SUPPORT OVERLAPPING $L_1$ CONTRACTIONS
AND EXACT NON-SINGULAR TRANSFORMATIONS

BY

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Dedicated to the memory of Anzelm Iwanik

Abstract. Let $T$ be a positive linear contraction of $L_1$ of a $\sigma$-finite measure space $(X, \Sigma, \mu)$ which overlaps supports. In general, $T$ need not be completely mixing, but it is in the following cases:

(i) $T$ is the Frobenius–Perron operator of a non-singular transformation $\phi$ (in which case complete mixing is equivalent to exactness of $\phi$).
(ii) $T$ is a Harris recurrent operator.
(iii) $T$ is a convolution operator on a compact group.
(iv) $T$ is a convolution operator on a LCA group.

Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space, and let $T$ be a positive contraction on $L_1(\mu)$ which preserves integrals. The study of the asymptotic behaviour of Markov chains leads to questions of convergence of $\{T^n f\}$ for $f \in L_1$. We call $T$ completely mixing if for any two non-negative functions $f, g \in L_1$ with $\int f \, d\mu = \int g \, d\mu$ we have $\lim_{n \to \infty} \|T^n f - T^n g\|_1 = 0$. When $T$ has a positive fixed point $h$ (normalized to $\int h \, d\mu = 1$), then complete mixing is equivalent to the convergence $\|T^n f - (\int f \, d\mu) h\|_1 \to 0$ for every $f \in L_1$ (asymptotic stability). For $L_1$ separable and $\mu$ non-atomic, residuality of completely mixing contractions was studied by Iwanik and Rębowski [IR].

We say that $T$ overlaps supports if for any two non-negative functions $f, g \in L_1(\mu)$ with positive integrals there exists a positive integer $n$ (depending on $f$ and $g$) such that $\int (T^n f \wedge T^n g) \, d\mu > 0$. A natural question (motivated also by the zero-two law of [OS], since support overlapping clearly implies $\lim_n \|T^n (f - T f)\|_1 < 2$ for every non-negative $f \in L_1$ with $\int f \, d\mu = 1$) is whether support overlapping (which is obviously necessary) is sufficient for complete mixing. It was noted in [R] that in general the answer is negative (the example has a strictly positive fixed point). Bartoszek and [515]
Brown [BBr] have shown that when $T$ has a strictly positive (normalized) fixed point $h$, support overlapping implies weak convergence of $\{T^n\}$, i.e.,

$$\langle T^n f, u \rangle \to \langle f d\mu(h, u) \rangle \text{ for every } u \in L_\infty.$$  

Thus, for $T$ with a strictly positive fixed point, support overlapping lies between complete mixing and mixing.

Let $\phi$ be a non-singular transformation of $(X, \Sigma, \mu)$ (we do not assume the existence of an invariant measure). Then there is a (unique) positive contraction $T$ in $L_1(\mu)$, called the Frobenius–Perron operator of $\phi$, such that $T^* u = u \circ \phi$ for $u \in L_\infty$ (see e.g. [K]). Note that the Frobenius–Perron operator depends on $\mu$, and is changed (through the Radon–Nikodym theorem) if $\mu$ is replaced by an equivalent measure. The transformation $\phi$ is called exact if the $\sigma$-algebra $\Sigma_\infty := \bigcap_{n=1}^{\infty} \phi^{-n} \Sigma$ is trivial modulo $\mu$ (i.e., if $A \in \Sigma_\infty$, then $\mu(A) = 0$ or $\mu(X - A) = 0$).

**Theorem 1.** Let $\phi$ be a non-singular transformation of a $\sigma$-finite measure space $(X, \Sigma, \mu)$. Then the following are equivalent:

(i) The transformation $\phi$ is exact.

(ii) The Frobenius–Perron operator of $\phi$ is completely mixing.

(iii) The Frobenius–Perron operator of $\phi$ overlaps supports.

**Proof.** The equivalence of (i) and (ii) is proved in [L], and clearly (ii) implies (iii).

We now assume that the Frobenius–Perron operator $T$ overlaps supports, and show that $\phi$ is exact. Let $A \in \bigcap_{n=1}^{\infty} \phi^{-n} \Sigma$, and assume that $A$ is not trivial (i.e., $\mu(A) > 0$ and $\mu(X - A) > 0$). Then for each $n$ there is a set $A_n \in \Sigma$ with $\phi^{-n} A_n = A$. Define $B = X - A$ and $B_n = X - A_n$, so $\phi^{-n} B_n = B$. Take $0 \leq f \in L_1(A)$ with $\int f \, d\mu = 1$ and $0 \leq g \in L_1(B)$ with $\int g \, d\mu = 1$. Then for every $n \geq 0$ we have

$$\|T^n f - T^n g\|_1 \geq |\langle T^n f - T^n g, 1_{A_n} - 1_{B_n} \rangle| = |\langle f - g, 1_{A_n} \circ \phi^n - 1_{B_n} \circ \phi^n \rangle|$$

$$= |\langle f - g, 1_A - 1_B \rangle| = \langle f, 1_A \rangle + \langle g, 1_B \rangle = 2.$$

This means that for every $n$ the norm one positive functions $T^n f$ and $T^n g$ have disjoint supports, contradicting the assumption that $T$ overlaps supports.

**Remark.** In the case where $\phi$ has an invariant probability equivalent to $\mu$, the theorem was proved by Bartoszek and Brown [BBr]. Zaharopol [Z] proved it when $\phi$ has an invariant probability absolutely continuous with respect to $\mu$, using the result of [BBr]. Our proof (including the result from [L]) is much simpler.

In order to construct examples of non-singular exact transformations which have only an infinite $\sigma$-finite invariant measure, or no invariant measure at all, we use the following result of Jamison and Orey [JO]: Let $P$
be a transition probability with \( \mu P \ll \mu \), and let \( \mathbf{P}_\mu \) be the probability of the Markov chain on the space of (one-sided) trajectories with initial measure \( \mu \). The operator \( T \) induced by (the pre-dual of) \( P \) in \( L_1(\mu) \) is completely mixing if and only if the one-sided Markov shift is exact (see [ALW] for a measure-theoretic proof). Thus, Markov shifts of aperiodic null-recurrent Markov matrices are exact with an infinite invariant measure. If we take \( \tau \) non-singular on \((S, \mathcal{A}, m)\) without any \( \sigma \)-finite invariant measure (a type III transformation) and define \( P(x, \cdot) = \frac{1}{2}(\delta_x + \delta_{\tau x}) \), then \( \|P^n(I - P)\| \to 0 \), and the \( L_1(\mu) \) pre-dual is completely mixing, without any invariant measure. Hence its one-sided Markov shift is exact and has no invariant measure.

**Proposition.** Let \( T \) be a Harris recurrent contraction of \( L_1(\mu) \). Then \( T \) is completely mixing if (and only if) it overlaps supports.

**Proof.** By the Jamison–Orey theorem (see e.g. [OS]), \( T \) is completely mixing if (and only if) it is aperiodic. When \( T \) overlaps supports, there cannot be a periodic set, so \( T \) is aperiodic.

**Remark.** In [BB] and [R], the proposition was proved under the assumption that \( T \) has a fixed point in \( L_1 \) (the positive recurrent case).

**Theorem 2.** Let \( T \) be a positive contraction of \( L_1(\mu) \) which preserves integrals. Then \( T \) is asymptotically stable if and only if it satisfies the following three conditions:

1. \( T \) is mean ergodic (i.e., \( n^{-1} \sum_{k=1}^n T^k f \) converges for every \( f \in L_1 \)).
2. \( T \) overlaps supports.
3. For every \( f \in L_1(\mu) \) with \( \int f \, d\mu = 0 \) there is a subsequence such that \( \{T^k f\} \) converges strongly.

**Proof.** The necessity of the three conditions is obvious.

Assume \( T \) satisfies the three conditions. Let \( C \) and \( D \) be the conservative and dissipative parts of \( T \). By Helmberg’s condition [K, p. 175], the mean ergodicity of \( T \) yields that \( \mu(C) > 0 \). \( T \) has an \( L_1 \) fixed point \( f_C \) which is supported on \( C \), and \( T^n 1_D \downarrow 0 \) a.e. Denote by \( T_C \) the restriction of \( T \) to the invariant subspace \( L_1(C) \). Then \( T_C \) has a strictly positive fixed point (on \( C \)), and obviously overlaps supports by (2). By Proposition 1 of [BB], \( \{T^n f\} \) converges weakly in \( L_1(C) \) for every \( f \in L_1(C) \). Condition (3) yields the strong convergence for \( f \in L_1(C) \) with \( \int f \, d\mu = 0 \). But \( T_C \) is ergodic since it overlaps supports, so we have asymptotic stability of \( T_C \), i.e., \( \{T^n f\} \) converges strongly to \( \{\int f \, d\mu/\int f \, d\mu\} f_C \) for every \( f \in L_1(C) \).

It remains to prove the convergence when \( f \in L_1(D) \), and we may assume \( f \geq 0 \). Fix \( \varepsilon > 0 \). Since \( \int_D T^n f \, d\mu = \int_X f \cdot T^n 1_D \, d\mu \to 0 \) by Lebesgue’s theorem, there is a \( j > 0 \) with \( \|1_D T^j f\| \ll \varepsilon \). Set \( g = 1_D T^j f \) and \( h = 1_CT^j f \). Then \( \|T^n g - T^{n+j} g\| \ll \|T^n h - T^{n+j} h\| + 2\varepsilon \) shows that \( \{T^n f\} \) is a Cauchy sequence in \( L_1 \), since \( \{T^n h\} \) is, by the first part. Because \( T \)
preserves integrals and its restriction to $L_1(C)$ is ergodic, $\{T^n f\}$ converges to the correct limit.

Remarks. 1. Asymptotic stability was proved in [B] with the additional assumption that $\mu(D) = 0$ (which is not necessary [Z]). This is equivalently stated there with condition (1) replaced by the assumption that $T$ has a strictly positive integrable fixed point.

2. Any two of the three conditions in Theorem 2 are not sufficient for asymptotic stability: (2) and (3) hold for completely mixing $T$ with no invariant measure; the example of [AkBo] mentioned in [Z] is not completely mixing though it satisfies (1) and (2); $T$ induced by a cyclic permutation of a finite set is periodic and satisfies (1) and (3).

Corollary. $T$ is asymptotically stable if and only if $T$ is almost periodic ($\{T^n f\}$ is conditionally compact for every $f \in L_1$) and overlaps supports.

Proof. Almost periodicity implies conditions (1) and (3) of Theorem 2.

Theorem 3. Let $G$ be a compact group with normalized Haar measure $\mu$, and let $\nu$ be a regular probability on $G$. The convolution operator $T f(x) = \int f(xy) \, d\nu(y)$, defined in $L_1(G, \mu)$, is asymptotically stable if (and only if) it overlaps supports.

Proof. $T^n f = \int f(xy) \, d\nu^{(n)}(y)$ (with $\nu^{(n)}$ the $n$th convolution power of $\nu$), so $\{T^n f\}$ is in the closed convex hull of the translation orbit $\{T_y f : y \in G\}$. This orbit is compact in $L_1$ by the continuity of the representation in $L_1$, and by the Banach–Mazur theorem also its closed convex hull is compact. Hence $T$ is almost periodic, and when it overlaps supports it is asymptotically stable by the previous Corollary (or by [B]).

For the next result, we need the following concept from [KL]: For a positive contraction $T$ of $L_1$, define $\Sigma_t(T) = \{A \in \Sigma : \text{for each } n \text{ there is } 0 \leq f_n \leq 1 \text{ with } T^{*n} f_n = 1_A\}$. It was proved in [KL] that if $T$ is non-disappearing (i.e., $T^* 1_A = 0$ a.e. implies $1_A = 0$ a.e.), then $\Sigma_t(T)$ is a $\sigma$-algebra (called the tail or asymptotic $\sigma$-algebra), and for $A \in \Sigma_t(T)$ the $f_n$ in the above definition are uniquely defined characteristic functions.

The proof of Theorem 1 shows in fact that if $T$ is non-disappearing and overlaps supports, then $\Sigma_t(T)$ is trivial mod $\mu$.

Theorem 4. Let $T$ be a positive contraction of $L_1(\mu)$ which preserves integrals, with $\mu$ invariant for $T$. Then $T$ is completely mixing if and only if it satisfies the following two conditions:

1. $T$ overlaps supports.

2. For every $f \in L_1(\mu)$ with $\int f \, d\mu = 0$ there is a subsequence such that $\{T^{k_n} f\}$ converges strongly.
Proof. The necessity is clear, so we assume that both conditions hold. If $\mu$ is finite, then $T$ is mean ergodic, and Theorem 2 yields asymptotic stability. Hence we assume that $\mu$ is infinite. Support overlapping implies that either $\mu(C) = 0$ or $\mu(D) = 0$ (since both sets are absorbing [K, p. 131]). If $\mu(C) = 0$, we have $T^nf \to 0$ a.e. for any $f \in L^1$ (by dissipativity), so when $\int f \, d\mu = 0$ condition (2) yields $\|T^{kj}f\|_1 \to 0$, so $\|T^n f\|_1 \to 0$ since $T$ is a contraction.

We now consider the conservative case. Since $\mu$ is invariant, $T$ is non-disappearing, and support overlapping implies that the tail $\sigma$-algebra $\Sigma_t(T)$ is trivial mod $\mu$. Theorem 2.1 of [KL] and the discussion on p. 68 there show that the isometric part of $T$ in $L^2(\mu)$ is trivial, and therefore also the automorphic part is trivial. By [F, p. 85] we deduce that $T^n f \to 0$ weakly in $L^2$ for every $f \in L^2(\mu)$. If $f \in L^1 \cap L^2$ with $\int f \, d\mu = 0$, then the weak convergence to 0 in $L^2$ and the strong convergence in $L^1$ of $\{T^{kj}f\}$ imply that the $L^1$ limit is zero, and therefore $\|T^n f\|_1 \to 0$. By standard approximation we conclude that $T$ is completely mixing.

Remark. The proof shows in fact the infinite measure analogue of Proposition 1 of [BBr]: If $\mu$ is an infinite invariant measure for $T$ and $T$ overlaps supports, then $T^n f \to 0$ weakly in $L^2$ for every $f \in L^2(\mu)$.

Theorem 5. Let $G$ be a locally compact $\sigma$-compact Abelian group with Haar measure $\mu$, and let $\nu$ be a regular probability on $G$. The convolution operator $Tf(x) = \int f(xy) \, d\nu(y)$, defined in $L_1(G, \mu)$, is completely mixing if (and only if) it overlaps supports.

Proof. For a convolution operator $T$, its dual $T^*$ is the convolution with the reflected probability $\check{\nu}(A) := \nu(A^{-1})$. The characterization of $\Sigma_t(T)$ which follows from Theorems 2.1 and 3.2 of [KL] yields $\Sigma_t(T) = \Sigma_t(T^*)$. If $T$ overlaps supports, then $\Sigma_t(T)$ is trivial mod $\mu$, and Theorem 3.2 of [KL] yields its complete mixing.

Remark. When $G$ is LCA non-compact, the Markov shifts of completely mixing convolution operators will be exact with infinite invariant measure.

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REFERENCES


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