COLLOQUIUM MATHEMATICUM

VOL. 84/85 2000 PART 2

ELLIS GROUPS OF QUASI-FACTORS OF MINIMAL FLOWS

BY

JOSEPH AUSLANDER (COLLEGE PARK, MD)

Dedicated to the memory of Anzelm Iwanik

Abstract. A quasi-factor of a minimal flow is a minimal subset of the induced flow on the space of closed subsets. We study a particular kind of quasi-factor (a "joining" quasi-factor) using the Galois theory of minimal flows. We also investigate the relation between factors and quasi-factors.

In this paper, we combine two important themes in topological dynamics. Quasi-factors, which are minimal subsets on the space of closed subsets of a flow, were introduced by Glasner ([G]). The Ellis groups ([A], [E]) are invariants which are the basis of the "Galois" theory of minimal flows. Here we consider the Ellis groups of a certain kind of quasi-factor.

A *flow* is a right topological action of a (discrete) group T on a compact Hausdorff space X: $(x,t) \mapsto xt$, $(x \in X,\ t \in T)$. The flow is *minimal* if every orbit is dense: $\overline{xT} = X$ for all $x \in X$.

If X is a compact Hausdorff space, then 2^X denotes the space of non-empty closed subsets of X, provided with the Hausdorff topology. An action of the group T on X induces an action of T on 2^X by $At = [at \mid a \in A]$ $(A \in 2^X,\ t \in T.)$ A *quasi-factor* of (X,T) is a minimal subset of the flow $(2^X, T)$.

In what follows, we suppose that (X,T) is a minimal flow, and focus on a particular kind of quasi-factors of X, namely one which arises as a "representation" of another minimal flow.

Such a quasi-factor is obtained by projecting the minimal flow (Y,T) on 2^X as follows. Let $X \vee Y$ be the "join" of X and Y, that is, the orbit closure of an almost periodic point (x,y) of the product flow $(X \times Y, T)$ (so the join depends on the choice of x and y). Let $K = [x' \in X \mid (x',y) \in X \vee Y]$. Equivalently, if $\pi_1 : X \vee Y \to X$ and $\pi_2 : X \vee Y \to Y$ are the projections, then $K = \pi_1(\pi_2^{-1}(y))$. Then $K \in 2^X$, and the *representation* of Y on X, written $\mathcal{X}_Y$, is defined to be the (unique) minimal subset of $\overline{KT}$ ([AG]). Another

2000 *Mathematics Subject Classification*: Primary 54H20.

description of such quasi-factors (also called "joining" quasi-factors) will be given below.

For our analysis of quasi-factors, we require two brief digressions: the algebraic theory of minimal flows, and maximal highly proximal flows and generators.

For details of the algebraic theory, see [A] and [E]. Whenever the (discrete) group acts on a compact Hausdorff space, there is an induced action of $\boldsymbol{\beta}T$, the Stone–Čech compactification of T. (If $p \in \boldsymbol{\beta}T$ with $t_i \to p$, then $xp = \lim xt_i$, for $x \in X$.) Moreover, T acts on $\boldsymbol{\beta}T$, and the map $p \mapsto xp$ (for $p \in \boldsymbol{\beta}T$ and $x \in X$) defines a flow homomorphism from $(\boldsymbol{\beta}T, T)$ to (X, T). The group operation on T extends to a semigroup structure on $\boldsymbol{\beta}T$; the maps $p \mapsto qp$ are continuous. The minimal subsets of the flow $(\boldsymbol{\beta}T, T)$ (all are isomorphic) coincide with the minimal right ideals of the semigroup $\boldsymbol{\beta}T$. These are universal minimal flows—every minimal flow is a homomorphic image. We fix a universal minimal flow (M, T), and let $J(M)$ denote the set of idempotents in M. Then $J(M)$ is non-empty; indeed, if (X, T) is a minimal flow and $x \in X$, there is a $u \in J(M)$ such that $xu = x$.

Now fix $u \in J(M)$, and let $G = Mu$. Then G is a group (with respect to the semigroup operation on M) which can be identified with the group of flow automorphisms of (M, T) via left multiplication. G can be provided with a compact T_1 (but not Hausdorff) topology, with respect to which multiplication is (separately) continuous, and inversion is continuous.

The Ellis groups, which are subgroups of G, are important invariants of minimal flows. If (X, T) is a minimal flow, and $x \in X$ is such that $xu = x$, the *Ellis group* of (X, T) (with respect to the basepoint x) is $\mathcal{G}(X) = [\alpha \in G \mid x\alpha = x]$. (Equivalently, if $\gamma : M \to X$ is the homomorphism $\gamma(p) = xp$ then $\mathcal{G}(X) = [\alpha \in G \mid \gamma\alpha = \gamma]$.) The groups $\mathcal{G}(X)$ are closed (and every closed subgroup of G is the Ellis group of some flow). This association of flows to groups is functorial—if (Y, T) is a factor of (X, T), then $\mathcal{G}(X) \subset \mathcal{G}(Y)$. The Ellis groups are proximal invariants of minimal flows. That is, two minimal flows are proximally equivalent (they have a common proximal extension) if and only if they have the same Ellis groups.

The action of T on 2^X extends to an action of $\boldsymbol{\beta}T$. This is described by the "circle" operation: if $K \in 2^X$ and $p \in \boldsymbol{\beta}T$, we write $K \circ p$ for the action of p on K. Note that $y \in K \circ p$ if and only if there are nets $\{x_n\}$ in K and $\{t_n\}$ in T with $t_n \to p$ and $x_n t_n \to y$. In general, $Kp = [xp \mid x \in K]$ is a proper subset of $K \circ p$.

The circle operation can be used to define the topology on G. This is accomplished by a closure operator: if F is a subset of G, the closure of F is $F \circ u \cap G$.

We use the circle operation, together with homomorphisms from M, to obtain an alternate description of representation quasi-factors.

Let (X,T) and (Y,T) be minimal flows, with $x \in X$, $y \in Y$ such that $xu = x$, $yu = y$. Let $\gamma : M \to X$, $\delta : M \to Y$ be the homomorphisms defined by $\gamma(p) = xp$, $\delta(p) = yp$. Then $\mathcal{X}_Y$ is the orbit closure in 2^X of $\gamma(\delta^{-1}(y)) \circ u$. Thus $\mathcal{X}_Y = [\gamma(\delta^{-1}(y) \circ p) \mid p \in M]$. (To see that our two descriptions of $\mathcal{X}_Y$ are the same, define $\theta : M \to X \vee Y$ by $\theta(p) = (x,y)p$. Then $\gamma = \pi_1\theta$ and $\delta = \pi_2\theta$ so $[\gamma\delta^{-1}(y) \circ p \mid p \in M] = [\pi_1\pi_2^{-1}(y) \circ p \mid p \in M]$ and the latter is easily seen to be $\mathcal{X}_Y$.)

The notion of high proximality was introduced in [AG] and further developed in [AW]. A homomorphism (extension) of minimal flows $\pi : X \to Y$ is said to be *highly proximal* if every fiber can be shrunk uniformly to a point: $\pi^{-1}(y)(t_n)(y) \to \{x\}$ for some net $\{t_n\}$ in T. Equivalent formulations are: every non-empty open subset of X contains a fiber, and for some (equivalently every) $y \in Y$ and every $p \in M$, $\pi^{-1}(y) \circ p = \{xp\}$ where $\pi(x) = y$. Clearly, a highly proximal extension is proximal, and in case X and Y are metric spaces, π is highly proximal if and only if it is an almost $1:1$ extension ([AG]).

A minimal flow is said to be *maximally highly proximal* if it has no non-trivial highly proximal extension. Every minimal flow has a maximally highly proximal extension. These can be described in terms of maximal highly proximal (MHP) generators: sets $C \in 2^M$ such that $u \in C$ and $C \circ p = C$ for all $p \in C$. The minimal flow $\overline{CT}$ is maximally highly proximal, and every MHP flow can be so represented. The MHP generator C can be decomposed as $C = BK$ where B is a closed subgroup of G and $K \subset J(M)$. It follows easily that the Ellis group of the minimal flow $\overline{CT}$ is B. It can also be shown that the sets $\{C \circ q\}$ $(q \in M)$ constitute a decomposition of M.

MHP generators are obtained via homomorphisms defined on M. Let (X,T) be a minimal flow, let $x \in X$ with $xu = x$, and let $\gamma : M \to X$ be a homomorphism. Then $C = \gamma^{-1}(x) \circ u$ is an MHP generator, and $\overline{CT}$ is the maximally highly proximal extension of X.

From this it follows that all representation quasi-factors are of the form $\overline{xCT}$, where C is an MHP generator. (Let $C = \delta^{-1}(y) \circ u$. Then $xC = \gamma(\delta^{-1}(y)) \circ u$.)

The following is an intrinsic characterization of the sets in 2^X of the form xC.

THEOREM 1. *Let (X,T) be a minimal flow, and let $x \in X$ be such that $xu = x$. Let $L \in 2^X$ be such that $x \in L$ and $L \circ u = L$. Let $\gamma : M \to X$ and $\delta : M \to \overline{LT}$ be the homomorphisms $\gamma(p) = xp$, $\delta(p) = L \circ p$. Then the following are equivalent*:

(i) $L = xC$ *where C is an MHP generator.*
(ii) $L = \gamma\delta^{-1}(L)$. (*That is*, $L = [xq \mid q \in M$ *such that* $L \circ q = L]$.)
(iii) $L = \gamma\delta^{-1}(L) \circ u$.

P r o o f. (i)⇒(ii). First note that $\gamma\delta^{-1}(L)\subset L$ since if $p\in\delta^{-1}(L)$, then $\gamma(p)=xp\in L\circ p=L$. Now, if $p\in C$, we have $L\circ p=xC\circ p=xC=L$, so $C\subset\delta^{-1}(L)$. Then $L=\gamma(C)\subset\gamma\delta^{-1}(L)\subset L$, so $L=\gamma\delta^{-1}(L)$.

(ii)⇒(iii). $L=L\circ u=\gamma\delta^{-1}(L)\circ u$.

(iii)⇒(i). $C=\delta^{-1}(L)\circ u$ is an MHP generator.

The next lemma will be used in our analysis of quasi-factors.

LEMMA 2. *Let $C=BK$ be an MHP generator, and let $p\in M$. Then $C\circ p=BpK_p$ where $K_p\subset J(M)$ is defined by $K_p=[w\in J(M)\mid rw=r$ for some $r\in C\circ p]$.*

P r o o f. Let $r\in C\circ p$ so $ru\in C\circ pu$ and $ru(pu)^{-1}\in C\circ u\cap G=C\cap G=B$ so $ru\in Bpu$. If $w\in J(M)$ is such that $rw=r$, then $r\in Bpuw=Bpw$. That is, $r\in BpK_p$ so $C\circ p\subset BpK_p$. Now let $v\in K_p$. We show that $C\circ pv=C\circ p$. There is an $r\in C\circ p$ such that $rv=r$. Then $r=rv\in C\circ pv$. Since the sets $\{C\circ q\}$ form a decomposition of M, it follows that $C\circ pv=C\circ p$. Now let $b\in B$. Then $bpv\in C\circ pv=C\circ p$.

The main thrust of this paper is the study of the Ellis groups of representation quasi-factors. We first obtain some of the elementary properties of these quasi-factors. Recall that the minimal flows (X,T) and (Y,T) are *disjoint* ($X\perp Y$) if the product flow $(X\times Y,T)$ is minimal. In fact, the next theorem indicates that representation quasi-factors are a measure of "non-disjointness" of two minimal flows.

THEOREM 3. *Let (X,T) and (Y,T) be minimal flows.*

(i) *(X,T) and (Y,T) are disjoint if and only if $\mathcal{X}_Y=\{X\}$ (the one-point flow).*

(ii) *If Y is a factor of X, then $\mathcal{X}_Y$ is a highly proximal extension of Y.*

(iii) *If X is a factor of Y, then $\mathcal{X}_Y=X$.*

(iv) *All common factors of X and Y are factors of $\mathcal{X}_Y$.*

P r o o f. For this proof, let $\gamma:M\to X$ and $\delta:M\to Y$ be the homomorphisms such that $\gamma(p)=xp$ and $\delta(p)=yp$, where $xu=x$ and $yu=y$.

(i) Note that X and Y are disjoint if and only if $\gamma\delta^{-1}(y)=[xp\mid p\in M$ with $yp=y]=X$. Thus if X and Y are disjoint, $\gamma(\delta^{-1}(y))\circ u=X\circ u=X$, and $\mathcal{X}_Y=1$. As for the converse, note that $\delta^{-1}(y)\circ u\subset\delta^{-1}(y)$ so $\gamma(\delta^{-1}(y))\circ u\subset\gamma(\delta^{-1}(y))$. Hence if X and Y are not disjoint, $\gamma(\delta^{-1}(y))\circ u\ne X$ so $\mathcal{X}_Y\ne\{X\}$.

(ii) In this case, let $\pi:X\to Y$ be a homomorphism, so $\delta=\pi\gamma$, and $\delta^{-1}=\gamma^{-1}\pi^{-1}$ and $\mathcal{X}_Y=[\pi^{-1}(y)\circ p\mid p\in M]$, which is a highly proximal extension of Y ([AG]).

(iii) Let $\psi:Y\to X$ be a homomorphism, with $\psi(y)=x$. Then $\gamma=\psi\delta$, $\gamma\delta^{-1}(y)=\psi(y)=x$, so $\mathcal{X}_Y=[\psi(yp)\mid p\in M]=[xp\mid p\in M]=X$.

(iv) Let (Z,T) be a common factor of (X,T) and (Y,T), and let $\pi : X \to Z$, $\psi : Y \to Z$ be homomorphisms with $\pi\gamma = \psi\delta$ and $\pi(x) = \psi(y) = z$. Let $C = \delta^{-1}(y) \circ u$ so $\mathcal{X}_Y = \overline{xCT}$. We define the homomorphism σ from $\mathcal{X}_Y$ to Z by $\sigma(xC \circ p) = zp$. To see that σ is well defined, it is sufficient to show that each element of $\mathcal{X}_Y$ is contained in a fiber of π, and for this note that $C \subset \delta^{-1}(y)$, so $xC \subset \pi^{-1}(z)$. It follows that $xC \circ p \subset \pi^{-1}(z) \circ p \subset \pi^{-1}(zp)$.

Let F be a subset of G. Define $\Delta(F) = [g \in G \mid Fg = F]$.

LEMMA 4. (i) *If F is a non-empty closed subset of G, then $\Delta(F)$ is a closed subgroup of G.*

(ii) *If A and B are closed subgroups of G, then $\Delta(AB)$ is the largest closed subgroup of G containing B which is contained in AB.*

(iii) $\Delta(AB) = (\Delta(AB) \cap A)B$.

Proof. (i) For any non-empty subset F of G, $\Delta(F)$ is a subgroup of G. Now, suppose F is closed, and let $g \in \overline{\Delta(F)}$. Then $F\Delta(F) = F$ so $F\overline{\Delta(F)} = \overline{F} = F$. Therefore $Fg \subset F$. But also (by the continuity of inversion) $g^{-1} \in \overline{\Delta(F)}$ so $Fg^{-1} \subset F$. Therefore $Fg = F$ and $g \in \Delta(F)$.

(ii) Let C be a closed subgroup of G with $B \subset C \subset AB$. If $c \in C$, then $c = ab$ for some $a \in A$, $b \in B$. Hence $a = cb^{-1} \in C$. Thus $Ba \subset Ca \subset C$, so $ABc = ABab \subset ACB \subset AABB = AB$. Therefore $ABc \subset AB$, and since C is a subgroup, $ABc^{-1} \subset AB$ so $c \in \Delta(AB)$.

The easy proof of (iii) is omitted.

From now on, we will write $\mathcal{G}(xC)$ for the Ellis group of the quasi-factor $\overline{xCT}$. The next theorem is our main result on the Ellis group of a representation quasi-factor.

THEOREM 5. *Let (X,T) be minimal, let $x \in X$ with $xu = x$, and let $A = \mathcal{G}(X)$. Let $C = BK$ be an MHP generator.*

(i) $B \subset \mathcal{G}(xC)$.

(ii) *$g \in \mathcal{G}(xC)$ if and only if $g \in \Delta(AB)$ and $xbK = xbK_g$ for all $b \in B$.*

Proof. (i) Since the quasi-factor determined by xC is a factor of $\overline{CT}$ whose Ellis group is B, we have $B \subset \mathcal{G}(xC)$.

(ii) Let $g \in \mathcal{G}(xC)$, let $a \in A$ and $b \in B$. Then $xabg = xbg \in xC \circ g = xC$ so $xabg \in xCu = xB$, $xabg = x\beta$ for some $\beta \in B$, $abg\beta^{-1} \in A$, $abg \in AB$. This shows that $ABg \subset AB$, and since $g^{-1} \in \mathcal{G}(xC)$ we have $abg^{-1} \in AB$ so $g \in \Delta(AB)$.

Let $b \in B$ and $v \in K$. Then $xBK = xC = xC \circ g = xBgK_g$ (Lemma 2), so $xbv = x\beta gw$ for some $\beta \in B$ and $w \in K_g$. Since $g \in \Delta(AB)$ we have $\beta g = \alpha\beta'$, where $\alpha \in A$ and $\beta' \in B$. Then $x\beta gw = x\alpha\beta' w = x\beta' w$. Thus $xbv = x\beta' w$ so $xb = x\beta'$, and $xbv = x\beta' w = xbw \in xbK_g$. That is, $xbK \subset xbK_g$. As for the opposite inclusion, let $w \in K_g$. Then, since $B \subset ABg$, there are $b' \in B$, $a \in A$ such that $xbw = xab'gw = xb'gw = x\beta v$ (where $\beta \in B$, $v \in K$). Then $xb = x\beta$ so $xbw = x\beta v = xbv \in xbK$.

Conversely, suppose $g \in \Delta(AB)$ and $xbK = xbK_g$ for all $b \in B$. Then $xC \circ g = xBgK_g = xABgK_g = xABK_g = xBK_g = xBK = xC$.

Using Theorem 5, we can determine $\mathcal{G}(xC)$ in a number of cases.

COROLLARY 6. *Let* (X,T) *be a distal minimal flow. Then* $\mathcal{G}(xC) = \Delta(AB)$.

Proof. Since (X,T) is distal, $x'v = x'$ for all $x' \in X$ and all $v \in J(M)$. Hence $xbK = \{xb\} = xbK_g$.

The minimal flow (X,T) is said to be *regular* if all almost periodic points of $(X \times X, T)$ are on graphs of automorphisms. That is, if (x,x') is almost periodic, there is an automorphism φ of (X,T) such that $x' = \varphi(x)$.

COROLLARY 7. *Let* (X,T) *be a regular minimal flow, and let* $g \in G$. *Then* $g \in \mathcal{G}(xC)$ *if and only if* $g \in \Delta(AB)$ *and* $xK = xK_g$.

Proof. Note that if $g \in G$, then (x, xg) is an almost periodic point. Then if $b \in B$, there is an automorphism φ of (X,T) such that $xb = \varphi(x)$, so if $xK = XK_g$ then $xbK = \varphi(x)K = \varphi(xK) = \varphi(xK_g) = xbK_g$.

COROLLARY 8. *If* $a^{-1}C \circ a = C$ *for all* $a \in A$, *then* $\mathcal{G}(xC) = AB$.

Proof. It is sufficient to show that $A \subset \mathcal{G}(xC)$. Let $a \in A$. Then $xC \circ a = xa(a^{-1}C \circ a) = xaC = xC$.

LEMMA 9. *Let* (X,T) *be a regular minimal flow, and let* (X',T) *be its maximal highly proximal extension. Then* (X',T) *is regular.*

Proof. Let $\pi : X' \to X$ be a homomorphism, let (x', y') be an almost periodic point in $X' \times X'$, and let $(x,y) = \pi(x',y')$. Then (x,y) is almost periodic, so by regularity of X, there is an automorphism φ such that $y = \varphi(x)$. By [AW], p. 392, there is an endomorphism φ' of X' such that $\pi\varphi' = \varphi\pi$. Now $(\varphi'(x'), y')$ is an almost periodic point, and $\pi(y') = \pi(\varphi'(x'))$. Since the homomorphism π is proximal, we must have $\varphi'(x') = y'$. Now apply the same argument to φ^{-1}, and obtain an endomorphism ψ' of X' such that $\psi'(y') = x'$. Then $\psi'\varphi'$ is the identity, so φ' is an automorphism.

COROLLARY 10. *Let* (X,T) *and* (Y,T) *be minimal flows with* (Y,T) *regular. Then* $\mathcal{G}(\mathcal{X}_Y) = AB$.

Proof. If $\delta : M \to Y$ and $C = \delta^{-1}(y) \circ u$, then $\overline{CT}$ is the maximal highly proximal extension of Y. By Lemma 9, $\overline{CT}$ is regular. It follows from [AW], Theorem 2.5(4), that $g^{-1}C \circ g = C$ for all $g \in G$. The conclusion now follows from Corollary 8.

LEMMA 11. *Let B be a closed subgroup of G. Then $B \circ u$ is an MHP generator with $B \circ u = BK$.*

Proof. If $b \in B$ then $B \circ u = Bb \circ u \subset B \circ b \circ u = B \circ b$. Since also $B \circ u \subset B \circ b^{-1}$ we have $B \circ b = B \circ u$. Now let $q \in B \circ u$, so there are nets $\{b_j\}$ in B and $\{s_j\}$ in T with $s_j \to u$ and $b_j s_j \to q$. Then $B \circ b_j s_j \to B \circ q$ and also $B \circ b_j s_j = B \circ u s_j \to B \circ u \circ u = B \circ u$. Thus $B \circ q = B \circ u$, and $B \circ u$ is an MHP generator. Since $B \circ u \cap G = B$, $B \circ u$ has "group part" B and $B \circ u = BK$.

THEOREM 12. *Let (X, T) be a minimal flow and let B be a closed subgroup of G. Then $\mathcal{G}(xB \circ u) = \Delta(AB)$.*

Proof. It follows from Theorem 5 and Lemma 11 that $\mathcal{G}(xB \circ u) \subset \Delta(AB)$. Now let $g \in \Delta(AB)$. Then $xB = xAB = xABg = xBg \subset xB \circ g$, so $xB \circ u \subset xB \circ g$. Since $g^{-1} \in \Delta(AB)$ we have $xB \circ u \subset xB \circ g^{-1}$. It follows that $xB \circ g = xB \circ u$ so $g \in \mathcal{G}(xB \circ u)$.

Theorem 12 implies that, for every closed subgroup B of G, the group $\Delta(AB)$ occurs as the Ellis group of a representation quasi-factor.

The next corollary indicates that there is a group theoretic obstruction to a quasi-factor being a factor.

COROLLARY 13. *Suppose $\mathcal{X}_Y$ is a factor of X. Then $\mathcal{G}(xC) = AB$ (in particular, AB is a group).*

Proof. If $\mathcal{X}_Y$ is a factor of X, then $A \subset \mathcal{G}(xC)$, and always $B \subset \mathcal{G}(xC)$, so we have $AB \subset \mathcal{G}(xC) \subset \Delta(AB) \subset AB$. Then $AB = \mathcal{G}(xC)$.

We conclude with two results concerning disjointness. If (X, T) and (Y, T) are minimal, they are said to be *disjoint over their common factor* (Z, T) if (for homomorphisms $\pi : X \to Z$ and $\psi : Y \to Z$) the relation $R_{\pi,\psi} = [(x, y) \mid \pi(x) = \psi(y)]$ is a minimal subset of $X \times Y$.

In the next theorem, we choose $x \in X$ and $y \in Y$ such that $xu = x$, $yu = y$, and $\pi(x) = \psi(y)$.

THEOREM 14. *Suppose the minimal flows (X, T) and (Y, T) are disjoint over (Z, T). Then $\mathcal{G}(\mathcal{X}_Y) = \mathcal{G}(X)\mathcal{G}(Y)$.*

Proof. Let $z = \pi(x) = \psi(y)$. We first show that $\pi^{-1}(z) = \gamma(\delta^{-1}(y))$. Always $\gamma(\delta^{-1}(y)) \subset \pi^{-1}(z)$. Let $x' \in \pi^{-1}(z)$, and let $r \in M$ be such that $x' = xr$. Now $\psi(yr) = \pi(xr) = z = \pi(x) = \psi(y)$, so $(x, y), (xr, y) \in R_{\pi,\psi}$. Since $R_{\pi,\psi}$ is minimal, $(xr, y) = (x, y)q$ for some $q \in M$. That is, $x' = xr = xq$ and $yq = y$, so $x' \in \gamma(\delta^{-1}(y))$.

Let $C = \delta^{-1}(y) \circ u$, so $\mathcal{X}_Y = \overline{xCT}$, and $\mathcal{G}(\mathcal{X}_Y) = \mathcal{G}(xC)$. Let $A = \mathcal{G}(X)$. It is sufficient (Theorem 5) to show that $A \subset \mathcal{G}(xC)$. Let $\alpha \in A$. Then $z\alpha = z$ and it follows easily that $\pi^{-1}(z) \circ \alpha \subset \pi^{-1}(z)$ so $\pi^{-1}(z) \circ \alpha \subset \pi^{-1}(z) \circ u$. Since also $z\alpha^{-1} = z$ we have $\pi^{-1}(z) \circ \alpha = \pi^{-1}(z) \circ u$. But $xC = \gamma(\delta^{-1}(y)) = \pi^{-1}(z)$ so $xC \circ \alpha = xC$, and $\alpha \in \mathcal{G}(xC)$.

It is elementary that disjoint flows cannot have a common (non-trivial) factor, but it is not known whether they can have a common quasi-factor. We rule this out in a special case. (Recall that a flow (X, T) cannot be disjoint from a non-trivial quasi-factor $\mathcal{Q}$, since $[(x, Q) \in X \times \mathcal{Q} \mid x \in Q]$ is a closed invariant proper subset of $X \times \mathcal{Q}$.)

THEOREM 15. *Let* (X, T) *and* (Y, T) *be disjoint minimal flows, with* (Y, T) *distal and regular. Let* $C = BK$ *and* $C' = B'K'$ *be MHP generators. Then* $\overline{xCT}$ *and* $\overline{yC'T}$ *are not proximally equivalent* (*unless they are trivial*).

Proof. Let $A = \mathcal{G}(X)$ and $F = \mathcal{G}(Y)$. Since X and Y are disjoint, we have $AF = G$. Now $\mathcal{G}(yC') = \Delta(FB') = FB'$. (The last equality holds since (Y, T) is regular, so F is normal in G.) If $\overline{xCT}$ and $\overline{yC'T}$ are proximally equivalent, then $\mathcal{G}(xC) = \mathcal{G}(yC')$, and $A\mathcal{G}(xC) = A\mathcal{G}(yC') = AFB' = GB' = G$. Thus X is disjoint from its quasi-factor $\overline{xCT}$, which is a contradiction unless the latter is trivial.

REFERENCES

[A] J. Auslander, *Minimal Flows and their Extensions*, North-Holland Math. Stud. 153, North-Holland, 1988.

[AG] J. Auslander and S. Glasner, *Distal and highly proximal extensions of minimal flows*, Indiana Univ. Math. J. 26 (1977), 731–749.

[AW] J. Auslander and J. van der Woude, *Maximal highly proximal generators of minimal flows*, Ergodic Theory Dynam. Systems 1 (1981), 389–412.

[E] R. Ellis, *Lectures in Topological Dynamics*, W. A. Benjamin, 1969.

[G] S. Glasner, *Compressibility properties in topological dynamics*, Amer. J. Math. 97 (1975), 148–171.

University of Maryland
College Park, MD 20742-4015
U.S.A.
E-mail: jna@math.umd.edu

Received 3 August 1999;
revised 4 January 2000
(3807)