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ERGODIC DECOMPOSITION OF QUASI-INVARIANT PROBABILITY MEASURES

BY

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This paper is dedicated to Anzelm Iwanik in memory of his contributions to the subject and his personal fortitude

Abstract. The purpose of this note is to prove various versions of the ergodic decomposition theorem for probability measures on standard Borel spaces which are quasi-invariant under a Borel action of a locally compact second countable group or a discrete nonsingular equivalence relation. In the process we obtain a simultaneous ergodic decomposition of all quasi-invariant probability measures with a prescribed Radon–Nikodym derivative, analogous to classical results about decomposition of invariant probability measures.

1. Introduction. Throughout this note we assume that $(X,\mathcal{S})$ is a standard Borel space, i.e. a measurable space which is isomorphic to the unit interval with its usual Borel structure. A *Borel action* T of a locally compact second countable group G on X is a group homomorphism $g \mapsto T^g$ from G into the group $\mathrm{Aut}(X,\mathcal{S})$ of Borel automorphisms of $(X,\mathcal{S})$ such that the map $(g,x)\mapsto T^g x$ from $G\times X$ to X is Borel.

Let T be a Borel action of a locally compact second countable group G on X, and let μ be a probability measure on $\mathcal{S}$. The measure μ is *quasi-invariant* under T if $\mu(T^g B)=0$ for every $g\in G$ and every $B\in\mathcal{S}$ with $\mu(B)=0$, and μ is *ergodic* under T if $\mu(B)\in\{0,1\}$ for every $B\in\mathcal{S}$ with $\mu(B\triangle T^g B)=0$ for every $g\in G$.

The following theorem is part of the mathematical folklore about group actions on measure spaces.

THEOREM 1.1 (Ergodic decomposition theorem). *Let T be a Borel action of a locally compact second countable group G on a standard Borel space $(X,\mathcal{S})$, and let μ be a probability measure on $\mathcal{S}$ which is quasi-invariant under T. Then there exist a standard Borel space $(Y,\mathcal{T})$, a probability measure*

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ν on $\mathcal{T}$ and a family $\{p_y : y \in Y\}$ of probability measures on $(X, \mathcal{S})$ with the following properties.

(1) *For every $B \in \mathcal{S}$, the map $y \mapsto p_y(B)$ is Borel on Y and*

$$\mu(B) = \int p_y(B)\, d\nu(y).$$

(2) *For every $y \in Y$, p_y is quasi-invariant and ergodic under T.*

(3) *If $y, y' \in Y$ and $y \neq y'$ then p_y and $p_{y'}$ are mutually singular.*

For $G = \mathbb{Z}$ or $G = \mathbb{R}$ there are many versions of Theorem 1.1 in the literature (cf. e.g. [5], [9], [8], [17], [22]). More general decomposition results can be found in [14]–[15], [16], [19]–[21] and [24]. However, none of these general results are stated and proved in a form particularly convenient for the purposes of general ergodic theory, and many specialists in ergodic theory do not seem aware of Theorem 1.1.

The purpose of this note is to give two reasonably elementary and self-contained proofs of Theorem 1.1 in the special case where the group G is countable. In the process we provide a little more information about the space Y and the measures p_y, $y \in Y$, appearing in the statement of the theorem. In the last section we sketch an extension of our method to actions of locally compact second countable groups (Theorem 5.2).

For the remainder of this section we fix a Borel action T of a countable group G on X and a probability measure μ on $\mathcal{S}$ which is quasi-invariant under T. There exists a Borel map $\varrho : G \times X \to \mathbb{R}$ such that

$$e^{\varrho(g,x)} = \frac{d\mu T^g}{d\mu}(x) \quad \text{for every } g \in G \text{ and } \mu\text{-a.e. } x \in X, \tag{1.1}$$

$$\varrho(gh, x) = \varrho(g, T^h x) + \varrho(h, x) \quad \text{for every } g, h \in G \text{ and } x \in X, \tag{1.2}$$

$$\varrho(g, x) = 0 \quad \text{for every } g \in G \text{ and } x \in X \text{ with } T^g x = x. \tag{1.3}$$

Definition 1.2. Let T be a Borel action of a countable group G on X. A *cocycle* of T is a Borel map $\varrho : G \times X \to \mathbb{R}$ satisfying (1.2). If ϱ satisfies both (1.2) and (1.3) it is called an *orbital cocycle* (cf. [16]).

If $\varrho : G \times X \to \mathbb{R}$ is a cocycle we call a probability measure μ on $\mathcal{S}$ *ϱ-admissible* if it satisfies (1.1). If the set M_ϱ^T of ϱ-admissible probability measures on $\mathcal{S}$ is nonempty then it is obviously convex, and we write $E_\varrho^T \subset M_\varrho^T$ for the set of extremal points in M_ϱ^T. Note that E_ϱ^T is precisely the set of ergodic elements in M_ϱ^T, and that distinct elements of E_ϱ^T are mutually singular.

Remark 1.3. If $\varrho : G \times X \to \mathbb{R}$ is a cocycle with $M_\varrho^T \neq \emptyset$ then

$$\mu(\{x \in X : T^g x = x \text{ and } \varrho(g, x) \neq 0\}) = 0 \tag{1.4}$$

for every $g \in G$. In particular, if G is countable, then there exists a T-invariant Borel set $N \subset X$ such that $\mu(N) = 0$ for every $\mu \in M_\varrho^T$ and the restriction of ϱ to $G \times (X \smallsetminus N)$ is orbital, and the orbital cocycle $\varrho' : G \times X \to \mathbb{R}$, defined by

$$\varrho'(g,x) = \begin{cases} \varrho(g,x) & \text{if } g \in G \text{ and } x \in X \smallsetminus N, \\ 0 & \text{if } g \in G \text{ and } x \in N, \end{cases}$$

satisfies $M_{\varrho'}^T = M_{\varrho}^T$. In other words, we may assume for our purposes that ϱ is orbital.

We denote by $M_1(X, \mathcal{S})$ the space of probability measures on $\mathcal{S}$ and by $\mathcal{B}_{M_1(X,\mathcal{S})}$ the smallest σ-algebra of subsets of $M_1(X,\mathcal{S})$ with respect to which the maps $\mu \mapsto \mu(B)$ from $M_1(X,\mathcal{S})$ to $\mathbb{R}$ are measurable for every $B \in \mathcal{S}$. Then $(M_1(X,\mathcal{S}), \mathcal{B}_{M_1(X,\mathcal{S})})$ is standard Borel and $M_\varrho^T \in \mathcal{B}_{M_1(X,\mathcal{S})}$. Note that E_ϱ^T could *a priori* be empty even if M_ϱ^T is nonempty.

With this terminology at hand we can formulate two closely related versions of Theorem 1.1 in the case where G is countable. The main feature of these results is that they yield a simultaneous ergodic decomposition for all quasi-invariant probability measures on X with a prescribed Radon–Nikodym derivative under T, analogous to the classical statements about simultaneous decomposition of all T-*invariant* probability measures.

THEOREM 1.4. *Let T be a Borel action of a countable group G on a standard Borel space $(X,\mathcal{S})$ and $\varrho : G \times X \to \mathbb{R}$ a cocycle of T. Then*

$$E_\varrho^T \in \mathcal{B}_{M_1(X,\mathcal{S})}, \tag{1.5}$$

and there exists, for every $\mu \in M_\varrho^T$, a unique probability measure ν_μ on $\mathcal{B}_{M_1(X,\mathcal{S})}$ with

$$\mu(B) = \int_{E_\varrho^T} \xi(B)\, d\nu_\mu(\xi) \qquad \text{for every } B \in \mathcal{S}. \tag{1.6}$$

THEOREM 1.5. *Let T be a Borel action of a countable group G on a standard Borel space $(X,\mathcal{S})$ and $\varrho : G \times X \to \mathbb{R}$ a cocycle of T with $M_\varrho^T \neq \emptyset$.*

(1) *There exists a Borel map $p : x \mapsto p_x$ from X to $E_\varrho^T \subset M_1(X,\mathcal{S})$ with the following properties.*

(a) *$p_x = p_{T^g x}$ for every $x \in X$ and $g \in G$.*
(b) *For every $\nu \in M_\varrho^T$ and every nonnegative Borel map $f : X \to \mathbb{R}$,*

$$\int f\, dp_x = E_\nu(f \mid \mathcal{S}^T)(x) \tag{1.7}$$

for ν-a.e. $x \in X$, where

$$\mathcal{S}^T = \{B \in \mathcal{S} : T^g B = B \text{ for every } g \in G\},$$

and where $E_\nu(\cdot \mid \cdot)$ denotes conditional expectation with respect to ν. In par-

ticular, by setting $f = 1_B$,

$$\int_C p_x(B)\, d\nu(x) = \nu(B \cap C)$$

for every $B \in \mathcal{S}$ *and* $C \in \mathcal{S}^T$.

(2) *If* $p' : x \mapsto p'_x$ *is another Borel map from* X *to* E^T_ϱ *with the properties* (a) *and* (b) *above, then*

$$\nu(\{x \in X : p_x \neq p'_x\}) = 0 \quad \textit{for every } \nu \in M^T_\varrho. \tag{1.8}$$

(3) *Let* $\mathcal{T} \subset \mathcal{S}^T$ *be the smallest* σ*-algebra such that the map* $x \mapsto p_x$ *from* X *to* E^T_ϱ *in* (1) *is* $\mathcal{T}$*-measurable. Then* $\mathcal{T}$ *is countably generated*,

$$\mathcal{T} = \mathcal{S}^T \pmod{\nu} \quad \textit{for every } \nu \in M^T_\varrho, \tag{1.9}$$

and

$$p_x([y]_{\mathcal{T}}) = \begin{cases} 1 & \textit{if } x \in [y]_{\mathcal{T}}, \\ 0 & \textit{otherwise}, \end{cases} \tag{1.10}$$

for every $x, y \in X$, *where* $[y]_{\mathcal{T}} = \bigcap_{y \in C \in \mathcal{T}} C$ *is the* atom *of a point* $y \in X$ *in* $\mathcal{T}$.

For a countable group G, Theorem 1.1 is an easy consequence of Theorems 1.4 or 1.5.

Proof of Theorem 1.1 for countable groups using Theorem 1.4. Let μ be a probability measure on $\mathcal{S}$ which is quasi-invariant under T and choose a Borel map $\varrho : G \times X \to \mathbb{R}$ satisfying (1.1)–(1.3). Then $\mu \in M^T_\varrho \neq \emptyset$, and Theorem 1.4 yields a probability measure ν on $Y = E^T_\varrho$ with the properties stated there. ∎

Proof of Theorem 1.1 for countable groups using Theorem 1.5. We choose a Borel map $\varrho : G \times X \to \mathbb{R}$ satisfying (1.1)–(1.3) as above and obtain a Borel map $p : x \mapsto p_x$ from X to E^T_ϱ with the properties (1) in Theorem 1.5. The probability measure $\nu = \mu p^{-1}$ again has the required properties. ∎

In Section 5 we prove a precise analogue of Theorem 1.5 for Borel actions of locally compact second countable groups (Theorem 5.2), which will yield Theorem 1.1.

We conclude this introduction with a corollary about a kind of unique ergodicity for quasi-invariant measures (cf. [18]).

COROLLARY 1.6. *Let* T *be a Borel action of a locally compact second countable group* G *on a standard Borel space* $(X, \mathcal{S})$ *and* $\varrho : G \times X \to \mathbb{R}$ *a cocycle of* T. *Then there exists*, *for every ergodic probability measure* $\nu \in M^T_\varrho$, *a Borel set* $B \subset X$ *with* $\nu(B) = 1$ *and* $\nu'(B) = 0$ *for every* $\nu' \in E^T_\varrho$ *with* $\nu \neq \nu'$.

Proof. If the group G is countable, then (1.7) in Theorem 1.4 implies that $\nu = p_x$ for some $x \in X$, and the set $[x]_{\mathcal{T}}$ has the required separation property. If G is uncountable we have to use the analogous statement in Theorem 5.2. ∎

This paper is organized as follows. Section 2 prepares the ground for the proofs of Theorems 1.4–1.5. Theorem 1.4 is proved in Section 3, and Theorem 1.5 in Section 4. In Section 5 it is shown how to generalize Theorem 1.5 to locally compact second countable groups and to prove Theorem 1.1.

2. Equivalence relations and group actions. Following [4] we call a Borel set $R \subset X \times X$ a *discrete Borel equivalence relation* on $(X, \mathcal{S})$ if R is an equivalence relation and every equivalence class

$$R(x) = \{y \in X : (x, y) \in R\} \tag{2.1}$$

is countable. If $R \subset X \times X$ is a discrete Borel equivalence relation on X then $R(B) \in \mathcal{S}$ for every $B \in \mathcal{S}$, where

$$R(B) = \bigcup_{x \in B} R(x) \tag{2.2}$$

is the *saturation* of B (cf. [4]). The group

$$[R] = \{V \in \mathrm{Aut}(X, \mathcal{S}) : (Vx, x) \in R \text{ for every } x \in X\}$$

is called the *full group* of R.

If T is a Borel action of a countable group G on X then

$$R_T = \{(T^g x, x) : x \in X,\ g \in G\} \tag{2.3}$$

is a discrete Borel equivalence relation on X, and a probability measure μ on $\mathcal{S}$ is quasi-invariant (resp. ergodic) with respect to R_T if it is so with respect to T. In [4] it was shown that *every* discrete Borel equivalence relation R on X is of the form R_T for some Borel action T of a countable group G on $(X, \mathcal{S})$.

Our task in this section is to replace the action T of the countable group G appearing in Theorems 1.4–1.5 by a Borel action of a possibly different countable group for which it will be easier to construct the desired ergodic decomposition.

Definition 2.1. Let R be a discrete Borel equivalence relation on X. A Borel map $\sigma : R \to \mathbb{R}$ is a *cocycle* on R if

$$\sigma(x, y) + \sigma(y, z) = \sigma(x, z)$$

for all $x, y, z \in X$ with $(x, y), (x, z) \in R$.

The following lemma is obvious.

LEMMA 2.2. *Let $\sigma : R \to \mathbb{R}$ be a cocycle on a discrete Borel equivalence relation R on X. Then σ defines a map $\overline{\sigma} : [R] \times X \to \mathbb{R}$ with*

$$\begin{aligned} &\overline{\sigma}(V,x) = \sigma(Vx,x) \quad \textit{for all } V \in [R],\ x \in X,\\ (2.4)\quad &\overline{\sigma}(VW,x) = \overline{\sigma}(V,Wx) + \overline{\sigma}(W,x) \quad \textit{for all } V,W \in [R],\ x \in X,\\ &\overline{\sigma}(V,x) = 0 \quad \textit{for every } V \in [R] \textit{ and } x \in X \textit{ with } Vx = x. \end{aligned}$$

In other words, $\overline{\sigma}$ is a cocycle in the sense of (1.2) *for the natural action of $[R]$ on X.*

In particular, if T is a Borel action of a countable group G on X with $R_T = R$ (cf. (2.3)), *then the map $\sigma_T : G \times X \to \mathbb{R}$, defined by*

$$\sigma_T(g,x) = \overline{\sigma}(T^g,x) \tag{2.5}$$

for every $g \in G$ and $x \in X$, is an orbital cocycle of T. Conversely, if ϱ is an orbital cocycle of T, and if

$$\sigma(T^g x,x) = \varrho(g,x) \tag{2.6}$$

for every $x \in X$ and $g \in G$, then the resulting map $\sigma : R \to \mathbb{R}$ is a cocycle on R.

PROPOSITION 2.3. *Let R be a discrete Borel equivalence relation on X and $\sigma : R \to \mathbb{R}$ a cocycle. Then there exists a countable set $\Gamma \subset [R]$ with the following properties.*

(1) $\gamma^2 = \mathrm{Id}_X$ *for every $\gamma \in \Gamma$, where Id_X is the identity map on X.*
(2) $\{(\gamma x,x) : x \in X,\ \gamma \in \Gamma\} = R$.
(3) *If $\Gamma^* \subset [R]$ is the group generated by Γ, then the map*

$$x \mapsto \sigma(\gamma x,x) = \overline{\sigma}(\gamma,x)$$

from X to $\mathbb{R}$ is bounded for every $\gamma \in \Gamma^*$.

Proof. This proof is a minor extension of the corresponding argument in [4]. The space X is Borel isomorphic to a Polish space, so we can find a sequence $\{A_k \times B_k : k \geq 1\}$ of measurable rectangles with

$$X^2 \setminus \Delta = \bigcup_{k \geq 1} A_k \times B_k,$$

where $\Delta = \{(x,x) : x \in X\} \subset X^2$. We denote by $\pi_i : X \times X \to X$ the two coordinate projections and write $l = \pi_1|_R$, $r = \pi_2|_R : R \to X$ for the restrictions of π_1 and π_2 to R. Since $l^{-1}(\{x\})$ is countable for every $x \in X$, Lusin's theorem [10] yields the existence of a countable Borel partition $\{C_m : m \geq 1\}$ of R such that l is injective on each C_m. Furthermore, r is injective on $\theta(C_m)$ for every $m \geq 1$, where $\theta : R \to R$ is the flip $\theta(x,y) = (y,x)$.

We set $D_{k,m,n} = (A_k \times B_k) \cap C_m \cap \theta(C_n)$ and conclude that both l and r are injective on each $D_{k,m,n}$. By Corollary I.3.3 of [12], both l and r are

Borel isomorphisms from $D_{k,m,n}$ onto their image, and the sets $l(D_{k,m,n})$ and $r(D_{k,m,n})$ are disjoint.

For every $k, m, n \geq 1$ and $x \in X$ we set

$$V_{k,m,n}(x) = \begin{cases} r(l^{-1}(x)) & \text{if } x \in l(D_{k,m,n}), \\ l(r^{-1}(x)) & \text{if } x \in r(D_{k,m,n}), \\ x & \text{otherwise,} \end{cases}$$

and note that this transformation is a Borel automorphism of order 2, and $(V_{k,m,n}x, x) \in R$ for every $x \in X$.

Next we set, for every $j, k, m, n \geq 1$ and $x \in X$,

$$W_{j,k,m,n}x = \begin{cases} V_{k,m,n}x & \text{if } |\sigma(V_{k,m,n}x, x)| \leq j, \\ x & \text{otherwise.} \end{cases}$$

Again $W_{j,k,m,n}$ is a Borel automorphism of X of order 2.

The set

$$\Gamma = \{W_{j,k,m,n} : j, k, m, n \geq 1\} \cup \{\mathrm{Id}_X\}$$

has the properties (1)–(3) in the statement of the proposition. If $\Gamma^* \subset \mathrm{Aut}(X, \mathcal{S})$ is the group generated by Γ then equalities (2.4) and (1.2) imply the boundedness of $\overline{\sigma}(\gamma, \cdot)$ for every $\gamma \in \Gamma^*$. ∎

A probability measure μ on $\mathcal{S}$ is *quasi-invariant* under a discrete Borel equivalence relation R if $\mu(R(B)) = 0$ for every $B \in \mathcal{S}$ with $\mu(B) = 0$, and *ergodic* if $\mu(R(B)) \in \{0, 1\}$ for every $B \in \mathcal{S}$.

LEMMA 2.4. *Let R be a discrete Borel equivalence relation on a standard Borel space $(X, \mathcal{S})$, μ a probability measure on $\mathcal{S}$ and T be a Borel action of a countable group G on X with $R = R_T$* (cf. (2.3)). *If μ is quasi-invariant under R then it is also quasi-invariant under every $V \in [R]$, and there exists a cocycle $\varrho_\mu : R \to \mathbb{R}$ with*

$$\varrho_\mu(Vx, x) = \log \frac{d\mu V}{d\mu}(x) \quad \text{for } \mu\text{-a.e. } x \in X, \text{ for every } V \in [R]. \tag{2.7}$$

In particular, μ is quasi-invariant under T. Conversely, if μ is quasi-invariant under T then it is quasi-invariant under R.

Proof. Since $R = R_T$ for some Borel action T of a countable group G on X there exists a countable subgroup $\Delta \subset [R]$ with $R = \{(Vx, x) : x \in X, V \in \Delta\}$. Since μ is quasi-invariant under every $V \in [R]$, we can choose, for every $V \in \Delta$, a Borel map $\varrho_V : X \to \mathbb{R}$ such that the following conditions hold.

(a) $\varrho_V = \log \frac{d\mu V}{d\mu} \pmod{\mu}$,
(b) $\varrho_V(x) = 0$ for every $x \in X$ with $Vx = x$,
(c) $\varrho_{VW}(x) = \varrho_V(Wx) + \varrho_W(x)$ for all $V, W \in \Delta$ and $x \in X$.

By setting $\varrho_\mu(Vx,x) = \varrho_V(x)$ for every $V \in \Delta$ and $x \in X$ we have defined consistently a cocycle $\varrho_\mu : R \to \mathbb{R}$ with the required properties. The remaining assertions are obvious. ∎

Lemma 2.4 suggests the following definition.

Definition 2.5. Let R be a discrete Borel equivalence relation on a standard probability space $(X, \mathcal{S})$, and let $\varrho : R \to \mathbb{R}$ be a cocycle. A probability measure μ on $\mathcal{S}$ is *ϱ-admissible* if it satisfies (2.7) with ϱ replacing ϱ_μ. The convex set of ϱ-admissible probability measures on $\mathcal{S}$ is denoted by $M_\varrho^R \in \mathcal{B}_{M_1(X,\mathcal{S})}$, and we write E_ϱ^R for the set of extreme points of M_ϱ^R.

By combining Lemmas 2.2 and 2.4 we obtain a final observation in this section.

Lemma 2.6. *Let R be a discrete Borel equivalence relation on a standard probability space $(X, \mathcal{S})$, $\sigma : R \to \mathbb{R}$ a cocycle, and T a Borel action of a countable group G with $R = R_T$. Then $M_\sigma^R = M_{\sigma_T}^T$ and $E_\sigma^R = E_{\sigma_T}^T$* (cf. (2.5)).

The discussion in this section, combined with Remark 1.3, shows that the following two statements are precisely equivalent to Theorems 1.4 and 1.5.

Theorem 2.7. *Let R be a discrete Borel equivalence relation on a standard probability space $(X, \mathcal{S})$ and $\varrho : R \to \mathbb{R}$ a cocycle. Then*

$$E_\varrho^R \in \mathcal{B}_{M_1(X,\mathcal{S})},$$

and there exists, for every $\mu \in M_\varrho^R$, a probability measure ν on $\mathcal{B}_{M_1(X,\mathcal{S})}$ with

$$\mu(B) = \int_{E_\varrho^R} \xi(B)\, d\nu(\xi) \qquad \text{for every } B \in \mathcal{S}.$$

Theorem 2.8. *Let R be a discrete Borel equivalence relation on a standard probability space $(X, \mathcal{S})$ and $\varrho : R \to \mathbb{R}$ a cocycle with $M_\varrho^R \neq \emptyset$.*

(1) *There exists a Borel map $p : x \mapsto p_x$ from X to $E_\varrho^R \subset M_1(X, \mathcal{S})$ with the following properties.*

(a) *$p_x = p_{x'}$ for all $(x, x') \in R$.*
(b) *For every $\nu \in M_\varrho^R$ and every nonnegative Borel map $f : X \to \mathbb{R}$,*

$$\int f\, dp_x = E_\nu(f \mid \mathcal{S}^R)(x)$$

for ν-a.e. $x \in X$; in particular, by setting $f = 1_B$,

$$\int_C p_x(B)\, d\nu(x) = \nu(B \cap C)$$

for every $B \in \mathcal{S}$ and $C \in \mathcal{S}^R$.

(2) *If* $p' : x \mapsto p'_x$ *is another Borel map from* X *to* E^R_ϱ *with the properties* (1) *above*, *then*

$$\nu(\{x \in X : p_x \neq p'_x\}) = 0 \quad \text{for every } \nu \in M^R_\varrho.$$

(3) *Let* $\mathcal{T} \subset \mathcal{S}^R$ *be the smallest* σ*-algebra such that the map* $x \mapsto p_x$ *from* X *to* E^R_ϱ *in* (1) *is* $\mathcal{T}$*-measurable. Then* $\mathcal{T}$ *is countably generated*, $\mathcal{T} = \mathcal{S}^R$ (mod ν) *for every* $\nu \in M^R_\varrho$, *and* $p_x([x]_\mathcal{T}) = 1$ *for every* $x \in X$.

3. Proof of the equivalent Theorems 1.4 and 2.7. The proof of Theorem 1.4 uses Choquet's theorem.

PROPOSITION 3.1 (Choquet's theorem). *Let* Y *be a metrizable compact convex subset of a locally convex space* W *which is a Choquet simplex. Then the set* $E \subset Y$ *of all extreme points of* Y *is a* G_δ *set. Furthermore there exists*, *for every* $y_0 \in Y$, *a unique probability measure* ν *on* $\mathcal{B}_Y$ *with* $\nu(E) = 1$ *and*

$$L(y_0) = \int_E L(y)\, d\nu(y) \quad \text{for every } L \in W^*, \tag{3.1}$$

where W^* *is the dual space of* W.

Proof. [13], Proposition 1.3, Theorem on p. 19, and Section 9. ∎

The following lemma will provide the topological setting necessary for applying Choquet's theorem.

LEMMA 3.2. *Let* R *be a discrete Borel equivalence relation on* X, $\sigma : R \to \mathbb{R}$ *a cocycle*, *and let* $\Gamma^* \subset [R]$ *be defined as in Proposition* 2.3. *Then there exist a compact metric space* Z, *an injective Borel map* $\phi : X \to Z$, *an action* V *of* Γ^* *by homeomorphisms of* Z, *and a cocycle* $\varrho' : \Gamma^* \times Z \to \mathbb{R}$ *with the following properties.*

(1) $\phi(\gamma x) = V^\gamma \phi(x)$ *for every* $\gamma \in \Gamma^*$ *and* $x \in X$.
(2) $\varrho'(\gamma, \phi(x)) = \sigma(\gamma x, x)$ *for every* $\gamma \in \Gamma^*$ *and* $x \in X$.
(3) *The map* $\varrho'(\gamma, \cdot)$ *from* Z *to* $\mathbb{R}$ *is continuous for every* $\gamma \in \Gamma^*$.

Proof. This is an elementary application of Gelfand theory. We choose a countable algebra $\mathcal{A} \subset \mathcal{S}$ which separates the points of X. Let $\mathcal{F}$ be the smallest algebra of bounded Borel functions $f : X \to \mathbb{C}$ with the following properties:

(a) for every $A \in \mathcal{A}$, the indicator function $1_A \in \mathcal{F}$,
(b) for every $\gamma \in \Gamma^*$, the map $x \mapsto \sigma(\gamma x, x)$ lies in $\mathcal{F}$,
(c) for every $f \in \mathcal{F}$, $c \in \mathbb{C}$ and $\gamma \in \Gamma^*$, $cf \in \mathcal{F}$ and $f \circ \gamma \in \mathcal{F}$,
(d) $\mathcal{F}$ is closed in the topology of uniform convergence.

It is clear that $\mathcal{F}$ is a separable complex Banach algebra under the maximum norm. We denote by Z the maximal ideal space of $\mathcal{F}$, i.e. the space of all algebra homomorphisms from $\mathcal{F}$ to $\mathbb{C}$. The space Z is compact and metrizable in the weak* topology, and the map $\phi : X \to Z$, defined by $\phi(x)(f) = f(x)$ for every $x \in X$ and $f \in \mathcal{F}$, is Borel and injective. For every $z \in Z$, $f \in \mathcal{F}$ and $\gamma \in \Gamma^*$ we set

$$(V^\gamma z)(f) = z(f \circ \gamma).$$

Then the properties (1)–(3) are obvious from these choices. ∎

We define $M_{\varrho'}^V \subset M_1(Z, \mathcal{B}_Z)$ as in Definition 1.2, where $\mathcal{B}_Z$ denotes the Borel field of Z.

LEMMA 3.3. *The convex set* $M_{\varrho'}^V \subset M_1(Z, \mathcal{B}_Z)$ *is a Choquet simplex* (cf. Proposition 3.1 and Lemma 3.2). *Furthermore*, *if* $\mu \in M_\sigma^R \subset M_1(X, \mathcal{S})$, *then* $\mu\phi^{-1} \in M_{\varrho'}^V \subset M_1(Z, \mathcal{B}_Z)$.

Proof. We denote by $C(Z, \mathbb{C})$ the space of continuous complex-valued functions on Z. The continuity of each $\varrho'(\gamma, \cdot) : Z \to \mathbb{R}$ implies that the set

$$M_{\varrho'}^V = \Big\{ \xi \in M_1(Z, \mathcal{B}_Z) : \int f \circ V^{\gamma^{-1}} \, d\xi = \int f \, d\xi \, V^\gamma = \int f(z) e^{\varrho'(\gamma, z)} \, d\xi(z) \\ \text{for every } \gamma \in \Gamma^* \text{ and } f \in C(Z, \mathbb{C}) \Big\}$$

is a closed convex subset of the weak*-compact metric space $M_1(Z, \mathcal{B}_Z)$.

The proof that $M_{\varrho'}^V$ is a Choquet simplex is essentially identical to that of Proposition 10.3 in [13], and the second assertion is an immediate consequence of Lemmas 2.4 and 3.2. ∎

Proof of Theorem 1.4. We assume the notation and hypotheses of Theorem 1.4, put $R = R_T$ (cf. (2.3)) and define a cocycle $\sigma : R \to \mathbb{R}$ by (2.6). By applying Lemma 3.2 we obtain a compact metric space Z, an injective Borel map $\phi : X \to Z$, a group $\Gamma^* \subset [R]$, an action V of Γ^* by homeomorphisms of Z, and a cocycle $\varrho' : \Gamma^* \times Z \to \mathbb{R}$ for V with the properties described there.

Lemma 3.3 and Proposition 3.1 show that $M_{\varrho'}^V$ is a Choquet simplex, that $E_{\varrho'}^V \subset M_{\varrho'}^V$ is a Borel set and that there exists, for every $\mu' \in M_{\varrho'}^V$, a unique probability measure $\nu'_{\mu'}$ on $\mathcal{B}_{M_1(Z, \mathcal{B}_Z)}$ with $\nu'_{\mu'}(E_{\varrho'}^V) = 1$ and

$$\int f \, d\mu' = \int_{E_{\varrho'}^V} \Big(\int f \, d\xi \Big) \, d\nu'_{\mu'}(\xi)$$

for every $f \in C(Z, \mathbb{C})$, and hence with

$$\mu'(B) = \int_{E_{\varrho'}^V} \xi(B) \, d\nu'_{\mu'}(\xi)$$

for every $B \in \mathcal{B}_Z$.

The set $Y' = \{\nu \in M_1(Z, \mathcal{B}_Z) : \nu(\phi(X)) = 1\}$ is Borel since $\phi(X) \in \mathcal{B}_Z$ (Corollary I.3.3 in [12]). If

$$\mu' \in M' = M_{\varrho'}^V \cap Y',$$

then

$$\int \xi(\phi(X))\, d\nu'_{\mu'}(\xi) = \mu'(\phi(X)) = 1,$$

so that $\nu'_{\mu'}(E') = 1$ for every $\mu' \in M'$, where

$$E' = E_{\varrho'}^V \cap Y'.$$

According to Lemma 2.6, $M_\varrho^T = M_\sigma^R$ and $E_\varrho^T = E_\sigma^R$. We define the equivalence relation R_V on Z and the cocycle $\sigma' : R_V \to \mathbb{R}$ by (2.3) and (2.6) with V and ϱ' replacing T and ϱ. Then the map $\nu \mapsto \nu' = \nu\phi^{-1}$ defines a Borel bijection between $M_\varrho^T = M_\sigma^R$ and $M_{\sigma'}^{R_V} \cap Y' = M'$ which carries E_ϱ^T to E'. It follows that $E_\varrho^T \subset M_\varrho^T$ is a Borel set, proving (1.5).

For every $\mu \in M_\varrho^T$ we put $\nu_\mu = \nu'_{\mu'}\phi$ (this definition makes sense since $\nu'_{\mu'}(\phi(E_\varrho^T)) = 1$) and obtain

$$\mu(B) = \int_{E_\varrho^T} \xi(B)\, d\nu_\mu(\xi) = \int_{E'} \xi'(\phi(B))\, d\nu'_{\mu'}(\xi') = \mu'(\phi(B))$$

for every $B \in \mathcal{S}$, which proves (1.6). The uniqueness of ν_μ follows from that of $\nu'_{\mu'}$. ∎

4. Proof of the equivalent Theorems 1.5 and 2.8. The proof of Theorem 1.5 follows the approach in [17]–[18] and uses the sufficiency of the σ-algebra $\mathcal{S}^T$ for the family of measures M_ϱ^T (cf. Definition 1.2 and e.g. [1]).

DEFINITION 4.1. Let $(X, \mathcal{S})$ be a standard Borel space and $M \subset M_1(X, \mathcal{S})$ a set of probability measures. A σ-algebra $\mathcal{T} \subset \mathcal{S}$ is *sufficient* for M if there exists, for every bounded Borel map $f : X \to \mathbb{R}$, a $\mathcal{T}$-measurable Borel map $\psi_f : X \to \mathbb{R}$ with

$$\psi_f = E_\nu(f \mid \mathcal{T}) \pmod{\nu}$$

for every $\nu \in M$.

PROPOSITION 4.2. *Let T be a Borel action of a countable group G on a standard Borel space $(X, \mathcal{S})$ and $\varrho : G \times X \to \mathbb{R}$ a cocycle of T with $M_\varrho^T \neq \emptyset$. Then the σ-algebra $\mathcal{S}^T$ is sufficient for the family $M_\varrho^T \subset M_1(X, \mathcal{S})$.*

For the proof of Proposition 4.2 we denote by $R = R_T$ and $\sigma : R \to \mathbb{R}$ the equivalence relation (2.3) and the cocycle (2.6), choose a countable subset $\Gamma \subset [R]$ with the properties described in Proposition 2.3 and write $\Gamma^* \subset [R]$ for the group generated by Γ. For every σ-algebra $\mathcal{T} \subset \mathcal{S}$ we denote by $L^\infty(X, \mathcal{T})$ the set of all bounded $\mathcal{T}$-measurable maps $f : X \to \mathbb{R}$ and set $\|f\|_\infty = \sup_{x \in X} |f(x)|$ for every $f \in L^\infty(X, \mathcal{T})$.

LEMMA 4.3. *For each* $\gamma \in \Gamma$ *we set* $\mathcal{S}^\gamma = \{B \in \mathcal{S} : \gamma B = B\}$. *Then the mapping* $P_\gamma : L^\infty(X, \mathcal{S}) \to L^\infty(X, \mathcal{S}^\gamma)$ *given by*

$$P_\gamma(f) = \frac{f + (f \circ \gamma) \cdot e^{\overline{\sigma}(\gamma,\cdot)}}{1 + e^{\overline{\sigma}(\gamma,\cdot)}} \tag{4.1}$$

satisfies

$$P_\gamma(f) = E_\nu(f \,|\, \mathcal{S}^\gamma) \pmod{\nu} \tag{4.2}$$

for every $\nu \in M_\varrho^T$.

Proof. Since $\gamma^2 = \mathrm{Id}_X$,

$$\begin{aligned} P_\gamma(f)(\gamma x) &= \frac{f(\gamma x) + f(x) \cdot e^{\overline{\sigma}(\gamma,\gamma x)}}{1 + e^{\overline{\sigma}(\gamma,\gamma x)}} \\ &= \frac{f(\gamma x) \cdot e^{\overline{\sigma}(\gamma,x)} + f(x) \cdot e^{\overline{\sigma}(\gamma,\gamma x) + \overline{\sigma}(\gamma,x)}}{e^{\overline{\sigma}(\gamma,x)} + e^{\overline{\sigma}(\gamma,\gamma x) + \overline{\sigma}(\gamma,x)}} = P_\gamma(f)(x) \end{aligned}$$

for every $x \in X$, by (2.4). The boundedness of $\overline{\sigma}(\gamma,\cdot)$ implies that $P_\gamma(f) \in L^\infty(X, \mathcal{S}^\gamma)$.

An elementary calculation shows that $\int_B P_\gamma(f)\, d\nu = \int_B f\, d\nu$ for every $\nu \in M_\varrho^T$, $B \in \mathcal{S}^\gamma$, and proves (4.2). ∎

We shall use the operators $P_\gamma, \gamma \in \Gamma$, to construct a common conditional expectation with respect to the σ-algebra $\mathcal{S}^R = \mathcal{S}^\Gamma = \bigcap_{\gamma \in \Gamma} \mathcal{S}^\gamma$. For this purpose we need the following classical ergodic theorem from [6], [2].

THEOREM 4.4. *Suppose that* $(X, \mathcal{S})$ *is a standard Borel space*, ν *a probability measure on* $\mathcal{S}$ *and* Q *a positive linear contraction on* $L^1(X, \mathcal{S}, \nu)$. *For every* $f \in L^1(X, \mathcal{S}, \nu)$, *the sequence*

$$S_n(f, Q) = \frac{1}{n} \sum_{k=0}^{n-1} Q^k f, \qquad n \geq 1, \tag{4.3}$$

converges ν*-a.e.*

We fix an enumeration $(\gamma_i,\ i \in \mathbb{N})$ of Γ and define inductively for every $f \in L^\infty(X, \mathcal{S})$ a sequence of maps

$$\begin{aligned} P_1(f)(x) &= \limsup_{n\to\infty} S_n(f, P_{\gamma_1}), \\ P_n(f)(x) &= \limsup_{n\to\infty} S_n(f, P_{\gamma_n} \circ P_{n-1}). \end{aligned} \tag{4.4}$$

LEMMA 4.5. *For every* $f \in L^\infty(X, \mathcal{S})$, $n \geq 1$ *and* $\nu \in M_\varrho^T$, $\|P_n(f)\|_\infty \leq \|f\|_\infty$ *and*

$$P_n(f) = E_\nu(f \,|\, \mathcal{S}_n) \pmod{\nu}, \tag{4.5}$$

where

$$\mathcal{S}_n = \bigcap_{k=1}^{n} \mathcal{S}^{\gamma_k}. \tag{4.6}$$

Proof. The proof is by induction. For $n = 1$ the assertion is clear from Lemma 4.3. The definition of P_γ implies that $P_\gamma(1) = 1$ for all $\gamma \in \Gamma$, and by induction we have $P_n(1) = 1$ for all $n \geq 1$. Since P_n is positive it follows that $\|P_n(f)\|_\infty \leq \|f\|_\infty$ for every $n \geq 1$.

Suppose that P_{n-1} has the required properties for some $n \geq 2$. In order to prove (4.5) for P_n we set $Q = P_{n-1} \circ P_{\gamma_n}$ and note that

$$\begin{aligned} &S_k(1) = 1, \\ &\|S_k(f,Q)\|_\infty \leq \|f\|_\infty, \\ &\|Q \circ S_k(f,Q) - S_k(f,Q)\|_\infty \leq \tfrac{2}{k}\|f\|_\infty \end{aligned} \tag{4.7}$$

for every $k \geq 1$.

We fix $\nu \in M_\varrho^T$. By applying the dominated convergence theorem and Theorem 4.4 we obtain

$$\lim_{k\to\infty} \|P_n(f) - S_k(f,Q)\|_1 = 0,$$

where $\|\cdot\|_1$ denotes the norm in $L^1(X,\mathcal{S},\nu)$, and (4.7) shows that

$$P_{n-1} \circ P_{\gamma_n} \circ P_n(f) = P_n(f) \pmod{\nu}. \tag{4.8}$$

According to Lemma 4.3 and our induction hypothesis, P_{n-1} and P_{γ_n} are conditional expectations and thus projections on $L^2(X,\mathcal{S},\nu)$. From (4.8) we conclude that $P_n(f)$ is invariant both under P_{n-1} and P_{γ_n} and therefore both $\mathcal{S}_{n-1}$- and $\mathcal{S}^{\gamma_n}$-measurable (mod ν).

Finally we use induction on k to see that

$$\int_B S_k(f,Q)\,d\nu = \int_B f\,d\nu$$

for all $k \in \mathbb{N}$ and $B \in \mathcal{S}_n$. The dominated convergence theorem implies that

$$\int_B P_n(f)\,d\nu = \lim_{k\to\infty} \int_B S_k(f,Q)\,d\nu = \int_B f\,d\nu,$$

and hence that $P_n(f) = E_\nu(f \mid \mathcal{S}_n) \pmod{\nu}$.

Since $\nu \in M_\varrho^T$ was arbitrary this proves the lemma. ■

Proof of Proposition 4.2. The sequence $(\mathcal{S}_n,\ n \geq 1)$ of σ-algebras in Lemma 4.5 decreases to $\mathcal{S}^T = \mathcal{S}^R = \mathcal{S}^\Gamma$, and the decreasing martingale theorem (cf. e.g. Theorem 2.3 in [11]) implies that

$$P_\infty(f) = \limsup_{n\to\infty} P_n(f) = E_\nu(f \mid \mathcal{S}^T) \pmod{\nu} \tag{4.9}$$

for every $f \in L^\infty(X,\mathcal{S})$ and $\nu \in M_\varrho^T$. ■

In order to apply Proposition 4.2 in the proof of Theorem 1.5 we require an elementary lemma.

LEMMA 4.6. *If $(X,\mathcal{S})$ is a standard Borel space then there exists a countable algebra $\mathcal{A} \subset \mathcal{S}$ with the following properties.*

(1) $\sigma(\mathcal{A}) = \mathcal{S}$, *where* $\sigma(\mathcal{A})$ *is the* σ*-algebra generated by* $\mathcal{A}$.

(2) *Every finitely additive set function* $\nu : \mathcal{A} \to [\prime, \infty]$ *is* σ*-additive and thus defines a unique probability measure* $\overline{\nu}$ *on* $\mathcal{S}$.

Proof. Since the zero-dimensional compact metric space $\mathbf{C} = \{0,1\}^{\mathbb{N}}$ is Borel isomorphic to X we may assume without loss of generality that $X = \mathbf{C}$ with its usual Borel field. Denote by $\mathcal{A} \subset \mathcal{S}$ the countable algebra of closed and open subsets of $\mathbf{C}$. If $A \in \mathcal{A}$ is the union of a sequence $(A_i,\, i \ge 1)$ of sets in $\mathcal{A}$, the compactness of A and the openness of each A_i imply that only finitely many A_i can be nonempty. This shows that $\mathcal{A}$ satisfies our requirements. ∎

Proof of Theorem 1.5. Let

$$\mathcal{N} = \{B \in \mathcal{S}^T : \nu(B) = 0 \text{ for every } \nu \in M_\varrho^T\}.$$

We choose a countable algebra $\mathcal{A} \subset \mathcal{S}$ according to Lemma 4.6 and define, for every $A \in \mathcal{A}$, $f_A = P_\infty(1_A)$ by (4.4) and (4.9). As $\mathcal{A}$ and G are countable and $f_A = E_\nu(1_A \,|\, \mathcal{S}^T)$ for every $\nu \in M_\varrho^T$, there exists a set $N \in \mathcal{N}$ with

$$f_A(T^g x) = f_A(x), \qquad f_{\bigcup_{k=1}^n A_k}(x) = \sum_{k=1}^n f_{A_k}(x),$$

for every $x \in X \setminus N$, $g \in G$ and every choice $A_1, \ldots, A_n$ of disjoint sets in $\mathcal{A}$. For every $x \in X \setminus N$, the finitely additive positive set function $A \mapsto f_A(x)$ on $\mathcal{A}$ extends to a probability measure q_x on $\mathcal{S} = \sigma(\mathcal{A})$ (cf. Lemma 4.6). Finally we fix an arbitrary point $x_0 \in X \setminus N$ and set $q_x = q_{x_0}$ for $x \in N$. Then $q_{T^g x} = q_x$ for every $g \in G$ and $x \in X$. Then $q : x \mapsto q_x$ is a Borel map from X to $M_1(X, \mathcal{S})$ with $q_x = q_{T^g x}$ for every $g \in G$ and $x \in X$.

Let $\nu \in M_\varrho^T$. By definition,

$$\{x \in X : P_\infty(1_A)(x) \ne q_x(A)\} \in \mathcal{N} \tag{4.10}$$

for every $A \in \mathcal{A}$. If $B \in \mathcal{S}$ is arbitrary, then we can find a sequence $(A_n,\, n \ge 1)$ in $\mathcal{A}$ with $\nu(B \,\triangle\, A_n) \to 0$ and hence with

$$\|E_\nu(1_B \,|\, \mathcal{S}^T) - E_\nu(1_{A_n} \,|\, \mathcal{S}^T)\|_1 \to 0.$$

Since $E_\nu(1_{A_n} \,|\, \mathcal{S}^T) = P_\infty(1_{A_n}) \pmod{\nu}$ and $P_\infty(1_{A_n})(x) = q_x(A_n)$ for every $n \ge 0$ and $x \in X \setminus N$, (4.10) holds for every $B \in \mathcal{S}$. By combining this with (4.9) we deduce (1.7) with $q : X \to M_\varrho^T$ replacing p.

We continue by showing that

$$N' = \{x \in X : q_x \notin E_\varrho^T\} \in \mathcal{N}.$$

For every $A \in \mathcal{A}$, $g \in G$, $C \in \mathcal{S}^T$ and $\nu \in M_\varrho^T$, (1.7) shows that

$$\begin{aligned}\int_C\int e^{\varrho(g,y)}1_A(y)\,dq_x(y)\,d\nu(x) &= \int_C e^{\varrho(g,x)}1_A(x)\,d\nu(x)\\ &= \int_{A\cap C} e^{\varrho(g,x)}\,d\nu(x) = \nu(T^g(A\cap C))\\ &= \int E_\nu(1_{T^gA}\cdot 1_C \mid \mathcal{S}^T)\,d\nu\\ &= \int_C E_\nu(1_{T^gA}\mid \mathcal{S}^T)\,d\nu = \int_C q_x(T^gA)\,d\nu(x).\end{aligned}$$

As the map $x \mapsto q_x$ is $\mathcal{S}^T$-measurable this shows that

$$\Big\{x\in X : \int e^{\varrho(g,y)}1_A(y)\,dq_x(y) \neq q_x(T^gA)\Big\} \in \mathcal{N}$$

for every $A \in \mathcal{A}$ and $g \in G$. The countability of $\mathcal{A}$ and G allows us to find a set $N' \in \mathcal{N}$ with

$$\int e^{\varrho(g,y)}1_A(y)\,dq_x(y) = q_x(T^gA) \tag{4.11}$$

for every $g \in G$, $A \in \mathcal{A}$ and $x \in X \smallsetminus N$, which is easily seen to imply (4.11) for every $g \in G$, $A \in \mathcal{S}$ and $x \in X \smallsetminus N$. This proves that $q_x \in M_\varrho^T$ for every $x \in X \smallsetminus N'$, as claimed.

We write $\mathcal{T}' \subset \mathcal{S}^T$ for the smallest σ-algebra with respect to which the map $q : X \to M_\varrho^T$ is measurable. Formula (1.7) implies that every $\mathcal{S}^T$-measurable map $f : X \to \mathbb{R}$ is $\mathcal{T}$-measurable $(\operatorname{mod}\nu)$ for every $\nu \in M_\varrho^T$, which proves (1.9) with $\mathcal{T}'$ replacing $\mathcal{T}$. As M_ϱ^T is standard Borel, $\mathcal{T}'$ is countably generated, and we choose a countable algebra $\mathcal{C} \subset \mathcal{T}'$ with $\sigma(\mathcal{C}) = \mathcal{T}'$.

For every $C \in \mathcal{C}$, $P_\infty(1_C) = 1_C$, and the validity of (4.10) for every $A \in \mathcal{S}$ allows us to increase the set $N' \in \mathcal{N}$ if necessary, and to assume that $q_x(C) = 1_C(x)$ for every $C \in \mathcal{C}$ and $x \in X \smallsetminus N'$.

If q_x were nonergodic for some $x \in X \smallsetminus N'$ we could find a set $B \in \mathcal{S}^T$ with $0 < q_x(B) < 1$, and (1.9) allows us to assume that $B \in \mathcal{T}'$. Since $\sigma(\mathcal{C}) = \mathcal{T}'$, there exists a sequence $(C_n,\ n \geq 1)$ in $\mathcal{C}$ with $\lim_{n\to\infty} q_x(C_n \triangle B) = 0$. However, as we have just checked, $q_x(C_n) \in \{0,1\}$ for every $n \geq 1$, which leads to a contradiction. This shows that $q_x \in E_\varrho^T$ for every $x \in X \smallsetminus N'$.

Finally we pick a point $x_1 \in X \smallsetminus N'$, set

$$p_x = \begin{cases} q_x & \text{if } x \in X \smallsetminus N',\\ q_{x_1} & \text{otherwise,}\end{cases}$$

and denote by $\mathcal{T} \subset \mathcal{T}'$ the smallest σ-algebra with respect to which $p : X \to E_\varrho^T$ is measurable. Since $\{x \in X : p_x \neq q_x\} \in \mathcal{N}$, the map p again has the properties (1) of Theorem 1.5, and $\mathcal{T}$ satisfies (1.9).

In order to prove (1.10) we fix $x \in X \smallsetminus N'$ and recall that $p_x(C) = q_x(C) = 1_C(x)$ for every $C \in \mathcal{C}$. Then

$$p_x([x]_{\mathcal{T}}) = p_x([x]_{\mathcal{T}'}) = p_x\Big(\bigcap_{x\in C\in\mathcal{C}} C\Big) = 1,$$

since $[x]_{\mathcal{T}'} \smallsetminus N' \subset [x]_{\mathcal{T}} \subset [x]_{\mathcal{T}'} \cup N'$ and $N' \in \mathcal{N}$.

For $x \in N'$, $[x]_{\mathcal{T}} = N' \cup [x_1]_{\mathcal{T}'}$ and

$$p_x([x]_{\mathcal{T}}) = q_{x_1}([x]_{\mathcal{T}}) = q_{x_1}([x_1]_{\mathcal{T}'}) = 1.$$

Finally, if $x, y \in X$ and $x \notin [y]_{\mathcal{T}}$, then there exists a set $B \in \mathcal{T}$ with $y \in B$ and $x \notin B$, and hence $p_x \neq p_y$. It follows that p_x and p_y are mutually singular, $p_x([x]_{\mathcal{T}}) = p_y([y]_{\mathcal{T}}) = 1$, $[x]_{\mathcal{T}} \cap [y]_{\mathcal{T}} = \emptyset$ and $p_x([y]_{\mathcal{T}}) = p_y([x]_{\mathcal{T}}) = 0$. This completes the proof of (1.10).

The uniqueness assertion (2) of Theorem 1.5 is clear from (1.7), applied to $f = 1_A$ for every $A \in \mathcal{A}$. ∎

5. Theorem 1.5 for locally compact groups. One way of extending Theorem 1.5 to Borel actions of locally compact second countable groups is to use lacunary sections (cf. e.g. [3] and [7]). Here we sketch a somewhat more elementary approach: if T is a Borel action of a locally compact second countable group G on X we restrict this action to a countable dense subgroup $\Delta \subset G$, apply Theorem 1.5 to the action of Δ, and complete the proof by showing that the resulting decomposition also works for the original G-action T.

If T is a Borel action of a locally compact second countable group G on X then [23] shows that there exists a G-equivariant embedding of X as a G-invariant Borel set in a compact G-space Y. We assume for simplicity that X itself is compact and metrizable, and that the map $(g, x) \mapsto T^g x$ from $G \times X$ to X is continuous. If μ is a probability measure on $\mathcal{S} = \mathcal{B}_X$ which is quasi-invariant under T then there exists a Borel map $\varrho : G \times X \to \mathbb{R}$ with

$$\begin{gathered}
e^{\varrho(g,x)} = \frac{d\mu T^g}{d\mu}(x) \qquad \text{for } \mu\text{-a.e. } x \in X,\\
\varrho(gh, x) = \varrho(g, T^h x) + \varrho(h, x) \qquad \text{for } \mu\text{-a.e. } x \in X,\\
\mu(\{x \in X : T^g x = x \text{ and } \varrho(g, x) \neq 0\}) = 0,
\end{gathered}$$

for every $g, h \in G$, and Theorem B.9 of [25] allows us to assume that ϱ satisfies (1.1)–(1.2) and (1.4). Again we call a Borel map $\varrho : G \times X \to \mathbb{R}$ satisfying (1.2) a *cocycle* of T and define the sets $E_\varrho^T \subset M_\varrho^T$ as in Definition 1.2.

We fix a cocycle $\varrho : G \times X \to \mathbb{R}$ with $M_\varrho^T \neq \emptyset$. Write $\mathcal{B}_G$ for the Borel field, λ for the right Haar measure, and 1_G for the identity element of G, and choose a strictly positive bounded continuous function $\eta \in L^1(G, \mathcal{B}_G, \lambda)$

with the following properties [1]:

$$\int \eta \, d\lambda = 1 \quad \text{and} \quad \lim_{h \to 1_G} \sup_{g \in G} \frac{|\eta(g) - \eta(gh)|}{\eta(g)} = 0. \tag{5.1}$$

For every $g \in G$ and $x \in X$ we set

$$\begin{aligned} a(x) &= \int \eta(h) \cdot e^{\varrho(h,x)} \, d\lambda(h), \\ b(x) &= \begin{cases} \log a(x) & \text{if } a(x) < \infty, \\ 0 & \text{otherwise,} \end{cases} \\ \widetilde{\varrho}(g,x) &= b(T^g x) + \varrho(g,x) - b(x). \end{aligned} \tag{5.2}$$

Then $\widetilde{\varrho} : G \times X \to \mathbb{R}$ is again a cocycle. For every $\nu \in M_\varrho^T$ we denote by $\widetilde{\nu}$ the probability measure defined by

$$\widetilde{\nu}(B) = \int_G \eta(g) \cdot \nu(T^g B) \, d\lambda(g) \qquad \text{for every } B \in \mathcal{S}, \tag{5.3}$$

or, equivalently, by

$$d\widetilde{\nu}(x) = e^{b(x)} d\nu(x). \tag{5.4}$$

From (5.3)–(5.4) we conclude that

$$e^{\widetilde{\varrho}(g,x)} = \frac{e^{\varrho(g,x)} a(T^g x)}{a(x)} = \frac{d\widetilde{\nu} T^g}{d\widetilde{\nu}}(x) = \frac{\int \eta(hg^{-1}) \cdot e^{\varrho(h,x)} \, d\lambda(h)}{\int \eta(h) \cdot e^{\varrho(h,x)} \, d\lambda(h)} \tag{5.5}$$

for every $\nu \in M_\varrho^T$, every $g \in G$, and ν-a.e. $x \in X$.

The properties of η in (5.1) imply that there exists, for every $\varepsilon > 0$, a neighbourhood $\mathcal{U}(\varepsilon)$ of the identity 1_G in G with

$$\sup_{g \in \mathcal{U}(\varepsilon)} \left\| \frac{d\widetilde{\nu} T^g}{d\widetilde{\nu}} - 1 \right\|_\infty < \varepsilon \tag{5.6}$$

for every $\nu \in M_\varrho^T$, where $\|\cdot\|_\infty$ denotes the L^∞-norm w.r.t. ν.

PROPOSITION 5.1. *Let T be a Borel action of a locally compact second countable group G on a standard Borel space $(X, \mathcal{S})$, and let $\varrho : G \times X \to \mathbb{R}$ be a cocycle of T with $M_\varrho^T \neq \emptyset$. Let furthermore $\Delta \subset G$ be a countable dense subgroup, and let T' and ϱ' be the restrictions of T and ϱ to Δ and $\Delta \times X$, respectively. Then*

[1] In order to find such a function we let $\mathcal{U} \subset \mathcal{G}$ be a compact symmetric neighbourhood of 1_G and cover the compact set $\mathcal{U}^{\in} = \{gg' : g, g' \in \mathcal{U}\}$ with k right translates $\mathcal{U}x_1, \ldots, \mathcal{U}x_k$ of $\mathcal{U}$. By induction, $\lambda(\mathcal{U}^n) \le k\lambda(\mathcal{U}^{n-1})$ for all $n \ge 2$. Then $H = \bigcup_{n \ge 1} \mathcal{U}^n$ is an open subgroup of G, and the space $H \backslash G$ of right cosets is finite or countable. Choose elements $y_i \in G$, $i \ge 1$, such that the cosets Hy_i, $i \ge 1$, form a partition of G, and set $h(x) = 2^{-i}/(k+1)^n$ for every $i \ge 1$ and $x \in \mathcal{U}^n y_i \smallsetminus \mathcal{U}^{n-1} y_i$. The resulting function $h : G \to \mathbb{R}$ is strictly positive, and $1/(k+1) \le h(x)/h(y) \le k+1$ for all $x, y \in G$ with $x \cdot y^{-1} \in \mathcal{U}$. The continuous map $\eta(x) = \int_{\mathcal{U}} h(yx) \, d\lambda(y) \le \lambda(\mathcal{U})(k+1)h(x)$ is integrable and meets our requirements after multiplication by a suitable constant.

$$(5.7)\qquad \begin{aligned} M' &= M_{\varrho'}^{T'} \cap \Big\{\nu : \int a\,d\nu < \infty\Big\} = M_{\varrho}^{T},\\ E' &= E_{\varrho'}^{T'} \cap \Big\{\nu : \int a\,d\nu < \infty\Big\} = E_{\varrho}^{T}\end{aligned}$$

(cf. Definition 1.2).

Proof. It is clear that $M_\varrho^T \subset M'$. In order to prove that $M' = M_\varrho^T$ we define, for every $\nu \in M'$, a probability measure $\widetilde{\nu}$ by $d\widetilde{\nu}(x) = c_\nu e^{b(x)} d\nu(x)$, where $c_\nu > 0$ is the normalizing constant (cf. (5.2)–(5.3)). Then $\widetilde{\nu}$ again satisfies (5.5) for every $g \in \Delta$ and ν-a.e. $x \in X$, i.e. $\widetilde{\nu} \in M_{\widetilde{\varrho}'}^{T'}$, where $\widetilde{\varrho}'$ is the restriction of $\widetilde{\varrho}$ to $\Delta \times X$. Hence there exists, for every $\varepsilon > 0$, a neighbourhood $\mathcal{U}(\varepsilon)$ satisfying (5.6), and the chain rule for Radon–Nikodym derivatives implies that $\widetilde{\nu}$ is quasi-invariant not only under T', but also under every $T^g, g \in G$, and that $d\widetilde{\nu}T^g/d\widetilde{\nu} = e^{\widetilde{\varrho(g,\cdot)}}$ (mod ν) for every $g \in G$. This implies that $\widetilde{\nu} \in M_{\widetilde{\varrho}}^{T}$ and completes the proof that $M' = M_\varrho^T$.

Every $\nu \in E'$ is obviously ergodic under T and hence lies in E_ϱ^T. Conversely, if $\nu \in E_\varrho^T$, then the continuity of the unitary representation U defined by T on $L^2(X, \mathcal{S}, \nu)$ implies that ν is also ergodic under the dense subgroup $\Delta \subset G$. ∎

THEOREM 5.2. *Let T be a Borel action of a locally compact second countable group G on a standard Borel space $(X, \mathcal{S})$ and $\varrho : G \times X \to \mathbb{R}$ a cocycle of T with $M_\varrho^T \neq \emptyset$.*

(1) *There exists a Borel map $p : x \mapsto p_x$ from X to $E_\varrho^T \subset M_1(X, \mathcal{S})$ with the following properties.*

(a) *$p_x = p_{T^g x}$ for every $x \in X$ and $g \in G$.*

(b) *For every $\nu \in M_\varrho^T$ and every nonnegative Borel map $f : X \to \mathbb{R}$,*

$$\int f\,dp_x = E_\nu(f \mid \mathcal{S}^T)(x)$$

for ν-a.e. $x \in X$, where

$$\mathcal{S}^T = \{B \in \mathcal{S} : T^g B = B \text{ for every } g \in G\}.$$

(2) *If $p' : x \mapsto p'_x$ is another Borel map from X to E_ϱ^T with the properties* (1), *then*

$$\nu(\{x \in X : p_x \neq p'_x\}) = 0 \qquad \text{for every } \nu \in M_\varrho^T.$$

(3) *Let $\mathcal{T} \subset \mathcal{S}^T$ be the smallest σ-algebra such that the map $x \mapsto p_x$ from X to E_ϱ^T in* (1) *is $\mathcal{T}$-measurable. Then $\mathcal{T}$ is countably generated,*

$$\mathcal{T} = \mathcal{S}^T \pmod{\nu} \qquad \text{for every } \nu \in M_\varrho^T,$$

and

$$p_x([y]_{\mathcal{T}}) = \begin{cases} 1 & \text{if } x \in [y]_{\mathcal{T}},\\ 0 & \text{otherwise,}\end{cases}$$

for every $x, y \in X$.

Proof. This is an almost immediate consequence of Remark 1.3, Theorem 1.5 and Proposition 5.1. The only point worth mentioning is that, according to (5.2) and (5.3)–(5.4), $\int a\,d\nu = 1$ for every $\nu \in M_\varrho^T$. In the notation of Theorem 1.5 we conclude that, for every $\nu \in M_\varrho^T \subset M_{\varrho'}^{T'}$, $\int a\,dp_x(y) < \infty$ for ν-a.e. $x \in X$, and hence that $\nu(\{x \in X : p_x \in M'\}) = 1$.

After modifying the measures $\mu_x \in M_{\varrho'}^{T'}$ for every x in a T-invariant Borel set $N \subset X$ with $\nu(N) = 0$ for every $\nu \in M_\varrho^T$ we may assume that $p_x \in M' = M_\varrho^T$ for *every* $x \in X$.

In order to verify condition (1.a) we note that $p_x = p_{T^g x}$ for every $x \in X$ and $g \in \Delta$, which is easily seen to imply that, for every $\nu \in M_\varrho^T$ and $g \in G$, $p_x = p_{T^g x}$ for ν-a.e. $x \in X$. A Fubini-type argument shows that there exists a T-invariant Borel set $N' \subset X$ with $\nu(N') = 0$ for every $\nu \in M_\varrho^T$ and $p_{T^g x} = p_x$ for every $x \in X \setminus N$ and $g \in G$. A final modification of $\{p_x : x \in N\}$ guarantees (1.a). ∎

Proof of Theorem 1.1. This is completely analogous to the proof in the case where G is countable. ∎

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