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LACUNARY SERIES ON COMPACT GROUPS

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Abstract. A theorem of Sidon concerning absolutely convergent Fourier series is extended to compact groups.

1. Introduction. Sidon [15] proved the following theorem concerning Fourier series of functions defined on the unit circle:

Let $E = \{n_k\}_{k=0}^{\infty}$ *be an infinite sequence of positive integers satisfying*

$$\inf_{k\ge 0} \frac{n_{k+1}}{n_k} > 1. \tag{1}$$

If $\sum(\alpha_k \exp(in_k x)+\beta_k \exp(-in_k x))$ *is the Fourier series of a bounded function then*

$$\sum_{k=0}^{\infty}(|\alpha_k| + |\beta_k|) < \infty.$$

Numerous authors have extended this result in a variety of ways. For example, the Hadamard lacunary condition (1) has gradually been recognized as a manifestation of the more natural group-theoretic criterion that E be expressible as a finite union of quasi-independent sets [16, 14, 6, 11, 10]. This has led to a generalization of Sidon's theorem with the unit circle replaced by an arbitrary compact abelian group G and the integers replaced by the group Γ dual to G [8(2.19), 10(2.13)]. In this setting, it has been discovered that the phrase "bounded function" can be replaced with "continuous function" [2] or even "randomly continuous function" [12] without changing the class of sets E.

For compact (possibly nonabelian) groups, however, generalizations of Sidon's theorem encounter fundamental obstacles; there exist compact groups whose only classical Sidon sets E are finite [5(37.21)(b), 1]. In this paper we extend Sidon's theorem to compact groups by showing that a certain class of infinite lacunary subsets always exists when the group is infinite (Theorem 1). When the group is abelian, these lacunary subsets reduce to classical Sidon sets [12]. Furthermore, for SU(2) we show that the lacunary

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subsets include all subsets in the dual which are expressible as a finite union of quasi-independent sets (Theorem 2).

2. The main lacunary result. Let G be a compact group with Haar integral $\int_G \ldots dx$. If $f \in L^1(G)$ then the Fourier series of f is

$$f(x) \sim \sum_{\sigma \in \widehat{G}} d(\sigma)\,\mathrm{tr}(\widehat{f}(\sigma)\sigma(x))$$

where $\widehat{G}$, the dual of G, is a maximal set of inequivalent continuous irreducible unitary representations of G, $d(\sigma)$ denotes the degree of the representation σ, and $\widehat{f}(\sigma)$ is the linear transformation

$$\widehat{f}(\sigma) = \int_G f(x)\sigma(x^{-1})\,dx.$$

If $\widehat{f}(\sigma) = 0$ for all $\sigma \notin E \subseteq \widehat{G}$ then f is called an *E-spectral function*.

For $\sigma \in \widehat{G}$, let $U(d(\sigma))$ denote the compact group of $d(\sigma)$-by-$d(\sigma)$ complex unitary matrices, and form the compact product group

$$\mathcal{U}^\infty = \prod_{\sigma \in \widehat{G}} U(d(\sigma)).$$

If F is a complex function on $\mathcal{U}^\infty$, let $\int_{\mathcal{U}^\infty} F(\mathbb{W})\,d\mathbb{W}$ denote its Haar integral.

If $f \in L^2(G)$ and $\mathbb{W} = \{W_\sigma\}_{\sigma \in \widehat{G}} \in \mathcal{U}^\infty$, let $f_\mathbb{W}$ be the function in $L^2(G)$ such that

$$\widehat{f_\mathbb{W}}(\sigma) = \widehat{f}(\sigma)W_\sigma \qquad (\sigma \in \widehat{G}).$$

A function $f \in L^2(G)$ is called *randomly continuous* on G provided $f_\mathbb{W}$ is a continuous function on G for almost every $\mathbb{W} \in \mathcal{U}^\infty$. Equipped with the norm

$$[\![f]\!] = \int_{\mathcal{U}^\infty} \sup_{x \in G} |f_\mathbb{W}(x)|\,d\mathbb{W},$$

the space of all randomly continuous functions on G forms a Banach space. For unexplained notation and results, see [5] or [9].

THEOREM 1. *If G is a compact group then every infinite set in $\widehat{G}$ contains an infinite subset E with the property that*

$$\sum_{\sigma \in E} \mathrm{tr}\,|\widehat{f}(\sigma)| < \infty \tag{2}$$

for all randomly continuous functions f on G.

Proof. Let A be an infinite subset of $\widehat{G}$. First, suppose $\sup\{d(\sigma) : \sigma \in A\} < \infty$. A theorem of Hutchinson [7] ensures that A contains an infinite Sidon set E, i.e. a set E such that

$$\sum_{\sigma\in E} d(\sigma)\,\mathrm{tr}\,|\widehat{g}(\sigma)| < \infty$$

for all E-spectral continuous functions g on G. Consequently [5(37.25)], there exists a constant $K > 0$ such that

$$\|f\|_{L^q(G)} \le K\sqrt{q}\,\|f\|_{L^2(G)}$$

for all E-spectral functions f in $L^2(G)$ and all $q > 2$. Let P denote the projection of $L^2(G)$ onto the subspace of E-spectral functions in $L^2(G)$. By the proof of Theorem VI.2.3 in [9], there exists a constant $L > 0$ such that

$$\sum_{\sigma\in E} d(\sigma)\,\mathrm{tr}\,|\widehat{f}(\sigma)| = \sum_{\sigma\in\widehat{G}} d(\sigma)\,\mathrm{tr}\,|\widehat{Pf}(\sigma)| \le L[\![f]\!]$$

for all randomly continuous functions f on G, and (2) follows.

Thus, we may suppose $\sup\{d(\sigma) : \sigma \in A\} = \infty$. In this case, an appeal to the following lemma concludes the proof of Theorem 1.

LEMMA 1. *Let $E = \{\sigma_j\}_{j=1}^{\infty}$ be a sequence of representations from $\widehat{G}$ with the property that $d(\sigma_j) \ge 2^j$ for $j \ge 1$. Then*

$$\sum_{j=1}^{\infty} \mathrm{tr}\,|\widehat{f}(\sigma_j)| < \infty$$

for all randomly continuous functions f on G.

It will be convenient to separate the demonstration into two distinct lemmas which together imply Lemma 1. Let M denote the central multiplier from $L^2(G)$ to $L^2(G)$ defined by

$$(Mf)^{\wedge}(\sigma) = \frac{1}{d(\sigma)}\widehat{f}(\sigma)$$

for $\sigma \in \widehat{G}$ and $f \in L^2(G)$.

LEMMA 2. *Let S be a subset of $\widehat{G}$ with the property that*

$$\|Mg\|_{L^q(G)} \le C\sqrt{q}\,\|g\|_{L^2(G)} \tag{3}$$

for some constant $C > 0$, all $q > 2$, and all S-spectral functions g in $L^2(G)$. Then there exists a constant $D > 0$ such that

$$\sum_{\sigma\in S} \mathrm{tr}\,|\widehat{f}(\sigma)| \le D[\![f]\!]$$

for all randomly continuous functions f on G.

LEMMA 3. *Let $E \subseteq \widehat{G}$ be as in Lemma 1. Then*

$$\|Mg\|_{L^q(G)} \le (4\ln(2))^{-1/2}\sqrt{q}\,\|g\|_{L^2(G)}$$

for all E-spectral functions g in $L^2(G)$ and all $q > 2$.

Proof of Lemma 2. It is well known that (3) is equivalent to the existence of a constant $B > 0$ such that

$$\|Mg\|_{L^\psi(G)} \le B\|g\|_{L^2(G)}$$

for all S-spectral functions g in $L^2(G)$; here $L^\psi(G)$ denotes the Orlicz space based on the function $\psi(t) = \exp(t^2) - 1$. A trivial modification of the argument given on pages 119–120 of [9] accomplishes the proof.

Proof of Lemma 3. Let g be an E-spectral function in $L^2(G)$, let $q > 2$, and let $p^{-1} + q^{-1} = 1$. The Hausdorff–Young–Riesz Theorem and Hölder's inequality imply

$$\begin{aligned} (4) \qquad \|Mg\|_{L^q(G)} &\le \Big(\sum_{\sigma\in E} d(\sigma)\operatorname{tr}\Big|\frac{1}{d(\sigma)}\widehat{g}(\sigma)\Big|^p\Big)^{1/p} \\ &\le \Big(\sum_{\sigma\in E} d(\sigma)\operatorname{tr}\Big|\Big(\frac{1}{d(\sigma)}\Big)^p I_{d(\sigma)}\Big|^{2/(2-p)}\Big)^{(2-p)/(2p)} \\ &\quad\times \Big(\sum_{\sigma\in E} d(\sigma)\operatorname{tr}|\widehat{g}(\sigma)|^2\Big)^{1/2} \\ &= \Big(\sum_{\sigma\in E} (d(\sigma))^{4/(2-q)}\Big)^{(q-2)/(2q)} \|g\|_{L^2(G)}. \end{aligned}$$

Since $d(\sigma_j) \ge 2^j$ for $j \ge 1$,

$$(5) \qquad \sum_{j=1}^{\infty} (d(\sigma_j))^{4/(2-q)} \le \sum_{j=1}^{\infty} (2^{4/(q-2)})^{-j} = (2^{4/(q-2)} - 1)^{-1}.$$

Combining (4) and (5), and noting that

$$F(x) = \frac{(2^{4/(x-2)} - 1)^{(2-x)/x}}{x}$$

is increasing on $(2, \infty)$ with $F(x) \to 1/(4 \ln 2)$ as $x \to \infty$, yields the desired conclusion.

3. Lacunarity and quasi-independence for SU(2)**.** For $\sigma \in \widehat{G}$, let $\chi_\sigma(x) = \operatorname{tr}(\sigma(x))$ $(x \in G)$ denote the character of σ. For $\varepsilon \in \{-1, 0, 1\}$ define χ_σ^ε to be χ_σ if $\varepsilon = 1$, 1 if $\varepsilon = 0$, and $\overline{\chi}_\sigma$ if $\varepsilon = -1$. Following [17], a subset E of $\widehat{G}$ is called *quasi-independent* if, for all finite subsets $\{\sigma_1, \dots, \sigma_n\}$ of E,

$$\int_G \chi_{\sigma_1}^{\varepsilon_1}(x) \dots \chi_{\sigma_n}^{\varepsilon_n}(x)\, dx > 0$$

for an n-tuple in $\{-1, 0, 1\}^n$ implies $\varepsilon_1 = \dots = \varepsilon_n = 0$. When G is abelian, this definition of quasi-independence agrees with Pisier's [10].

The special unitary group SU(2) is the compact group of 2-by-2 complex unitary matrices with determinant 1. Following a standard convention [5(29.27)], we parametrize the dual of SU(2) by the nonnegative half-integers:

$$\widehat{\mathrm{SU}(2)} = \{\sigma_0, \sigma_{1/2}, \sigma_1, \sigma_{3/2}, \ldots\}.$$

The representation σ_l has degree $d(\sigma_l) = 2l + 1$ and its character $\chi_l = \mathrm{tr}(\sigma_l)$ is real-valued:

$$\overline{\chi_l(x)} = \chi_l(x) \qquad (x \in \mathrm{SU}(2)). \tag{6}$$

The following consequence of the Clebsch–Gordon formula is useful in decomposing products of the irreducible characters of SU(2):

$$\chi_l \chi_{l'} = \sum_{j=0}^{2l} \chi_{l'-l+j} \tag{7}$$

for all half-integers $0 \le l \le l'$. The general orthogonality relations for irreducible characters imply

$$\int_{\mathrm{SU}(2)} \chi_l(x)\, dx = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

THEOREM 2. *If E is a finite union of quasi-independent subsets of the dual of* SU(2) *then*

$$\sum_{\sigma \in E} \mathrm{tr}\, |\widehat{f}(\sigma)| < \infty$$

for all randomly continuous functions f on SU(2).

Proof. Let A be a quasi-independent subset of the dual of SU(2) and write $A = \{\sigma_{l_j}\}$ where $0 < l_1 < l_2 < \ldots$ An elementary argument using (6), (7), (8), and the definition of quasi-independence shows that $l_{j+3} \ge l_{j+1} + l_j$ for all $j \ge 1$; consequently

$$d(\sigma_{l_{j+3}}) = 2l_{j+3} + 1 \ge 2(2l_j + 1) = 2d(\sigma_{l_j}).$$

It follows that $A = A_0 \cup A_1 \cup A_2$ where

$$A_i = \{\tau_j^{(i)}\}_{j \ge 1} = \{\sigma_{l_{3j-i}}\}_{j \ge 1}$$

and $d(\tau_j^{(i)}) \ge 2^j$ for $j \ge 1$.

Since E is a finite union of such sets A, there exists a finite collection $E_1, \ldots, E_N$ of subsets of the dual object of SU(2) with the following properties:

- $E = \bigcup_{m=1}^{N} E_m$;
- $E_m \cap E_n = \emptyset$ if $1 \le m < n \le N$;
- for each $1 \le m \le N$, $E_m = \{\tau_j^{(m)}\}_{j \ge 1}$; and
- $d(\tau_j^{(m)}) \ge 2^j$ for all $j \ge 1$.

Let f be an E-spectral function in $L^2(G)$ and write $f = f_1 + \ldots + f_N$ where each f_m is an E_m-spectral function in $L^2(G)$. For any $q > 2$, Lemma 3 yields

$$\begin{aligned}\|Mf\|_{L^q(G)} &\le \sum_{m=1}^{N} \|Mf_m\|_{L^q(G)} \\ &\le (4\ln 2)^{-1/2}\sqrt{q}\sum_{m=1}^{N}\|f_m\|_{L^2(G)} \le \left(\frac{N}{4\ln 2}\right)^{1/2}\sqrt{q}\,\|f\|_{L^2(G)}.\end{aligned}$$

Apply Lemma 2 to finish the proof.

4. Remarks. We know of no compact group G for which finite unions of quasi-independent sets in $\widehat{G}$ fail to have the property of Theorem 2. The second author wishes to express his thanks to David Wilson for his gracious hospitality during visits to the University of New South Wales in 1990 and 1992; the mathematical discussions of those visits ultimately bore fruit in this paper. We also wish to acknowledge our indebtedness to [4] and to discussions with both its authors. Properties of the lacunary sets of Theorem 1, and their generalizations, have been explored by the first author in a recent Ph.D. dissertation [0]. Finally, we would like to pose a question which we have been unable to resolve, even for the case $G = \mathrm{SU}(2)$:

Let G be an infinite compact group. Does $\widehat{G}$ contain an infinite subset E with the property that

$$\sum_{\sigma\in E} \operatorname{tr}|\widehat{f}(\sigma)| < \infty$$

for all E-spectral continuous functions f on G?

REFERENCES

[0] K. Adams, *Weighted Sidon sets*, Ph.D. dissertation, Univ. of Missouri at Rolla, 1998.

[1] C. Cecchini, *Lacunary Fourier series on compact Lie groups*, J. Funct. Anal. 11 (1972), 191–203.

[2] S. Banach, *Über einige Eigenschaften der lakunären trigonometrischen Reihen*, Studia Math. 2 (1930), 207–220.

[3] A. Figà-Talamanca and D. Rider, *A theorem of Littlewood and lacunary series for compact groups*, Pacific J. Math. 16 (1966), 505–514.

[4] K. Hare and D. Wilson, *Weighted p-Sidon sets*, J. Austral. Math. Soc. 61 (1996), 73–95.

[5] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vols. I and II, 2nd ed., Springer, New York, 1979.

[6] E. Hewitt and H. S. Zuckerman, *Some theorems on lacunary series with extensions to compact groups*, Trans. Amer. Math. Soc. 93 (1959), 1–19.

[7] M. F. Hutchinson, *Non-tall compact groups admit infinite Sidon sets*, J. Austral. Math. Soc. 23 (1977), 467–475.
[8] J. H. López and K. A. Ross, *Sidon Sets*, Marcel Dekker, New York, 1975.
[9] M. B. Marcus and G. Pisier, *Random Fourier Series with Applications to Harmonic Analysis*, Princeton Univ. Press, Princeton, 1981.
[10] G. Pisier, *Conditions d'entropie et caractérisations arithmétiques des ensembles de Sidon*, in: Topics in Modern Harmonic Analysis, vol. II, Ist. Naz. Alta Mat. F. Severi, Roma, 1983, 911–944.
[11] D. Rider, *Gap series on groups and spheres*, Canad. J. Math. 18 (1966), 389–398.
[12] —, *Randomly continuous functions and Sidon sets*, Duke Math. J. 42 (1975), 759–764.
[13] W. Rudin, *Trigonometric series with gaps*, J. Math. Mech. 9 (1960), 203–208.
[14] —, *Fourier Analysis on Groups*, Wiley, New York, 1962.
[15] S. Sidon, *Verallgemeinerung eines Satzes über die absolute Konvergenz von Fourierreihen mit Lücken*, Math. Ann. 97 (1927), 675–676.
[16] S. B. Stechkin, *On absolute convergence of Fourier series III*, Izv. Akad. Nauk SSSR Ser. Mat. 20 (1956), 385–412 (in Russian).
[17] D. Wilson, *On the structure of Sidon sets*, Monatsh. Math. 101 (1986), 67–74.

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