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SUPPORT OVERLAPPING L_1 CONTRACTIONS AND EXACT NON-SINGULAR TRANSFORMATIONS

BY

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Dedicated to the memory of Anzelm Iwanik

Abstract. Let T be a positive linear contraction of L_1 of a σ-finite measure space (X, Σ, μ) which overlaps supports. In general, T need not be completely mixing, but it is in the following cases:

(i) T is the Frobenius–Perron operator of a non-singular transformation ϕ (in which case complete mixing is equivalent to exactness of ϕ).

(ii) T is a Harris recurrent operator.

(iii) T is a convolution operator on a compact group.

(iv) T is a convolution operator on a LCA group.

Let (X, Σ, μ) be a σ-finite measure space, and let T be a positive contraction on $L_1(\mu)$ which preserves integrals. The study of the asymptotic behaviour of Markov chains leads to questions of convergence of $\{T^n f\}$ for $f \in L_1(\mu)$. We call T *completely mixing* if for any two non-negative functions $f, g \in L_1$ with $\int f\,d\mu = \int g\,d\mu$ we have $\lim_{n\to\infty} \|T^n f - T^n g\|_1 = 0$. When T has a positive fixed point h (normalized to $\int h\,d\mu = 1$), then complete mixing is equivalent to the convergence $\|T^n f - (\int f\,d\mu)h\|_1 \to 0$ for every $f \in L_1$ (*asymptotic stability*). For L_1 separable and μ non-atomic, residuality of completely mixing contractions was studied by Iwanik and Rębowski [IRe].

We say that T *overlaps supports* if for any two non-negative functions $f, g \in L_1(\mu)$ with positive integrals there exists a positive integer n (depending on f and g) such that $\int (T^n f \wedge T^n g)\,d\mu > 0$. A natural question (motivated also by the *zero-two law* of [OS], since support overlapping clearly implies $\lim_n \|T^n(f - Tf)\|_1 < 2$ for every non-negative $f \in L_1$ with $\int f\,d\mu = 1$) is whether support overlapping (which is obviously necessary) is sufficient for complete mixing. It was noted in [R] that in general the answer is negative (the example has a strictly positive fixed point). Bartoszek and

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Brown [BBr] have shown that when T has a strictly positive (normalized) fixed point h, support overlapping implies *weak* convergence of $\{T^n\}$, i.e., $\langle T^n f, u\rangle \to (\int f\,d\mu)\langle h, u\rangle$ for every $u \in L_\infty$. Thus, for T with a strictly positive fixed point, support overlapping lies between complete mixing and mixing.

Let ϕ be a non-singular transformation of (X, Σ, μ) (we do not assume the existence of an invariant measure). Then there is a (unique) positive contraction T in $L_1(\mu)$, called the *Frobenius–Perron operator* of ϕ, such that $T^*u = u \circ \phi$ for $u \in L_\infty$ (see e.g. [K]). Note that the Frobenius–Perron operator depends on μ, and is changed (through the Radon–Nikodym theorem) if μ is replaced by an equivalent measure. The transformation ϕ is called *exact* if the σ-algebra $\Sigma_\infty := \bigcap_{n=1}^\infty \phi^{-n}\Sigma$ is trivial modulo μ (i.e., if $A \in \Sigma_\infty$, then $\mu(A) = 0$ or $\mu(X - A) = 0$).

THEOREM 1. *Let ϕ be a non-singular transformation of a σ-finite measure space (X, Σ, μ). Then the following are equivalent*:

(i) *The transformation ϕ is exact.*
(ii) *The Frobenius–Perron operator of ϕ is completely mixing.*
(iii) *The Frobenius–Perron operator of ϕ overlaps supports.*

Proof. The equivalence of (i) and (ii) is proved in [L], and clearly (ii) implies (iii).

We now assume that the Frobenius–Perron operator T overlaps supports, and show that ϕ is exact. Let $A \in \bigcap_{n=1}^\infty \phi^{-n}\Sigma$, and assume that A is not trivial (i.e., $\mu(A) > 0$ and $\mu(X - A) > 0$). Then for each n there is a set $A_n \in \Sigma$ with $\phi^{-n}A_n = A$. Define $B = X - A$ and $B_n = X - A_n$, so $\phi^{-n}B_n = B$. Take $0 \le f \in L_1(A)$ with $\int f\,d\mu = 1$ and $0 \le g \in L_1(B)$ with $\int g\,d\mu = 1$. Then for every $n \ge 0$ we have

$$\begin{aligned}\|T^n f - T^n g\|_1 &\ge |\langle T^n f - T^n g, 1_{A_n} - 1_{B_n}\rangle| = |\langle f - g, 1_{A_n}\circ\phi^n - 1_{B_n}\circ\phi^n\rangle| \\ &= |\langle f - g, 1_A - 1_B\rangle| = \langle f, 1_A\rangle + \langle g, 1_B\rangle = 2.\end{aligned}$$

This means that for every n the norm one positive functions $T^n f$ and $T^n g$ have disjoint supports, contradicting the assumption that T overlaps supports.

REMARK. In the case where ϕ has an invariant probability equivalent to μ, the theorem was proved by Bartoszek and Brown [BBr]. Zaharopol [Z] proved it when ϕ has an invariant probability absolutely continuous with respect to μ, using the result of [BBr]. Our proof (including the result from [L]) is much simpler.

In order to construct examples of non-singular exact transformations which have only an infinite σ-finite invariant measure, or no invariant measure at all, we use the following result of Jamison and Orey [JOr]: *Let P*

be a transition probability with $\mu P \ll \mu$, *and let* $\mathbf{P}_\mu$ *be the probability of the Markov chain on the space of* (*one-sided*) *trajectories with initial measure* μ. *The operator* T *induced by* (*the pre-dual of*) P *in* $L_1(\mu)$ *is completely mixing if and only if the one-sided Markov shift is exact* (see [ALW] for a measure-theoretic proof). Thus, Markov shifts of aperiodic null-recurrent Markov matrices are exact with an infinite invariant measure. If we take τ non-singular on $(S, \mathcal{A}, m)$ without any σ-finite invariant measure (a type III transformation) and define $P(x, \cdot) = \frac{1}{2}(\delta_x + \delta_{\tau x})$, then $\|P^n(I - P)\| \to 0$, and the $L_1(\mu)$ pre-dual is completely mixing, without any invariant measure. Hence its one-sided Markov shift is exact and has no invariant measure.

PROPOSITION. *Let* T *be a Harris recurrent contraction of* $L_1(\mu)$. *Then* T *is completely mixing if* (*and only if*) *it overlaps supports.*

Proof. By the Jamison–Orey theorem (see e.g. [OS]), T is completely mixing if (and only if) it is aperiodic. When T overlaps supports, there cannot be a periodic set, so T is aperiodic.

REMARK. In [BBr] and [R], the proposition was proved under the assumption that T has a fixed point in L_1 (the positive recurrent case).

THEOREM 2. *Let* T *be a positive contraction of* $L_1(\mu)$ *which preserves integrals. Then* T *is asymptotically stable if and only if it satisfies the following three conditions*:

(1) T *is mean ergodic* (*i.e.*, $n^{-1}\sum_{k=1}^{n} T^k f$ *converges for every* $f \in L_1$).
(2) T *overlaps supports.*
(3) *For every* $f \in L_1(\mu)$ *with* $\int f\,d\mu = 0$ *there is a subsequence such that* $\{T^{k_j} f\}$ *converges strongly.*

Proof. The necessity of the three conditions is obvious.

Assume T satisfies the three conditions. Let C and D be the conservative and dissipative parts of T. By Helmberg's condition [K, p. 175], the mean ergodicity of T yields that $\mu(C) > 0$, T has an L_1 fixed point f_C which is supported on C, and $T^{*n} 1_D \downarrow 0$ a.e. Denote by T_C the restriction of T to the invariant subspace $L_1(C)$. Then T_C has a strictly positive fixed point (on C), and obviously overlaps supports by (2). By Proposition 1 of [BBr], $\{T_C^n f\}$ converges weakly in $L_1(C)$ for every $f \in L_1(C)$. Condition (3) yields the strong convergence for $f \in L_1(C)$ with $\int f\,d\mu = 0$. But T_C is ergodic since it overlaps supports, so we have asymptotic stability of T_C, i.e., $\{T^n f\}$ converges strongly to $(\int f\,d\mu / \int f_C\,d\mu) f_C$ for every $f \in L_1(C)$.

It remains to prove the convergence when $f \in L_1(D)$, and we may assume $f \ge 0$. Fix $\varepsilon > 0$. Since $\int_D T^n f\,d\mu = \int_X f \cdot T^{*n} 1_D\,d\mu \to 0$ by Lebesgue's theorem, there is a $j > 0$ with $\|1_D T^j f\|_1 < \varepsilon$. Set $g = 1_D T^j f$ and $h = 1_C T^j f$. Then $\|T^{n+j} f - T^{k+j} f\|_1 \le \|T^n h - T^k h\| + 2\varepsilon$ shows that $\{T^n f\}$ is a Cauchy sequence in L_1, since $\{T^n h\}$ is, by the first part. Because T

preserves integrals and its restriction to $L_1(C)$ is ergodic, $\{T^n f\}$ converges to the correct limit.

REMARKS. 1. Asymptotic stability was proved in [B] with the additional assumption that $\mu(D) = 0$ (which is *not* necessary [Z]). This is equivalently stated there with condition (1) replaced by the assumption that T has a strictly positive integrable fixed point.

2. Any two of the three conditions in Theorem 2 are not sufficient for asymptotic stability: (2) and (3) hold for completely mixing T with no invariant measure; the example of [AkBo] mentioned in [Z] is not completely mixing though it satisfies (1) and (2); T induced by a cyclic permutation of a finite set is periodic and satisfies (1) and (3).

COROLLARY. *T is asymptotically stable if and only if T is almost periodic* ($\{T^n f\}$ *is conditionally compact for every* $f \in L_1$) *and overlaps supports.*

Proof. Almost periodicity implies conditions (1) and (3) of Theorem 2.

THEOREM 3. *Let G be a compact group with normalized Haar measure μ, and let ν be a regular probability on G. The convolution operator $Tf(x) = \int f(xy)\,d\nu(y)$, defined in $L_1(G,\mu)$, is asymptotically stable if* (*and only if*) *it overlaps supports.*

Proof. $T^n f = \int f(xy)\,d\nu^{(n)}(y)$ (with $\nu^{(n)}$ the nth convolution power of ν), so $\{T^n f\}$ is in the closed convex hull of the translation orbit $\{T_y f : y \in G\}$. This orbit is compact in L_1 by the continuity of the representation in L_1, and by the Banach–Mazur theorem also its closed convex hull is compact. Hence T is almost periodic, and when it overlaps supports it is asymptotically stable by the previous Corollary (or by [B]).

For the next result, we need the following concept from [KL]: For a positive contraction T of L_1, define $\Sigma_t(T) = \{A \in \Sigma : \text{for each } n \text{ there is } 0 \le f_n \le 1 \text{ with } T^{*n} f_n = 1_A\}$. It was proved in [KL] that if T is non-disappearing (i.e., $T^* 1_A = 0$ a.e. implies $1_A = 0$ a.e.), then $\Sigma_t(T)$ is a σ-algebra (called the *tail* or *asymptotic σ-algebra*), and for $A \in \Sigma_t(T)$ the f_n in the above definition are uniquely defined characteristic functions.

The proof of Theorem 1 shows in fact that if T is non-disappearing and overlaps supports, then $\Sigma_t(T)$ is trivial mod μ.

THEOREM 4. *Let T be a positive contraction of $L_1(\mu)$ which preserves integrals*, *with μ invariant for T. Then T is completely mixing if and only if it satisfies the following two conditions*:

(1) *T overlaps supports.*

(2) *For every $f \in L_1(\mu)$ with $\int f\,d\mu = 0$ there is a subsequence such that $\{T^{k_j} f\}$ converges strongly.*

Proof. The necessity is clear, so we assume that both conditions hold. If μ is finite, then T is mean ergodic, and Theorem 2 yields asymptotic stability. Hence we assume that μ is infinite. Support overlapping implies that either $\mu(C)=0$ or $\mu(D)=0$ (since both sets are absorbing [K, p. 131]). If $\mu(C)=0$, we have $T^n f \to 0$ a.e. for any $f \in L_1$ (by dissipativity), so when $\int f\,d\mu = 0$ condition (2) yields $\|T^{k_j} f\|_1 \to 0$, so $\|T^n f\|_1 \to 0$ since T is a contraction.

We now consider the conservative case. Since μ is invariant, T is non-disappearing, and support overlapping implies that the tail σ-algebra $\Sigma_t(T)$ is trivial mod μ. Theorem 2.1 of [KL] and the discussion on p. 68 there show that the isometric part of T in $L_2(\mu)$ is trivial, and therefore also the automorphic part is trivial. By [F, p. 85] we deduce that $T^n f \to 0$ weakly in L_2 for every $f \in L_2(\mu)$. If $f \in L_1 \cap L_2$ with $\int f\,d\mu = 0$, then the weak convergence to 0 in L_2 and the strong convergence in L_1 of $\{T^{k_j} f\}$ imply that the L_1 limit is zero, and therefore $\|T^n f\|_1 \to 0$. By standard approximation we conclude that T is completely mixing.

Remark. The proof shows in fact the infinite measure analogue of Proposition 1 of [BBr]: *If μ is an infinite invariant measure for T and T overlaps supports*, *then $T^n f \to 0$ weakly in L_2 for every $f \in L_2(\mu)$.*

Theorem 5. *Let G be a locally compact σ-compact Abelian group with Haar measure μ, and let ν be a regular probability on G. The convolution operator $Tf(x) = \int f(xy)\,d\nu(y)$, defined in $L_1(G,\mu)$, is completely mixing if (and only if) it overlaps supports.*

Proof. For a convolution operator T, its dual T^* is the convolution with the reflected probability $\check{\nu}(A) := \nu(A^{-1})$. The characterization of $\Sigma_t(T)$ which follows from Theorems 2.1 and 3.2 of [KL] yields $\Sigma_t(T) = \Sigma_t(T^*)$. If T overlaps supports, then $\Sigma_t(T)$ is trivial mod μ, and Theorem 3.2 of [KL] yields its complete mixing.

Remark. When G is LCA non-compact, the Markov shifts of completely mixing convolution operators will be exact with infinite invariant measure.

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