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A NOTE ON THE CONSTRUCTION OF NONSINGULAR GIBBS MEASURES

BY

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Dedicated to the memory of Prof. Anzelm Iwanik

Abstract. We give a sufficient condition for the construction of Markov fibred systems using countable Markov partitions with locally bounded distortion.

0. Introduction. Let X be a compact metric space with metric d and $T : X \to X$ be a noninvertible piecewise C^0-invertible map, i.e. there exists a finite or countable partition $X = \bigcup_{i\in I} X_i$ such that $\bigcup_{i\in I} \operatorname{int} X_i$ is dense in X and

(1) For each $i \in I$ with $\operatorname{int} X_i \neq \emptyset$, $T|_{\operatorname{int} X_i} : \operatorname{int} X_i \to T(\operatorname{int} X_i)$ is a homeomorphism and $(T|_{\operatorname{int} X_i})^{-1}$ extends to a homeomorphism v_i on $\operatorname{cl}(T(\operatorname{int} X_i))$.

(2) $T(\bigcup_{\operatorname{int} X_i=\emptyset} X_i) \subset \bigcup_{\operatorname{int} X_i=\emptyset} X_i$.

(3) $\{X_i\}_{i\in I}$ generates $\mathcal{F}$ with respect to T, where $\mathcal{F}$ is the σ-algebra of Borel subsets of X.

We set $\overline{A} = \operatorname{cl}(\operatorname{int} A)$ $(A \subset X)$ and define $\alpha = \{\overline{X}_i\}_{i\in I}$. Then α is a finite or countable partition of a dense subset of X which is not necessarily a disjoint family. We impose the Markov property on α:

(4) $\operatorname{int}(\overline{X}_i \cap \overline{TX_j}) \neq \emptyset$ implies $\overline{TX_j} \supset \overline{X}_i$.

Let $\mathcal{A}$ denote the set of all admissible sequences with respect to (T, α), i.e. $\forall \underline{i} = (i_1 \ldots i_n) \in \mathcal{A}, \operatorname{int}(v_{i_1} \circ \ldots \circ v_{i_n}(\overline{TX_{i_n}})) \neq \emptyset$. We write $v_{i_1} \circ \ldots \circ v_{i_n} = v_{i_1\ldots i_n}$ and $v_{i_1} \circ \ldots \circ v_{i_n}(\overline{TX_{i_n}}) = \overline{X}_{\underline{i}}$ for $\underline{i} \in \mathcal{A}$. Finally we let $|\underline{i}| = n$.

A measure m on X is called *locally nonsingular* if it is nonsingular with respect to the maps $v_i^{-1} : \overline{X}_i \to \overline{TX_i}$ for each $\overline{X}_i \in \alpha$ and if the boundary of α has measure 0. If m is finite, the system $(X, \mathcal{F}, T, m, \alpha)$ is called a *Markov map* (*Markov fibred system*) (cf. [2] or [4]). There are some canonical examples for this notion: Markov shifts and maps of the interval (e.g.

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continued fraction algorithm, Jacobi's algorithm [8]), maps originating from higher dimensional flows (e.g. [3]), parabolic rational functions ([4], [5]) or real piecewise differentiable maps of $\mathbb{R}^2$ (see [11]–[14]). In many cases, the measure m is Lebesgue measure. More general examples are obtained in [7] when the partition α is Bernoulli (i.e. $T\overline{X}_i = X$ for all $\overline{X}_i \in \alpha$). Considering this system as an iterated function system one can show that the Hausdorff measure is a good candidate for such a measure.

No general method seems to be known to construct Markov maps as described above. Here we show that for piecewise C^0-invertible maps there exist such measures in quite general situations. In fact, for every Hölder continuous function $\phi : X \to \mathbb{R}_+$ satisfying some regularity condition (see §1) we construct a measure with the property that the Jacobian $d(m \circ T)/dm$ of the measure is $\exp[P(\phi) - \phi]$, where $P(\phi)$ denotes the topological pressure of ϕ (as defined in §1). In [6] these measures were called conformal. It may be more convenient to call them (non-invariant) *Gibbs measures*. In addition, we shall prove that these measures have the local bounded distortion property (which is sometimes called the Schweiger property) in case T is conservative. Let $v'_{\underline{i}} = d(m \circ v_{\underline{i}})/dm$. Then $(X, \mathcal{F}, T, m, \alpha)$ has the *Schweiger property* if for some constant $C \geq 1$ the system of sets

$$\mathcal{R} = \{\overline{X}_{\underline{i}} : \underline{i} \in \mathcal{A},\ v'_{\underline{i}}(x)/v'_{\underline{i}}(y) \leq C \ m \times m \text{ a.e. } x, y \in \overline{T^{|\underline{i}|} X_{\underline{i}}}\}$$

has the strong playback property and generation property (see [1], pp. 143 ff., [8] or [4]).

1. Main Theorem. In this section we assume in addition to (1)–(4) that the Markov partition α is irreducible and that

(5) $\{v_i\}_{i \in I}$ is an equicontinuous family of partially defined uniformly continuous maps.

For $A \in \alpha$ with $\operatorname{int} A \neq \emptyset$, let ψ denote the first return time to A, i.e.

$$\psi(x) = \begin{cases} \inf\{n \geq 1 : T^n(x) \in A\} & \text{if exists,} \\ \infty & \text{otherwise,} \end{cases} \qquad x \in A.$$

Let $T_A = T^{\psi}$ denote the induced transformation on $\{\psi < \infty\} \subset A$. By the Markov property there exists a partition of the set $A_k = \{x \in A : \psi(x) = k\}$ for each $k \geq 1$ so that T^k, restricted to the interior of each element of the partition, is a homeomorphism onto its image $\operatorname{int} A$. Let I_A denote the set of all indices corresponding to such elements of the partition of $\bigcup_{k \geq 1} A_k$. Then $\{v_{\underline{j}} : \underline{j} \in I_A\}$ is a family of extensions of local inverses of T_A. We shall identify $\underline{j} \in I_A^n$ with elements of $\mathcal{A}$. The next condition can be easily verified for some parabolic examples (e.g., inhomogeneous diophantine transformation [14], Brun's map [13], parabolic rational maps [5], and complex continued fractions (see §3)):

(6) there are $0 < \gamma < 1$, $0 < \Gamma < \infty$ such that $\sup_{\underline{j}\in I_A^n} \operatorname{diam} \overline{X}_{\underline{j}} \le \Gamma\gamma^n$.

For a given piecewise Hölder continuous potential $\phi : X \to \mathbb{R}$ (with exponent θ) with respect to α, define the *topological pressure* for ϕ by

$$P_{\mathrm{top}}(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{(i_1\ldots i_n)\in\mathcal{A}} \sup_{x\in X} \exp\Big[\sum_{k=0}^{n-1} \phi(v_{i_{k+1}\ldots i_n}(x))\Big].$$

For $s \in \mathbb{R}$, $\underline{j} \in I_A$, and $x \in A$ define

$$\phi_A^{(s)}(v_{\underline{j}}(x)) = \sum_{i=0}^{|\underline{j}|-1} \phi(v_{j_{i+1}} \circ \ldots \circ v_{j_{|\underline{j}|}}(x)) - s|\underline{j}|.$$

Then the topological pressure for $\phi_A^{(s)}$ is

$$P_{\mathrm{top}}(\phi_A^{(s)}) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{(\underline{j}_1\ldots\underline{j}_n)\in I_A^n} \sup_{x\in A} \exp\Big[\sum_{k=0}^{n-1} \phi_A^{(s)}(v_{\underline{j}_{k+1}\ldots\underline{j}_n}(x))\Big].$$

The next condition gives a weak Hölder type condition on $\phi_A^{(s)}$:

(7) (Local bounded distortion with respect to α) For all $\underline{j} = (j_1 \ldots j_{|\underline{j}|}) \in I_A$ and all $0 \le i < |\underline{j}|$ there is $0 < L_\phi(\underline{j}, i) < \infty$ satisfying

$$|\phi(v_{j_{i+1}\ldots j_{\underline{j}}}(x)) - \phi(v_{j_{i+1}\ldots j_{\underline{j}}}(y))| \le L_\phi(\underline{j}, i) d(x, y)^\theta \quad (\forall x, y \in A),$$

$$\sup_{\underline{j}\in I_A} \sum_{i=0}^{|\underline{j}|-1} L_\phi(\underline{j}, i) < \infty.$$

Define

$$\widehat{T}_\phi f(x) = \sum_{i\in I} f(v_i(x)) \exp[\phi(v_i(x))], \quad x \in X,$$

whenever the series converges for $f : X \to \mathbb{R}$ and define

$$\widehat{T}_{\phi_A^{(s)}} g(x) = \sum_{\underline{j}\in I_A} g(v_{\underline{j}}(x)) \exp[\phi_A^{(s)}(v_{\underline{j}}(x))], \quad x \in A,$$

whenever the series converges for $g : A \to \mathbb{R}$.

We shall prove the following theorem.

THEOREM. *Let $T : X \to X$ be a piecewise C^0-invertible map on a compact metric space satisfying* (1)–(5). *Suppose that the Markov partition α is irreducible. Let $\phi : X \to \mathbb{R}$ be a piecewise Hölder continuous potential* (*with exponent θ*) *with respect to α such that $P_{\mathrm{top}}(\phi) < \infty$. Suppose that there is $A \in \alpha$ satisfying* (6) *and* (7). *Then for all $s \in \mathbb{R}$ with $\widehat{T}_{\phi_A^{(s)}} 1 \in C(A)$ and $P_{\mathrm{top}}(\phi_A^{(s)}) = 0$ there exists a σ-finite measure m on X*

with the Schweiger property such that $\widehat{T}^*_\phi m = (\exp s)m$. *In particular*, *if* m *is finite*, $(X, \mathcal{B}, T, m, \alpha)$ *is a Markov map with the Schweiger property*, *and if* $P_{\mathrm{top}}(\phi_A^{(P_{\mathrm{top}}(\phi))}) = 0$, *then* $\widehat{T}^*_\phi m = (\exp P_{\mathrm{top}}(\phi))m$.

Remarks. (1) If m is a probability measure and $\inf\{m(TA) : A \in \alpha\} > 0$, then there exists an absolutely continuous invariant measure.

(2) m is exact (see [4]).

2. Proof of the main theorem

Lemma (cf. [13]). *There exists* $0 < D < \infty$ *such that for all* $x, y \in A$ *and* $\underline{j} \in I_A$,

$$|\phi_A^{(s)}(v_{\underline{j}}(x)) - \phi_A^{(s)}(v_{\underline{j}}(y))| \le Dd(x,y)^\theta.$$

Proof. A direct computation shows that it suffices to choose

$$D = \sup_{\underline{j} \in I_A} \sum_{i=0}^{|\underline{j}|-1} L_\phi(\underline{j}, i) < \infty.$$

Proof of Theorem. It follows from the Lemma that there exists $C \ge 1$ such that

$$\sup_n \sup_{\underline{j}_1 \ldots \underline{j}_n \in I_A^n} \sup_{x,y \in A} \frac{\exp[\sum_{k=0}^{n-1} \phi_A^{(s)}(v_{\underline{j}_{k+1} \ldots \underline{j}_n}(x))]}{\exp[\sum_{k=0}^{n-1} \phi_A^{(s)}(v_{\underline{j}_{k+1} \ldots \underline{j}_n}(y))]} \le C.$$

Therefore $\{\phi_A^{(s)} \circ v_{\underline{j}} : \underline{j} \in I_A\}$ forms a strong Hölder family of order $-\log\gamma$ (cf. (6)) in the sense of [7]. Now $\widehat{T}_{\phi_A^{(s)}}$ acts on all continuous functions on A and so $\widehat{T}^*_{\phi_A^{(s)}}$ acts on $C(A)^*$. Hence there is an eigenvalue λ and a probability μ on $\{\psi < \infty\}$ satisfying

$$\widehat{T}^*_{\phi_A^{(s)}} \mu = \lambda\mu$$

and by Lemma 2.4 of [7] we have $\log\lambda = P_{\mathrm{top}}(\phi_A^{(s)})$. Then our assumption gives $\lambda = 1$.

Applying [10], Lemma 9, we obtain $\mu(\mathrm{int}\, A) = 1$ (alternatively use Lemma 2.1 of [4]). Since μ is nonsingular, it follows that the boundary of $\overline{A} \cap \alpha_0^n$ is a null set with respect to μ.

Let σ denote the shift, i.e., $\sigma(i_1 \ldots i_n) = (i_2 \ldots i_n)$ and $\sigma^k(i_1 \ldots i_n) = (i_{k+1} \ldots i_n)$ for $k = 1, \ldots, n-1$. For $k = n$ we define $\sigma^k(i_1 \ldots i_n) = \emptyset$. Let $\mathcal{A}^*$ be the subset of $\mathcal{A}$ defined by $\mathcal{A}^* = \{\underline{i} \in \mathcal{A} : A \cap v_{\sigma^k \underline{i}}(A) = \emptyset$ $(k = 0, \ldots, |\underline{i}| - 1)\}$. For $\underline{i} \in \mathcal{A}$, we define

$$\phi^{(\underline{i},s)}(x) = \sum_{k=0}^{|\underline{i}|-1} \phi(v_{i_{k+1} \ldots i_{|\underline{i}|}}(x)) - |\underline{i}|s.$$

In particular, if $|\underline{i}|$ is the empty word, we put $\phi^{(\underline{i},s)} = 0$. We define a measure m (which may be infinite, but σ-finite) on X via μ as follows:

$$\int f(x)\, m(dx) = \sum_{\underline{i}\in\mathcal{A}^*} \int_A f(v_{\underline{i}}(x)) \exp[\phi^{(\underline{i},s)}(x)]\, \mu(dx) + \int_A f(x)\, \mu(dx)$$

where f is a continuous function on X.

The Perron–Frobenius operator for T and m is defined by

$$\widehat{T}_\phi f(x) = \sum_{T(y)=x} f(y) \exp(\phi(y) - s) = \sum_{l\in I} f(v_l(x)) \exp(\phi(v_l(x)) - s) 1_{\overline{TX_l}}(x).$$

In fact we shall show that $\int \widehat{T}_\phi f\, dm = \int f\, dm$ so that

$$\frac{d(m\circ v_l)}{dm}(x) = \exp[\phi(v_l(x)) - s] \quad \text{for a.e. } x \in X.$$

We have

$$\begin{aligned}
\int \widehat{T}_\phi f(x)\, dm(x) &= \sum_{\underline{i}\in\mathcal{A}^*} \int_A \widehat{T}_\phi f(v_{\underline{i}}(x)) \exp[\phi^{(\underline{i},s)}(x)]\, \mu(dx) + \int_A \widehat{T}_\phi f(x)\, \mu(dx) \\
&= \sum_{\underline{i}\in\mathcal{A}^*} \int_A \sum_{l\in I} f(v_l(v_{\underline{i}}(x))) \exp[\phi(v_l(v_{\underline{i}}(x))) - s] \\
&\quad \times 1_{\overline{TX_l}}(v_{\underline{i}}(x)) \exp[\phi^{(\underline{i},s)}(x)]\, \mu(dx) \\
&\quad + \int_A \sum_{l\in I} f(v_l(x)) \exp[\phi(v_l(x)) - s] 1_{\overline{TX_l}}(x)\, \mu(dx) \\
&= \int_A \sum_{\underline{j}\in I_A} f(v_{\underline{j}}(x)) \exp[\phi_A^{(s)}(v_{\underline{j}}(x))]\, \mu(dx) \\
&\quad + \sum_{\underline{i}\in\mathcal{A}^*} \int_A f(v_{\underline{i}}(x)) \exp[\phi^{(\underline{i},s)}(x)]\, \mu(dx).
\end{aligned}$$

Since

$$\int_A \sum_{\underline{j}\in I_A} f(v_{\underline{j}}(x)) \exp[\phi_A^{(s)}(v_{\underline{j}}(x))]\, \mu(dx) = \int_A \widehat{T}_{\phi_A^{(s)}} f\, d\mu = \int_A f\, d\mu,$$

we have

$$\begin{aligned}
\int \widehat{T}_\phi f(x)\, dm(x) &= \int_A f\, d\mu + \sum_{\underline{i}\in\mathcal{A}^*} \int_A f(v_{\underline{i}}(x)) \exp[\phi^{(\underline{i},s)}(x)]\, d\mu(x) \\
&= \int_X f(x)\, m(dx).
\end{aligned}$$

The Schweiger property follows from irreducibility and (6) and (7).

3. Examples

Example 1 (A real two-dimensional map which is related to a complex continued fraction expansion defined in [9]). Let $\alpha = 1+i$. We set $X = \{z = x_1\alpha + x_2\overline{\alpha} : -1/2 \le x_1, x_2 \le 1/2\}$ and define $T : X \to X$ by $Tz = 1/z - [1/z]_1$, where $[z]_1$ denotes $[x_1+1/2]\alpha + [x_2+1/2]\overline{\alpha}$ for a complex number $z = x_1\alpha + x_2\overline{\alpha}$. (Here $[x] = \max\{n \in \mathbb{Z} : n \le x\}$ $(x \in \mathbb{N})$ and $[x] = \max\{n \in \mathbb{Z} : n < x\}$ $(x \in \mathbb{Z}\setminus\mathbb{N})$.) The index set is $I = \{n\alpha + m\overline{\alpha} : m, n \in \mathbb{Z}\}\setminus\{0\}$. For each $n\alpha + m\overline{\alpha} \in I$, we define

$$X_{n\alpha+m\overline{\alpha}} = \{z \in X : [1/z]_1 = n\alpha + m\overline{\alpha}\}.$$

Then we have a countable partition $\alpha = \{X_a\}_{a\in I}$ of X which is a topologically mixing Markov partition. The map T induces a continued fraction like expansion of $z \in X$,

$$z = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots \cfrac{1}{a_n + \ldots}}}},$$

where each a_i is contained in I. Now T has indifferent fixed points $\pm i$ and indifferent periodic points ± 1 of periodic 2. All conditions (1)–(5) were established in [9], [11], and [12].

Put

$$p_{-1} = \alpha, \quad p_0 = 0, \quad p_n = a_n p_{n-1} + p_{n-2} \quad (n \ge 1)$$
$$q_{-1} = 0, \quad q_0 = \alpha, \quad q_n = a_n q_{n-1} + q_{n-2} \quad (n \ge 1).$$

Then

$$v_{a_1\ldots a_n}(z) = \frac{p_n + z p_{n-1}}{q_n + z q_{n-1}}.$$

Let A be a cylinder away from the indifferent periodic points. Then (6) can be verified by observing the following facts.

(1) $|v'_{a_1\ldots a_n}(z)| = |q_n + z q_{n-1}|^{-2}$.
(2) $|q_{n-1}/q_n| \le 1$ for all $n > 0$.
(3) For $X_{a_1\ldots a_n}$ such that X_{a_n} does not contain the indifferent periodic points, $|q_{n-1}/q_n| < 2/3$.

Thus our theorem applies to T.

Example 2. Let $T : S^2 \to S^2$ be a parabolic rational map of the Riemann sphere (see e.g. [5] for a definition). We restrict the action of T to its Julia set J. Then by [5] there is a finite Markov partition α satisfying $A \subset \mathrm{cl}(\mathrm{int}\, A)$ for every $A \in \alpha$. Moreover, for each $A \in \alpha$, away from the rationally indifferent periodic points, the Koebe distortion theorem applies to balls centred in A and all analytic inverse branches (since the forward orbits of critical points only accumulate at parabolic periodic points). It follows that (6) and (7) are satisfied (see [5]). The main theorem shows that one can obtain conformal measures in more general situations that those previously known: These known results are concerned with potentials ϕ satisfying

$$P(\phi) > \sup_{z\in J} \phi(z),$$

where $P(\phi)$ denotes the pressure of ϕ as in [10], or with the potential $\phi = h\log|T'|$, where h denotes the Hausdorff dimension of J.

REFERENCES

[1] J. Aaronson, *An Introduction to Infinite Ergodic Theory*, Amer. Math. Soc., 1997.

[2] J. Aaronson and M. Denker, *Local limit theorems for Gibbs–Markov maps*, preprint, Math. Gottingensis 1 (1997).

[3] —, —, *The Poincaré series of* $\mathbb{C}\setminus\mathbb{Z}$, Ergodic Theory Dynam. Systems 19 (1999), 1–20.

[4] J. Aaronson, M. Denker and M. Urbański, *Ergodic theory for Markov fibred systems and parabolic rational maps*, Trans. Amer. Math. Soc. 337 (1993), 495–548.

[5] M. Denker and M. Urbański, *Absolutely continuous invariant measures for expansive rational maps with rationally indifferent periodic points*, Forum Math. 3 (1991), 561–579.

[6] —, —, *On the existence of conformal measures*, Trans. Amer. Math. Soc. 76 (1991), 193–214.

[7] P. Hanus, R. D. Mauldin and M. Urbański, *Thermodynamic formalism and multi-fractal analysis of conformal infinite iterated functional systems*, preprint, IHES, 1999.

[8] F. Schweiger, *Ergodic Theory of Fibred Systems and Metric Number Theory*, Oxford Univ. Press, Oxford, 1995.

[9] S. Tanaka, *A complex continued fraction transformation and its ergodic properties*, Tokyo J. Math. 8 (1985), 191–214.

[10] P. Walters, *Invariant measures and equilibrium states for some mappings which expand distances*, Trans. Amer. Math. Soc. 236 (1978), 121–153.

[11] M. Yuri, *On a Bernoulli property for multi-dimensional mappings with finite range structure*, Tokyo J. Math. 9 (1986), 457–485.

[12] —, *On the convergence to equilibrium states for certain non-hyperbolic systems*, Ergodic Theory Dynam. Systems 17 (1997), 977–1000.

[13] —, *Thermodynamic formalism for certain nonhyperbolic maps*, ibid. 19 (1999), 1365–1378.

[14] —, *Statistical properties for nonhyperbolic maps with finite range structure*, Trans. Amer. Math. Soc., to appear.

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