A NOTE ON THE CONSTRUCTION
OF NONSINGULAR GIBBS MEASURES

BY

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Dedicated to the memory of Prof. Anzelm Iwanik

Abstract. We give a sufficient condition for the construction of Markov fibred systems using countable Markov partitions with locally bounded distortion.

0. Introduction. Let $X$ be a compact metric space with metric $d$ and $T : X \to X$ be a noninvertible piecewise $C^0$-invertible map, i.e. there exists a finite or countable partition $X = \bigcup_{i \in I} X_i$ such that $\bigcup_{i \in I} \operatorname{int} X_i$ is dense in $X$ and

1. For each $i \in I$ with $\operatorname{int} X_i \neq \emptyset$, $T|_{\operatorname{int} X_i} : \operatorname{int} X_i \to T(\operatorname{int} X_i)$ is a homeomorphism and $(T|_{\operatorname{int} X_i})^{-1}$ extends to a homeomorphism $v_i$ on $\overline{\operatorname{cl}(T(\operatorname{int} X_i))}$.

2. $T(\bigcup_{i \in I} \operatorname{int} X_i = \emptyset X_i) \subset \bigcup_{i \in I} \operatorname{int} X_i = \emptyset X_i$.

3. $\{X_i\}_{i \in I}$ generates $\mathcal{F}$ with respect to $T$, where $\mathcal{F}$ is the $\sigma$-algebra of Borel subsets of $X$.

We set $\overline{A} = \overline{\operatorname{cl}(int A)}$ ($A \subset X$) and define $\alpha = \{\overline{X_i}\}_{i \in I}$. Then $\alpha$ is a finite or countable partition of a dense subset of $X$ which is not necessarily a disjoint family. We impose the Markov property on $\alpha$:

4. $\operatorname{int}(\overline{X_i} \cap \overline{TX_j}) \neq \emptyset$ implies $\overline{TX_j} \supset \overline{X_i}$.

Let $\mathcal{A}$ denote the set of all admissible sequences with respect to $(T, \alpha)$, i.e. $\forall \underline{i} = (i_1 \ldots i_n) \in \mathcal{A}, \operatorname{int}(v_{i_1} \circ \ldots \circ v_{i_n}(TX_{i_n})) \neq \emptyset$. We write $v_{i_1} \circ \ldots \circ v_{i_n} = v_{i_1} \circ \ldots \circ v_{i_n} (TX_{i_n}) = \overline{X_{\underline{i}}}$ for $\underline{i} \in \mathcal{A}$. Finally we let $|\underline{i}| = n$.

A measure $m$ on $X$ is called locally nonsingular if it is nonsingular with respect to the maps $v_{\underline{i}}^{-1} : \overline{X_{\underline{i}}} \to \overline{TX_{\underline{i}}}$ for each $\underline{i} \in \mathcal{A}$ and if the boundary of $\alpha$ has measure 0. If $m$ is finite, the system $(X, \mathcal{F}, T, m, \alpha)$ is called a Markov map (Markov fibred system) (cf. [2] or [4]). There are some canonical examples for this notion: Markov shifts and maps of the interval (e.g.

continued fraction algorithm, Jacobi’s algorithm [8]), maps originating from higher dimensional flows (e.g. [3]), parabolic rational functions ([4], [5]) or real piecewise differentiable maps of \( \mathbb{R}^2 \) (see [11]–[14]). In many cases, the measure \( m \) is Lebesgue measure. More general examples are obtained in [7] when the partition \( \alpha \) is Bernoulli (i.e. \( T \mathbf{X}_i = \mathbf{X} \) for all \( \mathbf{X}_i \in \alpha \)). Considering this system as an iterated function system one can show that the Hausdorff measure is a good candidate for such a measure.

No general method seems to be known to construct Markov maps as described above. Here we show that for piecewise \( C^0 \)-invertible maps there exist such measures in quite general situations. In fact, for every Hölder continuous function \( \phi : X \to \mathbb{R}^+ \) satisfying some regularity condition (see §1) we construct a measure with the property that the Jacobian \( \frac{d(m \circ T)}{dm} \) of the measure is \( \exp[P(\phi) - \phi] \), where \( P(\phi) \) denotes the topological pressure of \( \phi \) (as defined in §1). In [6] these measures were called conformal. It may be more convenient to call them (non-invariant) Gibbs measures. In addition, we shall prove that these measures have the local bounded distortion property (which is sometimes called the Schweiger property) in case \( T \) is conservative. Let \( \psi'_j = \frac{d(m \circ \psi_j)}{dm} \). Then \((X, F, T, m, \alpha)\) has the Schweiger property if for some constant \( C \geq 1 \) the system of sets

\[
\mathcal{R} = \{ \mathbf{X}_i : i \in A, \psi'_i(x) / \psi'_i(y) \leq C \ \text{m} \times \text{m} \ a.e. \ x, y \in \bigcup \mathcal{X}_i \}
\]

has the strong playback property and generation property (see [1], pp. 143 ff., [8] or [4]).

1. Main Theorem. In this section we assume in addition to (1)–(4) that the Markov partition \( \alpha \) is irreducible and that

\[
\{\psi_j\}_{j \in I} \text{ is an equicontinuous family of partially defined uniformly continuous maps.}
\]

For \( A \in \alpha \) with \( \text{int} A \neq \emptyset \), let \( \psi \) denote the first return time to \( A \), i.e.

\[
\psi(x) = \begin{cases} 
\inf \{ n \geq 1 : T^n(x) \in A \} & \text{if exists,} \\
\infty & \text{otherwise,}
\end{cases} \quad x \in A.
\]

Let \( T_\psi = T^{\psi} \) denote the induced transformation on \( \{ \psi < \infty \} \subset A \). By the Markov property there exists a partition of the set \( A_1 = \{ x \in A : \psi(x) = 1 \} \) for each \( k \geq 1 \) so that \( T^k \), restricted to the interior of each element of the partition, is a homeomorphism onto its image \( \text{int} A \). Let \( I_A \) denote the set of all indices corresponding to such elements of the partition of \( \bigcup_{k \geq 1} A_k \). Then \( \{\psi_j : j \in I_A\} \) is a family of extensions of local inverses of \( T_A \). We shall identify \( j \in I_A \) with elements of \( A \). The next condition can be easily verified for some parabolic examples (e.g., inhomogeneous diophantine transformation [14], Brun’s map [13], parabolic rational maps [5], and complex continued fractions (see §3)):
(6) there are $0 < \gamma < 1$, $0 < \Gamma < \infty$ such that $\sup \text{diam} \mathcal{X}_n \leq \Gamma \gamma^n$.

For a given piecewise Hölder continuous potential $\phi : X \to \mathbb{R}$ (with exponent $\theta$) with respect to $\alpha$, define the topological pressure for $\phi$ by

$$P_{\text{top}}(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{(i_1, \ldots, i_n) \in A} \sup_{x \in A} \left[ \sum_{k=0}^{n-1} \phi(v_{i_{k+1}} \ldots i_n(x)) \right].$$

For $s \in \mathbb{R}$, $j \in I_A$, and $x \in A$ define

$$\phi_A^{(s)}(v_j(x)) = \sum_{i=0}^{|j|-1} \phi(v_{j_{i+1}} \ldots i_j(x)) - s|j|.$$

Then the topological pressure for $\phi_A^{(s)}$ is

$$P_{\text{top}}(\phi_A^{(s)}) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{(j_1 \ldots j_n) \in I_A} \sup_{x \in A} \left[ \sum_{k=0}^{n-1} \phi_A^{(s)}(v_{j_{k+1}} \ldots j_n(x)) \right].$$

The next condition gives a weak Hölder type condition on $\phi_A^{(s)}$:

(7) (Local bounded distortion with respect to $\alpha$) For all $j = (j_1 \ldots j_n) \in I_A$ and all $0 \leq i < |j|$ there is $0 < L_{\phi}(j, i) < \infty$ satisfying

$$|\phi(v_{j_{i+1}} \ldots j_n(x)) - \phi(v_{j_{i+1}} \ldots j_n(y))| \leq L_{\phi}(j, i) d(x, y)^\theta \quad (\forall x, y \in A),$$

$$\sup_{j \in I_A} \sum_{i=0}^{|j|-1} L_{\phi}(j, i) < \infty.$$

Define

$$\hat{T}_\phi f(x) = \sum_{i \in I} f(v_i(x)) \exp[\phi(v_i(x))], \quad x \in X,$$

whenever the series converges for $f : X \to \mathbb{R}$ and define

$$\hat{T}_{\phi_A^{(s)}} g(x) = \sum_{j \in I_A} g(v_j(x)) \exp[\phi_A^{(s)}(v_j(x))], \quad x \in A,$$

whenever the series converges for $g : A \to \mathbb{R}$.

We shall prove the following theorem.

**Theorem.** Let $T : X \to X$ be a piecewise $C^0$-invertible map on a compact metric space satisfying (1)–(5). Suppose that the Markov partition $\alpha$ is irreducible. Let $\phi : X \to \mathbb{R}$ be a piecewise Hölder continuous potential (with exponent $\theta$) with respect to $\alpha$ such that $P_{\text{top}}(\phi) < \infty$. Suppose that there is $A \in \alpha$ satisfying (6) and (7). Then for all $s \in \mathbb{R}$ with $\hat{T}_{\phi_A^{(s)}} 1 \in C(A)$ and $P_{\text{top}}(\phi_A^{(s)}) = 0$ there exists a $\sigma$-finite measure $m$ on $X$. 
with the Schweiger property such that $\hat{T}_s m = (\exp s) m$. In particular, if $m$ is finite, $(X, B, T, m, \alpha)$ is a Markov map with the Schweiger property, and if $P_{\text{top}}(\phi_A^{(P_{\text{top}}(\phi))}) = 0$, then $\hat{T}_s m = (\exp P_{\text{top}}(\phi)) m$.

Remarks. (1) If $m$ is a probability measure and $\inf\{m(TA) : A \in \alpha\} > 0$, then there exists an absolutely continuous invariant measure.

(2) $m$ is exact (see [4]).

2. Proof of the main theorem

Lemma (cf. [13]). There exists $0 < D < \infty$ such that for all $x, y \in A$ and $j \in I_A$,

$$|\phi_A^{(s)}(v_j(x)) - \phi_A^{(s)}(v_j(y))| \leq D d(x, y)^\theta.$$

Proof. A direct computation shows that it suffices to choose

$$D = \sup_{j \in I_A} \sum_{i=0}^{\lfloor |j| - 1 \rfloor} L_\phi(j, i) < \infty.$$

Proof of Theorem. It follows from the Lemma that there exists $C \geq 1$ such that

$$\sup_n \sup_{j, \ldots, j_n \in I_A} \sup_{x, y \in A} \exp[\sum_{k=0}^{n-1} \phi_A^{(s)}(v_{j_k+1} \ldots j_n(x))] \leq C.$$

Therefore $\{\phi_A^{(s)} \circ v_j : j \in I_A\}$ forms a strong Hölder family of order $-\log \gamma$ (cf. (6)) in the sense of [7]. Now $\hat{T}_s^{\phi_A^{(s)}}$ acts on all continuous functions on $A$ and so $\hat{T}_s^{\phi_A^{(s)}}$ acts on $C(A)^\ast$. Hence there is an eigenvalue $\lambda$ and a probability $\mu$ on $\{\psi < \infty\}$ satisfying

$$\hat{T}_s^{\phi_A^{(s)}} \mu = \lambda \mu$$

and by Lemma 2.4 of [7] we have $\log \lambda = P_{\text{top}}(\phi_A^{(s)})$. Then our assumption gives $\lambda = 1$.

Applying [10], Lemma 9, we obtain $\mu(\text{int } A) = 1$ (alternatively use Lemma 2.1 of [4]). Since $\mu$ is nonsingular, it follows that the boundary of $\overline{A} \cap \alpha_0^s$ is a null set with respect to $\mu$.

Let $\sigma$ denote the shift, i.e., $\sigma(i_1 \ldots i_n) = (i_2 \ldots i_n)$ and $\sigma^k(i_1 \ldots i_n) = (i_{k+1} \ldots i_n)$ for $k = 1, \ldots, n - 1$. For $k = n$ we define $\sigma^k(i_1 \ldots i_n) = \emptyset$. Let $A^\ast$ be the subset of $A$ defined by $A^\ast = \{i \in A : A \cap v_{\sigma^k i}(A) = \emptyset \} (k = 0, \ldots, |i| - 1)$. For $i \in A$, we define

$$\phi_A^{(s)}(x) = \sum_{k=0}^{\lfloor |i| - 1 \rfloor} \phi(v_{i_k+1} \ldots i_{i})(x) - |i| s.$$
In particular, if $|i|$ is the empty word, we put $\phi^{(s)} = 0$. We define a measure $m$ (which may be infinite, but $\sigma$-finite) on $X$ via $\mu$ as follows:

$$\int f(x) \, m(dx) = \sum_{i \in A^*} \int f(v_i(x)) \exp[\phi^{(s)}(x)] \mu(dx) + \int f(x) \, \mu(dx)$$

where $f$ is a continuous function on $X$.

The Perron–Frobenius operator for $T$ and $m$ is defined by

$$\hat{T}_\phi f(x) = \sum_{T(y) = x} f(y) \exp(\phi(y) - s) = \sum_{l \in I} f(v_l(x)) \exp(\phi(v_l(x)) - s) 1_{TX_l}(x).$$

In fact we shall show that $\int \hat{T}_\phi f \, dm = \int f \, dm$ so that

$$\frac{d(\mu \circ v_l)}{dm}(x) = \exp[\phi(v_l(x)) - s] \quad \text{for a.e. } x \in X.$$

We have

$$\int \hat{T}_\phi f(x) \, dm(x) = \sum_{i \in A^*} \int \hat{T}_\phi f(v_i(x)) \exp[\phi^{(s)}(x)] \mu(dx) + \int \hat{T}_\phi f(x) \, \mu(dx)$$

$$= \sum_{i \in A^*} \sum_{l \in I} f(v_l(x)) \exp[\phi(v_l(x)) - s]$$

$$\times 1_{TX_l}(x) \exp[\phi^{(s)}(x)] \mu(dx)$$

$$+ \int \sum_{l \in I} f(v_l(x)) \exp[\phi(v_l(x)) - s] 1_{TX_l}(x) \mu(dx)$$

$$= \sum_{A \, i \in I_A} f(v_i(x)) \exp[\phi^{(s)}(A)(v_i(x))] \mu(dx)$$

$$+ \sum_{A \, i \in I_A} f(v_i(x)) \exp[\phi^{(s)}(x)] \mu(dx).$$

Since

$$\int \sum_{A \, i \in I_A} f(v_i(x)) \exp[\phi^{(s)}(A)(v_i(x))] \mu(dx) = \int \hat{T}_\phi f \, d\mu = \int f \, d\mu,$$

we have

$$\int \hat{T}_\phi f(x) \, dm(x) = \int f \, d\mu + \sum_{i \in A^*} \int f(v_i(x)) \exp[\phi^{(s)}(x)] \, d\mu(x)$$

$$= \int f(x) \, m(dx).$$

The Schweiger property follows from irreducibility and (6) and (7).
3. Examples

**Example 1** (A real two-dimensional map which is related to a complex continued fraction expansion defined in [9]). Let $\alpha = 1 + i$. We set $X = \{ z = x_1 \alpha + x_2 \overline{\alpha} : -1/2 \leq x_1, x_2 \leq 1/2 \}$ and define $T : X \to X$ by $Tz = 1/z - [1/z]_1$, where $[z]$ denotes $[x_1 + 1/2] \alpha + [x_2 + 1/2] \overline{\alpha}$ for a complex number $z = x_1 \alpha + x_2 \overline{\alpha}$. (Here $[x] = \max\{ n \in \mathbb{Z} : n \leq x \}$ ($x \in \mathbb{N}$) and $[x] = \max\{ n \in \mathbb{Z} : n < x \}$ ($x \in \mathbb{Z} \setminus \mathbb{N}$).) The index set is $I = \{ m \alpha + n \overline{\alpha} : m, n \in \mathbb{Z} \} \setminus \{ 0 \}$. For each $m \alpha + n \overline{\alpha} \in I$, we define $X_{m \alpha + n \overline{\alpha}} = \{ z \in X : [1/z]_1 = m \alpha + n \overline{\alpha} \}$.

Then we have a countable partition $\alpha = \{ X_a \}_{a \in I}$ of $X$ which is a topologically mixing Markov partition. The map $T$ induces a continued fraction like expansion of $z \in X$,

$$ z = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ldots + \frac{1}{a_n + \ldots}}}} $$

where each $a_i$ is contained in $I$. Now $T$ has indifferent fixed points $\pm i$ and indifferent periodic points $\pm 1$ of periodic 2. All conditions (1)–(5) were established in [9], [11], and [12].

Put $p_{-1} = \alpha, \quad p_0 = 0, \quad p_n = anp_{n-1} + p_{n-2} \quad (n \geq 1), \quad q_{-1} = 0, \quad q_0 = \alpha, \quad q_n = a_nq_{n-1} + q_{n-2} \quad (n \geq 1)$.

Then $v_{a_1 \ldots a_n}(z) = \frac{p_n + zp_{n-1}}{q_n + zq_{n-1}}$.

Let $A$ be a cylinder away from the indifferent periodic points. Then (6) can be verified by observing the following facts.

(1) $|v'_{a_1 \ldots a_n}(z)| = |q_n + zq_{n-1}|^{-2}$.

(2) $|q_{n-1}/q_n| \leq 1$ for all $n > 0$.

(3) For $X_{a_1 \ldots a_n}$ such that $X_{a_n}$ does not contain the indifferent periodic points, $|q_{n-1}/q_n| < 2/3$.

Thus our theorem applies to $T$.

**Example 2.** Let $T : S^2 \to S^2$ be a parabolic rational map of the Riemann sphere (see e.g. [5] for a definition). We restrict the action of $T$ to its Julia set $J$. Then by [5] there is a finite Markov partition $\alpha$ satisfying $A \subset \text{cl}(\text{int} A)$ for every $A \in \alpha$. Moreover, for each $A \in \alpha$, away from the rationally indifferent periodic points, the Koebe distortion theorem applies to balls centred in $A$ and all analytic inverse branches (since the forward orbits of critical points only accumulate at parabolic periodic points). It follows
that (6) and (7) are satisfied (see [5]). The main theorem shows that one can obtain conformal measures in more general situations than those previously known: These known results are concerned with potentials \( \phi \) satisfying

\[
P(\phi) > \sup_{z \in J} \phi(z),
\]

where \( P(\phi) \) denotes the pressure of \( \phi \) as in [10], or with the potential \( \phi = h \log |T'| \), where \( h \) denotes the Hausdorff dimension of \( J \).

REFERENCES


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