inequivalent, unitary

2000 *Mathematics Subject Classification*: Primary 43A46.

Key words and phrases: Sidon sets, compact non-abelian groups.

This research was partially supported by NSERC.

representations. The degree of $\sigma \in \widehat{G}$ will be denoted by d_σ. The Fourier transform of an integrable function f is given by $\widehat{f}(\sigma) = \int_G f(x)\overline{\sigma(x)}\,dx$ for $\sigma \in \widehat{G}$ where the measure is Haar measure on G. When $E \subset \widehat{G}$, we will write $\mathrm{Trig}_E(G)$ for the set of all trigonometric polynomials on G whose Fourier transform is non-zero only on E.

DEFINITION 2.1. Let $w : \mathbb{N} \to \mathbb{R}^+$. We will say $E \subset \widehat{G}$ is a (w,p) *Sidon set* if there exists a constant $c = c(w,p)$ such that

$$\Big(\sum_\sigma w(d_\sigma)\,\mathrm{Tr}\,|\widehat{f}(\sigma)|^p\Big)^{1/p} \le c\|f\|_\infty$$

for all $f \in \mathrm{Trig}_E(G)$. We will call E a *local* (w,p) *Sidon set* if there exists a constant $c = c(w,p)$ such that

$$(w(d_\sigma)\,\mathrm{Tr}\,|A_\sigma|^p)^{1/p} \le c\|d_\sigma\,\mathrm{Tr}\,A_\sigma\sigma\|_\infty$$

where $\sigma \in E$ and A_σ is any $d_\sigma \times d_\sigma$ matrix.

The (a,p) Sidon sets of [6] are the special case when the weight function $w(n) = n^a$. In this paper we will restrict our attention to the case $p = 1$ and refer to $(w,1)$ Sidon sets as simply *w-Sidon sets*, or *weighted Sidon sets* if we do not wish to specify the weight function w. Notice that the classical definition of Sidonicity is the case when the weight function is the identity.

When the set E consists of representations of bounded degree then there is no distinction between the classes of weighted Sidon sets. Furthermore, it is known that an infinite set of representations of bounded degree always contains an infinite Sidon subset [10], thus our interest is primarily in the case when E consists of representations of unbounded degree.

The concept of a random Sidon set was important in solving the union problem for Sidon sets [12]. To define random Sidon sets we need further notation: Let $U(n)$ be the set of $n \times n$ unitary matrices and $U^\infty = \prod_{\sigma\in\widehat{G}} U(d_\sigma)$. If $f \in L^2(G)$ and $W = (W_\sigma)_{\sigma\in\widehat{G}} \in U^\infty$, then we will write f_W for the L^2 function whose Fourier transform is given by $\widehat{f_W}(\sigma) = \widehat{f}(\sigma)W_\sigma$. A function $f \in L^2$ is called *randomly continuous* if f_W is continuous for almost every $W \in U^\infty$. Equipped with the norm

$$[[f]] = \int_{U^\infty} \|f_W\|_\infty\,dW,$$

the space of all randomly continuous functions on G forms a Banach space.

DEFINITION 2.2. We will say $E \subset \widehat{G}$ is a (*local*) *random w-Sidon set* if there exists a constant $c = c(w)$ such that

$$\sum_\sigma w(d_\sigma)\,\mathrm{Tr}\,|\widehat{f}(\sigma)| \le c[[f]]$$

for all $f \in \mathrm{Trig}_E(G)$ (respectively, for all $f = d_\sigma\,\mathrm{Tr}\,A_\sigma\sigma,\ \sigma \in E$).

Notice that $[[f]] \geq \inf\{\|f_W\|_\infty : f_W$ is continuous$\}|$, consequently any w-Sidon set is a random w-Sidon set. The converse is known to be true in the classical case when the weight function is the identity [12], but otherwise is open.

Pisier's characterization of Sidon sets, in terms of the size of the L^p norms of certain functions, generalizes to the following characterization which is key to our work.

THEOREM 2.3. *A subset E of $\widehat{G}$ is a* (*local*) *random w-Sidon set if and only if there exists a constant C such that for every $p \geq 2$ and for every $f \in \mathrm{Trig}_E(G)$* (*respectively*, $f = d_\sigma \operatorname{Tr} A_\sigma \sigma,\ \sigma \in E$),

$$\|f\|_p \leq C\sqrt{p}\bigg(\sum \frac{d_\sigma^3}{w(d_\sigma)^2} \operatorname{Tr} |\widehat{f}(\sigma)|^2\bigg)^{1/2}.$$

Proof. The proof for Sidon sets can be found in [9], and by making the obvious modifications the result for general weights can be obtained. The details can be found in [1].

3. Existence of weighted Sidon sets. Clearly, it is easier to be a (random) w-Sidon set as w decreases, and any (random) w-Sidon set is local (random) w-Sidon. Because

$$\operatorname{Tr} |\widehat{f}(\sigma)| \leq (d_\sigma \operatorname{Tr} |\widehat{f}(\sigma)|^2)^{1/2} = \|d_\sigma \operatorname{Tr} \widehat{f}(\sigma)\sigma\|_2 \leq \|d_\sigma \operatorname{Tr} \widehat{f}(\sigma)\sigma\|_\infty,$$

the entire set $\widehat{G}$ is local 1-Sidon. It is also known that every infinite subset of $\widehat{G}$ contains an infinite $n^{-\varepsilon}$-Sidon set for any $\varepsilon > 0$; and if G is a compact, connected, simple Lie group, then every local n^{ε}-Sidon set, for $\varepsilon > 0$, is finite (see [6]).

To partially bridge this gap, Adams and Grow [2] have shown that any infinite subset of the dual of any compact group contains an infinite random 1-Sidon set. Our goal is to prove that this assertion remains true if the weight $w = 1$ is replaced by $w(n) = O(\sqrt{\log n})$, and that this is optimal.

First, we will prove that the assertion is not (universally) true for any larger weight function.

PROPOSITION 3.1. *Suppose G is a compact*, *connected*, *simple Lie group and $E \subset \widehat{G}$ is a local random w-Sidon set. Then there exists a constant C such that $w(n) \leq C\sqrt{\log n}$ for all $n \in \{d_\sigma : \sigma \in E\}$.*

Proof. Suppose E is a local random w-Sidon set. The characterization theorem implies that for all $p \geq 2$ and $\sigma \in E$,

$$\|\operatorname{Tr} \sigma\|_p \leq C\sqrt{p}\, \frac{d_\sigma}{w(d_\sigma)}.$$

On the other hand, it is known ([5]) that if G is a compact, connected, simple Lie group, then for all $\sigma \in \widehat{G}$ and $p \ge 2$ we have

$$\|\operatorname{Tr}\sigma\|_p \ge C_G d_\sigma^{1-\dim G/p},$$

where C_G is a constant which depends only on the group G, and $\dim G$ is the dimension of the Lie group G.

Taking $p = (\dim G)(\log d_\sigma)$ gives the conclusion that $w(d_\sigma) \le C_1\sqrt{\log d_\sigma}$ for $C_1 = Ce\sqrt{\dim G}/C_G$.

It is also straightforward to prove that the entire dual object $\widehat{G}$ is always local random $\sqrt{\log n}$-Sidon.

PROPOSITION 3.2. *If G is any compact group then $\widehat{G}$ is local random w-Sidon for any weight function w satisfying $w(n) \le C\sqrt{\log n}$.*

Proof. If A_σ is a $d_\sigma \times d_\sigma$ matrix and $q < 2$ then

$$(\operatorname{Tr}|A_\sigma|^q)^{1/q} \le d_\sigma^{1/q-1/2}(\operatorname{Tr}|A_\sigma|^2)^{1/2}.$$

Applying this inequality with $q = p'$, the conjugate index to p, together with the Hausdorff–Young inequality, shows that for any $p \ge 2$,

$$\|d_\sigma \operatorname{Tr} A_\sigma\sigma\|_p \le (d_\sigma^{4/p'-1}\operatorname{Tr}|A_\sigma|^2)^{1/2} = (d_\sigma^{3-4/p}\operatorname{Tr}|A_\sigma|^2)^{1/2}.$$

By the characterization theorem it suffices to establish that

$$\|d_\sigma \operatorname{Tr} A_\sigma\sigma\|_p \le C\sqrt{p}\left(\frac{d_\sigma^3}{\log d_\sigma}\operatorname{Tr}|A_\sigma|^2\right)^{1/2}$$

for some constant C. Thus we need only verify the inequality

$$\frac{\log x}{x^{4/p}} \le Cp \quad \text{for all } x \ge 1,$$

and this is a routine calculus exercise.

More interesting is to find examples of random $\sqrt{\log n}$-Sidon sets. We will show that any set of representations whose degrees form a suitably sparse set of integers is such a set.

THEOREM 3.3. *Let G be any compact group and E any subset of $\widehat{G}$ consisting of representations of unbounded degrees. Then E contains an infinite random $\sqrt{\log n}$-Sidon set.*

Proof. The infinite subset of E will be chosen according to the following procedure. Select any $\sigma_1 \in E$ and assume inductively that $\sigma_1, \ldots, \sigma_{k-1}$ have been picked. Choose any $\sigma_k \in E$ with degree sufficiently large to satisfy the following three conditions:

(1) if $d_{\sigma_{k-1}} \in (2^{2^{n-1}}, 2^{2^n}]$ then $d_{\sigma_k} \ge 2^{2^{n+1}}$;
(2) $d_{\sigma_k} \ge e^{\sqrt{2^k}/4}$; and
(3) if $d_{\sigma_{k-1}} = e^{\sqrt{p_1}/4}$ then $d_{\sigma_k} \ge e^{p_1}$.

Condition (2) ensures that

$$|\{\sigma_j : d_{\sigma_j} \leq e^{\sqrt{2^k}/4}\}| \leq k \quad \text{for all } k,$$

and the reader can readily verify that (3) guarantees that for any p,

$$|[e^{\sqrt{p}/4}, e^p) \cap \{d_{\sigma_k}\}_{k=1}^{\infty}| \leq 1.$$

It is convenient for the proof to analyze the behaviour of the function

$$g_p(x) = x^{-4/(p-2)}(\log x)^{p/(p-2)} \quad \text{for } p \geq 4.$$

One can easily check that g_p increases to its maximum value of $g_p(e^{p/4}) = e^{-p/(p-2)}(p/4)^{p/(p-2)}$, and then decreases. Since $p^{2/(p-2)} \to 1$ as $p \to \infty$, it follows that

$$g_p(x) \leq g_p(e^{p/4}) \leq C_1 p$$

for some constant C_1. One can similarly show that $g_p(x) \leq C_2\sqrt{p}$ if $x \leq e^{\sqrt{p}/4}$.

Next, temporarily fix $p \geq 4$ and suppose $d_{\sigma_k} \geq e^p$. Suppose also that $d_{\sigma_k} \in (2^{2^{n(k)-1}}, 2^{2^{n(k)}}]$. Then $d_{\sigma_{k+1}} \geq 2^{2^{n(k)+1}}$, and as g_p is a decreasing function for $x \geq e^{p/4}$ we have $g_p(d_{\sigma_k}) \geq g_p(2^{2^{n(k)}})$ and $g_p(d_{\sigma_{k+1}}) \leq g_p(2^{2^{n(k)+1}})$. Hence

$$\begin{aligned}
\frac{g_p(d_{\sigma_k})}{g_p(d_{\sigma_{k+1}})} &\geq \frac{(2^{-2^{n(k)}})^{4/(p-2)}(\log 2^{2^{n(k)-1}})^{p/(p-2)}}{(2^{-2^{n(k)+1}})^{4/(p-2)}(\log 2^{2^{n(k)+1}})^{p/(p-2)}} \\
&= (2^{2^{n(k)+1}-2^{n(k)}})^{4/(p-2)}\left(\frac{2^{n(k)-1}}{2^{n(k)+1}}\right)^{p/(p-2)} \\
&\geq 2^{2^{n(k)}4/(p-2)}2^{-2p/(p-2)}.
\end{aligned}$$

Since $2^{2^{n(k)}} \geq d_{\sigma_k} \geq e^p$,

$$2^{2^{n(k)}4/(p-2)}2^{-2p/(p-2)} \geq e^{4p/(p-2)}2^{-2p/(p-2)} \geq 4.$$

This means $\sum_{d_{\sigma_k} \geq e^p} g_p(d_{\sigma_k})$ is a geometric series with ratio at most $1/4$, and consequently

$$\sum_{d_{\sigma_k} \geq e^p} g_p(d_{\sigma_k}) \leq 2 \max g_p(x) \leq 2C_1 p.$$

We are now ready to establish that $\{\sigma_k\}$ is a random $\sqrt{\log n}$-Sidon set. We first remark that it clearly suffices to prove

$$\|f\|_p \leq C\sqrt{p}\left(\sum \frac{d_\sigma^3}{w(d_\sigma)^2} \operatorname{Tr}|\widehat{f}(\sigma)|^2\right)^{1/2}$$

for $p = 2^n$ and all $n \geq 2$. Fix $p = 2^n$ and let q be the conjugate index. We will use the Cauchy–Schwarz inequality in the form

$$\sum d_i \operatorname{Tr}|A_i B_i| \leq \left(\sum d_i \operatorname{Tr}|A_i|^{2/q}\right)^{q/2}\left(\sum d_i \operatorname{Tr}|B_i|^{(2/q)'}\right)^{1/(2/q)'}.$$

Together with the Hausdorff–Young inequality this gives

$$\begin{aligned}\Big\|\sum d_{\sigma_k}\operatorname{Tr}A_k\sigma_k\Big\|_p &\le \Big(\sum d_{\sigma_k}\operatorname{Tr}|A_k|^q\Big)^{1/q}\\ &= \left(\sum \frac{d_{\sigma_k}^3}{\log d_{\sigma_k}}\operatorname{Tr}\left(\frac{\log d_{\sigma_k}}{d_{\sigma_k}^2}|A_k|^q\right)\right)^{1/q}\\ &\le \left(\sum \frac{d_{\sigma_k}^3}{\log d_{\sigma_k}}\operatorname{Tr}|A_k|^2\right)^{1/2}\\ &\quad\times\left(\sum \frac{d_{\sigma_k}^3}{\log d_{\sigma_k}}\operatorname{Tr}\left|\frac{\log d_{\sigma_k}}{d_{\sigma_k}^2}I_{d_{\sigma_k}}\right|^{2/(2-q)}\right)^{(2-q)/2q}.\end{aligned}$$

Simplifying, we obtain

$$\begin{aligned}\Big\|\sum d_{\sigma_k}TrA_k\sigma_k\Big\|_p \le \left(\sum \frac{d_{\sigma_k}^3}{\log d_{\sigma_k}}\operatorname{Tr}|A_k|^2\right)^{1/2}\\ \times\Big(\sum d_{\sigma_k}^{-4/(p-2)}(\log d_{\sigma_k})^{p/(p-2)}\Big)^{(p-2)/2p}.\end{aligned}$$

Because $p^{-1/p}$ is bounded, if we can prove

$$\sum d_{\sigma_k}^{-4/(p-2)}(\log d_{\sigma_k})^{p/(p-2)} \le Cp, \tag{1}$$

then we would have the desired inequality

$$\Big(\sum d_{\sigma_k}^{-4/(p-2)}(\log d_{\sigma_k})^{p/(p-2)}\Big)^{(p-2)/2p} \le (Cp)^{1/2-1/p} \le C_0\sqrt{p}.$$

It is in estimating this sum that our study of the function g_p is helpful, for what we need to show in order to prove (1) is that $\sum g_p(d_{\sigma_k}) \le Cp$. We write this sum as

$$\sum_{d_{\sigma_k}\le e^{\sqrt{p}/4}} g_p(d_{\sigma_k}) + \sum_{d_{\sigma_k}\in(e^{\sqrt{p}/4},\, e^p)} g_p(d_{\sigma_k}) + \sum_{d_{\sigma_k}\ge e^p} g_p(d_{\sigma_k}).$$

Recall that $|\{\sigma_j : d_{\sigma_j} \le e^{\sqrt{2^n}/4}\}| \le n$ and that $p = 2^n$. Thus

$$\sum_{d_{\sigma_k}\le e^{\sqrt{p}/4}} g_p(d_{\sigma_k}) \le n\max\{g_p(x) : x \le e^{\sqrt{p}/4}\} \le nC_2\sqrt{p} \le C2^n.$$

Since $|[e^{\sqrt{p}/4}, e^p) \cap \{d_{\sigma_k}\}| \le 1$, the second summand is at most $\max g_p \le C_1 2^n$. The third summand is the geometric series which we saw above summed to at most $2C_1 2^n$. Combining these three parts gives the desired result, completing the proof that this subset of E is a random $\sqrt{\log n}$-Sidon set.

An easy consequence of this theorem is to demonstrate that random $\sqrt{\log n}$-Sidon sets are as plentiful as Sidon sets are in the abelian setting.

COROLLARY 3.4. *Let G be any compact group. Then every infinite subset of $\widehat{G}$ contains an infinite random w-Sidon set for $w(n) = \sqrt{\log n}$.*

Proof. Let E be an infinite subset of $\widehat{G}$. If E contains a infinite subset of representations of bounded degree then E contains a Sidon set [10]. Otherwise, E consists of representations of unbounded degrees and we may appeal to the previous result.

REFERENCES

[1] K. Adams, *Weighted Sidon sets*, Ph.D. dissertation, Univ. of Missouri, 1998.
[2] K. Adams and D. Grow, *Lacunary series on compact groups*, this issue, 1–7.
[3] A. Dooley, *Central lacunary sets for Lie groups*, J. Austral. Math. Soc. 45 (1988), 30–45.
[4] A. Dooley and P. Soardi, *Local p-Sidon sets for Lie groups*, Proc. Amer. Math. Soc. 72 (1978), 125–126.
[5] K. Hare, *The size of characters of compact Lie groups*, Studia Math. 129 (1998), 1–18.
[6] K. Hare and D. Wilson, *Weighted p-Sidon sets*, J. Austral. Math. Soc. 61 (1996), 73–95.
[7] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Springer, New York, 1979.
[8] J. López and K. Ross, *Sidon Sets*, Marcel Dekker, New York, 1975.
[9] M. Marcus and G. Pisier, *Random Fourier Series with Applications to Harmonic Analysis*, Ann. of Math. Stud. 101, Princeton Univ. Press, Princeton, 1981.
[10] W. Parker, *Central Sidon and central $\Lambda(p)$ sets*, J. Austral. Math. Soc. 14 (1972), 62–74.
[11] D. Ragozin, *Central measures on compact simple Lie groups*, J. Funct. Anal. 10 (1972), 212–229.
[12] D. Rider, *Randomly continuous functions and Sidon sets*, Duke Math. J. 42 (1975), 759–764.
[13] S. Sidon, *Verallgemeinerung eines Satzes über die absolute Konvergenz von Fourierreihen mit Lücken*, Math. Ann. 97 (1927), 675–676.

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario
N2L 3G1 Canada
E-mail: kehare@uwaterloo.ca

Received 28 September 1999 (3866)