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ON MEASURE THEORETICAL ANALOGUES OF THE TAKESAKI STRUCTURE THEOREM FOR TYPE III FACTORS

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Dedicated to the memory of Anzelm Iwanik

Abstract. The orbit equivalence of type III_0 ergodic equivalence relations is considered. We show that it is equivalent to the outer conjugacy problem for the natural trace-scaling action of a countable dense $\mathbb{R}$-subgroup by automorphisms of the Radon–Nikodym skew product extensions of these relations. A similar result holds for the weak equivalence of arbitrary type III_0 cocycles with values in Abelian groups.

0. Introduction. In 1973 M. Takesaki made enormous progress in understanding the structure and classification of von Neumann algebras. He demonstrated that every type III factor is isomorphic to the cross product of a type II_∞ von Neumann algebra by an $\mathbb{R}$-action scaling the canonical trace [Ta]. This result proved to be extremely important not only for classification of von Neumann algebras but also for the orbit equivalence of measured nonsingular actions (for a detailed discussion on the interplay between ergodic theory and operator algebras we refer to the surveys [Mo] and [Sc2]). Our objective here is to find a measure theoretical analogue of the Takesaki theorem.

Given a type III_0 ergodic countable transformation group Γ acting on a standard probability space $(X, \mathfrak{B}_X, \mu)$, we consider the skew product extension $\widetilde{\Gamma} = \{\widetilde{\gamma}\}_{\gamma\in\Gamma}$ of Γ acting on the product space $X \times \mathbb{R}$ as

$$\widetilde{\gamma}(x, y) = \left(\gamma x, y + \log \frac{d\mu\circ\gamma}{d\mu}(x)\right).$$

If we furnish $\mathbb{R}$ with the measure λ, $d\lambda(t) = \exp(-t)dt$, then $\widetilde{\Gamma}$ is infinite

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measure preserving. Since the natural $\mathbb{R}$-action $V = \{V(t)\}_{t\in\mathbb{R}}$ on $X \times \mathbb{R}$ by translations along the second coordinate commutes with $\widetilde{\Gamma}$, it induces an $\mathbb{R}$-action on the space of $\widetilde{\Gamma}$-ergodic components. This new action is called the *associated flow* of Γ. Notice that $\widetilde{\Gamma}$ and V are analogues of the type II_∞ von Neumann algebra and the $\mathbb{R}$-action scaling the trace respectively in the Takesaki structure theorem. The associated flow is then the restriction of the flow of weights for the cross product to the center of the type II_∞ subalgebra.

Let R_0 be a countable dense subgroup of $\mathbb{R}$.

THEOREM 0.1. *Let Γ_1, Γ_2 be two transformation groups as above acting on standard probability spaces $(X_1, \mathfrak{B}_{X_1}, \mu_1)$ and $(X_2, \mathfrak{B}_{X_2}, \mu_2)$ respectively. Then the following are equivalent*:

(i) *Γ_1 and Γ_2 are orbit equivalent*,

(ii) *there is a measure space isomorphism $\theta : X_1 \times \mathbb{R} \to X_2 \times \mathbb{R}$ such that $\theta(\widetilde{\Gamma}_1 z) = \widetilde{\Gamma}_2\theta(z)$ and $\theta(V_1(t)z) \in \widetilde{\Gamma}_2 V_2(t)\theta(z)$ for a.e. $z \in X_1 \times \mathbb{R}$ and each $t \in R_0$.*

We remark that the implication (i)⇒(ii) is trivial and the condition (ii) implies

(iii) the associated flows of Γ_1 and Γ_2 are conjugate.

In general, the implication (iii)⇒(i) does not hold. To see this, take for instance Γ_1 amenable and set $\Gamma_2 := \Gamma_1 \times \Gamma_3$, where Γ_3 is a Bernoullian—finite measure preserving—action of a nonamenable group. However, if Γ_1 and Γ_2 are both amenable then by the Krieger theorem [Kr], (iii)⇔(i) and Theorem 0.1 follows. Hence our result can be regarded as a step toward extending Krieger's theorem to the nonamenable case via the ideas embodied in Takesaki's theorem.

We also remark that Theorem 0.1 is only a partial analogue of Takesaki's theorem since—at this stage—we are unable to settle the following conjecture:

Γ is orbit equivalent to the action generated by $\widetilde{\Gamma}$ and $V(R_0)$.

Clearly, it is true in the amenable case by the Krieger theorem.

Theorem 0.1 can be generalized naturally to the general setup of cocycles with values in locally compact Abelian groups. We find it more convenient to use orbital concepts like equivalence relation, orbital cocycle, etc. instead of their dynamical counterparts: action, dynamical cocycle, etc. respectively. Let G be a locally compact second countable Abelian group and $G_0 \subset G$ a countable dense subgroup.

THEOREM 0.2. *Let $\mathcal{R}_i$ be an ergodic discrete equivalence relation of infinite type on a standard probability space $(X_i, \mathfrak{B}_i, \mu_i)$ and $\alpha_i : \mathcal{R}_i \to G$ a* *recurrent cocycle with* $\overline{r}(\alpha_i) = \{0, \infty\}$, $i = 1, 2$. *Denote by* $\mathcal{R}_i(\alpha_i)$ *the* α_i-*skew product extension of* $\mathcal{R}_i$ *and by* $\mathcal{F}_i$ *the equivalence relation on* $X_i \times G$ *generated by* $\mathcal{R}_i(\alpha_i)$ *and the* G_0-*action by translations along the second coordinate. Then the pairs* $(\mathcal{R}_1, \alpha_1)$ *and* $(\mathcal{R}_2, \alpha_2)$ *are weakly equivalent if and only if there exists a measure space isomorphism* $\theta : X_1 \times G \to X_2 \times G$ *such that the following are satisfied*:

(a) $(\theta \times \theta)\mathcal{F}_1 = \mathcal{F}_2 \bmod 0$,
(b) $(\theta \times \theta)(\mathcal{R}_1(\alpha_1)) = \mathcal{R}_2(\alpha_2) \bmod 0$,
(c) *the* $\mathcal{F}_1$-*cocycles* $\alpha_1 \otimes 1$ *and* $(\alpha_2 \otimes 1) \circ \theta$ *are cohomologous.*

We refer the reader to §1 for the definition of the weak equivalence and to §2 (just before the proof of Theorem 2) for the definition of $\alpha_i \otimes 1$. Note that $\mathcal{R}_i(\alpha_i)$ need not be of type II_∞. Hence to make the analogy with Takesaki's theorem more apparent one needs to replace α_i by the double cocycle $\alpha_i \otimes \varrho_{\mu_i}$ with values in $G \times \mathbb{R}$, where ϱ_{μ_i} is the Radon–Nikodym cocycle of $\mathcal{R}_i$. Then $\mathcal{R}_i(\alpha_i \otimes \varrho_{\mu_i})$ is of type II_∞. The hypotheses of Theorem 0.2 hold for the double cocycles. For $\mathcal{R}_i$ hyperfinite, Theorem 0.2 (with the double cocycles) follows from [BG], where an appropriate extension of Krieger's theorem was proved.

Originally the second named author proved Theorem 0.1. After reading his manuscript the first named author wrote a different (shorter) proof. It works in the general case (Theorem 0.2) and appears here.

1. Preliminaries. Let $\mathcal{R}$ be a discrete (countable) Borel equivalence relation on a standard measure space $(X, \mathfrak{B}, \mu)$ (see [FM]). $\mathcal{R}$ is called *nonsingular* if the $\mathcal{R}$-saturation of every μ-null subset is also μ-null. It is *ergodic* if every $\mathcal{R}$-saturated subset is either μ-null or μ-conull. It is well known that every μ-nonsingular (ergodic) countable equivalence relation is generated by a nonsingular (ergodic) action of a countable transformation group [FM]. $\mathcal{R}$ is *hyperfinite* if it is generated by a single transformation (i.e. by a $\mathbb{Z}$-action). We denote the trivial—diagonal—equivalence relation by $\mathcal{D}$.

A Borel map $\alpha : \mathcal{R} \to G$ is a *cocycle* of $\mathcal{R}$ if there is a μ-conull subset $A \subset X$ with

$$\alpha(x, y) + \alpha(y, z) = \alpha(x, z)$$

for all $x, y, z \in A$ with $z \sim_{\mathcal{R}} y \sim_{\mathcal{R}} z$. Let $\operatorname{Ker} \alpha := \{(x, y) \in \mathcal{R} \mid \alpha(x, y) = 0\}$. It is easy to see that $\operatorname{Ker} \alpha$ is a subrelation of $\mathcal{R}$. Let Γ be a countable transformation group generating $\mathcal{R}$. The *Radon–Nikodym cocycle* $\varrho_\mu : \mathcal{R} \to \mathbb{R}$ of $\mathcal{R}$ is given by

$$\varrho_\mu(x, \gamma x) := \log \frac{d\mu \circ \gamma}{d\mu}(x), \quad x \in X, \ \gamma \in \Gamma.$$

Two cocycles $\alpha, \beta : \mathcal{R} \to G$ are *cohomologous* if there are a Borel map $\phi : X \to G$ and a μ-conull subset A with

$$\alpha(x, y) = -\phi(x) + \beta(x, y) + \phi(y)$$

for all $(x, y) \in \mathcal{R} \cap (A \times A)$. We denote this by $\alpha \approx_\phi \beta$.

An element g of the one-point compactification G^* of G is an *essential value* of α if for every neighborhood U of g in G^* and every pair of subsets $A, B \subset X$ of positive measure, there exists a subset $A' \subset A$ of positive measure and a measurable map $\gamma : A' \to B$ such that $(x, \gamma x) \in \mathcal{R}$ and $\alpha(x, \gamma x) \in U$ for all $x \in A'$. The set of essential values of α is denoted by $\overline{r}(\alpha)$. If $\overline{r}(\varrho_\mu) = \{0, \infty\}$ then $\mathcal{R}$ is of *type* III_0.

We associate with α a nonsingular equivalence relation $\mathcal{R}(\alpha)$ on the product measure space $(X \times G, \mu \times \lambda_G)$ by setting

$$(x, g) \sim_{\mathcal{R}(\alpha)} (x', g') \Leftrightarrow (x, x') \in \mathcal{R} \text{ and } g' = g + \alpha(x, x').$$

Here λ_G stands for a Haar measure on G. Then α is called *transient* if $\mathcal{R}(\alpha)$ is of type I, i.e. the corresponding $\mathcal{R}(\alpha)$-orbit partition of $X \times G$ is measurable. Otherwise α is *recurrent.* If U is a neighborhood of 0 in G such that $\alpha(\mathcal{R} \cap (A \times A)) \cap U = \{0\}$ for a μ-conull subset A then α is called *U-lacunary.*

Let $\mathcal{R}_i$ be a nonsingular equivalence relation on a standard probability space $(X_i, \mathfrak{B}_{X_i}, \mu_i)$ and $\alpha_i : \mathcal{R}_i \to G$ a cocycle, $i = 1, 2$. The pairs $(\mathcal{R}_1, \alpha_1)$ and $(\mathcal{R}_2, \alpha_2)$ are *weakly equivalent* if there is a measure space isomorphism $\theta : X_1 \to X_2$ such that $(\theta \times \theta)\mathcal{R}_1 = \mathcal{R}_2 \bmod 0$ and α_1 is cohomologous to the cocycle $\alpha_2 \circ \theta$ given by

$$\alpha_2 \circ \theta(x, y) = \alpha_2(\theta x, \theta y), \qquad (x, y) \in \mathcal{R}.$$

Let $A_i \subset X_i$ be a subset of positive measure. Denote by $(\mathcal{R}_i)_{A_i} := \mathcal{R}_i \cap (A_i \times A_i)$ the induced equivalence relation on $(A_i, \mathfrak{B}_{X_i} \restriction A_i, \mu \restriction A_i)$. It is well known (see [Sc1], [BG]) that if $\mathcal{R}_1$ and $\mathcal{R}_2$ are of infinite type (i.e. not of type II_1) and the induced pairs $((\mathcal{R}_1)_{A_1}, \alpha_1)$ and $((\mathcal{R}_2)_{A_2}, \alpha_2)$ are weakly equivalent then $(\mathcal{R}_1, \alpha_1)$ and $(\mathcal{R}_2, \alpha_2)$ are also weakly equivalent.

2. Proofs of the main results.

Here we prove Theorem 0.2 and deduce Theorem 0.1 from it.

PROPOSITION 2.1 (cf. [GS, Proposition 1.2] and [Da2, Proposition 1.6]). *Let $\mathcal{R}$ be a hyperfinite ergodic equivalence relation on $(X, \mathfrak{B}, \mu)$ and $\alpha : \mathcal{R} \to G$ a cocycle. Then for a neighborhood U of 0 in G, there exists a cocycle β of $\mathcal{R}$ such that $\beta \approx \alpha$, $\beta(\mathcal{R}) \subset G_0$ and $\beta(x, y) - \alpha(x, y) \in U$ for all $(x, y) \in \mathcal{R}$.*

Proof. Since $\mathcal{R}$ is hyperfinite, we may assume without loss of generality that $X = (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$, $\mathfrak{B}_X$ is the product Borel structure on X and $\mathcal{R}$ is the tail equivalence relation on X, i.e. two points $x = (x_n), y = (y_n)$ of X are $\mathcal{R}$-equivalent if $x_n = y_n$ eventually. Let π be the left shift map on X:

$$\pi(x_1, x_2, \ldots) = (x_2, x_3, \ldots).$$

By [Go, §1, Theorem 2] and [Sc1, Theorem 9.1], there is a sequence of Borel maps $a_n : X \to G$ such that

$$\alpha(x, y) = \sum_{n=1}^{\infty}(a_n(\pi^n x) - a_n(\pi^n y)).$$

Take a sequence $(U_n)_{n\geq 0}$ of symmetric neighborhoods of 0 in G such that $U_n + U_n \subset U_{n-1}$, $U_0 + U_0 \subset U$ and $\bigcap_n U_n = \{0\}$. There are Borel maps $b_n : X \to G_0$ such that $b_n(x) - a_n(x) \in U_n$ for all $x \in X$. We define a cocycle $\beta : \mathcal{R} \to G$ by setting

$$\beta(x, y) := \sum_{n=1}^{\infty}(b_n(\pi^n x) - b_n(\pi^n y)).$$

Clearly, $\beta(\mathcal{R}) \subset G_0$. Next, for each $N \in \mathbb{N}$ we have

$$\sum_{n=1}^{N}(b_n(\pi^n x) - a_n(\pi^n x) - b_n(\pi^n y) + a_n(\pi^n y)) \in \sum_{n=1}^{N}(U_n - U_n) \subset \sum_{n=1}^{N} U_{n-1} \subset U_0 + U_0 \subset U.$$

Hence $\alpha(x, y) - \beta(x, y) \in U$ for all $(x, y) \in \mathcal{R}$. We define a Borel map $\phi : X \to G$ by setting $\phi(x) := \sum_{n=1}^{\infty}(b_n(\pi^n x) - a_n(\pi^n x))$. It is easy to see that ϕ is well defined. Clearly, $\alpha \approx_\phi \beta$. ■

Corollary 2.2. *If α is lacunary then there exists a lacunary cocycle β such that $\beta \approx \alpha$ and $\beta(\mathcal{R}) \subset G_0$.*

Denote by $\mathcal{T}$ the orbit equivalence relation for the natural G_0-action on (G, λ_G) by translations. Clearly, $\mathcal{T}$ is ergodic and hyperfinite. Recall the notation from the statement of Theorem 0.2. The equivalence relation on $(X_i \times G, \mu_i \times \lambda_G)$ generated by $\mathcal{R}_i(\alpha_i)$ and $\mathcal{D} \otimes \mathcal{T}$ is denoted by $\mathcal{F}_i$. Clearly, it is ergodic. The cocycle $\alpha_i \otimes 1$ of $\mathcal{F}_i$ is defined by

$$\alpha_i \otimes 1((x, g), (x', g')) := \alpha_i(x, x').$$

Proof of Theorem 0.2. The "only if" part is obvious. In what follows we prove the converse.

Since $\overline{r}(\alpha_i) = \{0, \infty\}$, it follows from [Sc, Theorem 7.22] that there are:

- a measure space projection π_i of $(X_i, \mathfrak{B}_{X_i}, \mu_i)$ onto a standard probability space $(Y_i, \mathfrak{B}_{Y_i}, \nu_i)$,

- an (ergodic) ν_i-nonsingular equivalence relation $\mathcal{Q}_i$ on Y_i such that $(\pi_i \times \pi_i)\mathcal{R}_i = \mathcal{Q}_i \bmod 0$,
- a transient lacunary cocycle $\beta_i : \mathcal{Q}_i \to G$ such that the cocycle $\widetilde{\alpha}_i := \beta_i \circ \pi_i$ is cohomologous to α_i.

Since β_i is transient, $\operatorname{Ker}\beta_i$ is a type I subrelation of $\mathcal{Q}_i$. Passing, if necessary, to the $(\operatorname{Ker}\beta_i)$-ergodic decomposition we may assume without loss of generality that $\operatorname{Ker}\beta_i$ is trivial. Since β_i is transient and G is Abelian, $\mathcal{Q}$ is hyperfinite by [FHM, Corollary 7.11]. Hence we may apply Corollary 2.2 to the pair $(\mathcal{Q}, \beta)$.

Let $\mathcal{S}_i := \operatorname{Ker}\widetilde{\alpha}_i$. Since $\widetilde{\alpha}_i$ is cohomologous to α_i, it is also recurrent. Hence $\mathcal{S}_i$ is conservative, i.e. the corresponding conditional measures on the $\mathcal{S}_i$-ergodic components are nonatomic. Therefore these components are isomorphic as measure spaces to a standard probability space $(Z_i, \mathfrak{B}_{Z_i}, \kappa_i)$ with κ_i nonatomic.

Summarizing the above we may assume that the following are satisfied:

(1) $(X_i, \mu_i) = (Z_i \times Y_i, \kappa_i \times \nu_i)$.
(2) $\pi_i(z_i, y_i) = y_i$ for all $(z_i, y_i) \in X_i$.
(3) π_i is the $\mathcal{S}_i$-ergodic decomposition of X_i.
(4) There is a neighborhood U of 0 in G such that β_i is U-lacunary.
(5) $\beta_i(\mathcal{Q}) \subset G_0$.

Next we notice that the properties (a)–(c) hold if we replace α_i by $\widetilde{\alpha}_i$, $i = 1, 2$. Hence by (c) there is a measurable map $\phi : X_1 \times G \to G$ such that

(2-1) $$(\widetilde{\alpha}_2 \otimes 1) \circ \theta \approx_\phi \widetilde{\alpha}_1 \otimes 1.$$

Let V be a neighborhood of 0 in G with $V - V \subset U$. Take a Borel subset $A_1 \subset X_1 \times G$ of positive measure and an element $g \in G$ with $\phi(A_1) \subset g + V$. Then it follows from (2-1) that

(2-2) $$(\widetilde{\alpha}_2 \otimes 1) \circ \theta - \widetilde{\alpha}_1 \otimes 1 \in U \quad \text{everywhere on } (\mathcal{F}_1)_{A_1}.$$

Since $\operatorname{Ker}(\widetilde{\alpha}_i \otimes 1) = \mathcal{S}_i \otimes \mathcal{T}$ by (4), we deduce from (a) and (2-2) that

(2-3) $$(\theta \times \theta)((\mathcal{S}_1 \otimes \mathcal{T})_{A_1}) = (\mathcal{S}_2 \otimes \mathcal{T})_{A_2},$$

where $A_2 := \theta A_1$. The property (b) with $\widetilde{\alpha}_i$ instead of α_i yields

(2-4) $$(\theta \times \theta)((\mathcal{R}_1(\widetilde{\alpha}_1))_{A_1}) = (\mathcal{R}_2(\widetilde{\alpha}_2))_{A_2}.$$

Since $(\mathcal{S}_i \otimes \mathcal{T}_i) \cap \mathcal{R}_i(\widetilde{\alpha}_i) = \mathcal{S}_i \otimes \mathcal{D}$, we deduce from (2-3) and (2-4) that

$$(\theta \times \theta)((\mathcal{S}_1 \otimes \mathcal{D})_{A_1}) = (\mathcal{S}_2 \otimes \mathcal{D})_{A_2}.$$

Hence θ intertwines $(\mathcal{S}_1 \otimes \mathcal{D})_{A_1}$-ergodic components with $(\mathcal{S}_2 \otimes \mathcal{D})_{A_2}$-ergodic components. We set $B_i := \{(y_i, g) \in Y_i \times G \mid \kappa(\{z_i \mid (z_i, y_i, g) \in A_i\}) > 0\}$, $i = 1, 2$. Then there is a measure space isomorphism $\vartheta : B_1 \to B_2$ such that the diagram

(2-5)
$$\begin{array}{ccc} A_1 & \xrightarrow{\theta} & A_2 \\ {\scriptstyle \pi_{Y_1\times G}}\downarrow & & \downarrow{\scriptstyle \pi_{Y_2\times G}} \\ B_1 & \xrightarrow[\vartheta]{} & B_2 \end{array}$$

commutes, where $\pi_{Y_i\times G}$ is the natural projection onto $Y_i \times G$, $i = 1, 2$. Of course,

$$(\pi_{Y_i\times G} \times \pi_{Y_i\times G})(\mathcal{S}_i \otimes \mathcal{T}) = \mathcal{D} \otimes \mathcal{T}.$$

Then it follows from (2-3) and (2-5) that

$$(\vartheta \times \vartheta)((\mathcal{D} \otimes \mathcal{T})_{B_1}) = (\mathcal{D} \otimes \mathcal{T})_{B_2}.$$

Hence ϑ intertwines $(\mathcal{D}\otimes\mathcal{T})_{B_1}$-ergodic components with $(\mathcal{D}\otimes\mathcal{T})_{B_2}$-ergodic components. We set $C_i := \{g \in Y_i \mid \lambda_G(\{g \in G \mid (y, g) \in A_i\}) > 0\}$. It is easy to see that the natural projection $\pi_{Y_i} : B_i \ni (y, g) \mapsto y \in C_i$ is just the $(\mathcal{D} \otimes \mathcal{T})_{B_i}$-ergodic decomposition. Hence there exists a measure space isomorphism $\psi : C_1 \to C_2$ such that the diagram

(2-6)
$$\begin{array}{ccc} B_1 & \xrightarrow{\vartheta} & B_2 \\ {\scriptstyle \pi_{Y_1}}\downarrow & & \downarrow{\scriptstyle \pi_{Y_2}} \\ C_1 & \xrightarrow[\psi]{} & C_2 \end{array}$$

commutes. We deduce from (2-5) and (2-6) that θ has the following form:

(2-7) $$\theta(z, y, t) = (\zeta_{y,t}(z), \psi(y), \tau_y(t)) \quad \text{for a.e. } (z, y, t) \in A_1,$$

where $(\zeta_{y,t})_{(y,t)\in B_1}$ and $(\tau_y)_{y\in C_1}$ are measurable fields of measure space isomorphisms between the corresponding fibers of A_1 and A_2. Without loss of generality we may assume that (2-7) holds everywhere on A_1. Take $g_0 \in G$ in such a way that the subset $E_1 := \{(z_1, y_1) \in X_1 \mid (z_1, y_1, g_0) \in A_1\}$ is of positive measure and define a map $\eta : E_1 \to \eta(E_1) \subset X_2$ by setting

$$\eta(z_1, y_1) := (\zeta_{y_1,g_0}(z_1), \psi(y_1)).$$

Clearly, $\eta : E_1 \to E_2$ is a nonsingular isomorphism, where $E_2 := \eta(E_1)$. We claim that $(\eta \times \eta)((\mathcal{R}_1)_{E_1}) = (\mathcal{R}_2)_{E_2}$. Actually, for (z, y) and (z', y') from E_1, we have

$$\begin{aligned} (z, y) \sim_{\mathcal{R}_1} (z', y') &\Leftrightarrow (z, y, g_0) \sim_{\mathcal{R}_1(\tilde{\alpha}_1)} (z', y', g_0 + \beta(y, y')) \\ &\overset{(5)}{\Leftrightarrow} (z, y, g_0) \sim_{\mathcal{F}_1} (z', y', g_0) \\ &\Leftrightarrow \theta(z, y, g_0) \sim_{\mathcal{F}_2} \theta(z', y', g_0) \\ &\Leftrightarrow \eta(z, y) \sim_{\mathcal{R}_2} \eta(z', y') \end{aligned}$$

as desired. Moreover,

$$
\begin{aligned}
&\widetilde{\alpha}_2 \circ \eta((z,y),(z',y')) \\
&\qquad = (\widetilde{\alpha}_2 \otimes 1) \circ \theta((z,y,t_0),(z',y',t_0)) \\
&\qquad \overset{(2\text{-}1)}{=} -\phi(z,y,t_0) + (\widetilde{\alpha}_1 \otimes 1)((z,y,t_0),(z',y',t_0)) + \phi(z',y',t_0) \\
&\qquad = -\phi_0(z,y) + \widetilde{\alpha}_1((z,y),(z',y')) + \phi_0(z',y'),
\end{aligned}
$$

where $\phi_0(z,y) := \phi(z,y,t_0)$. Thus the pairs $((\mathcal{R}_1)_{E_1}, \widetilde{\alpha}_1)$ and $((\mathcal{R}_2)_{E_2}, \widetilde{\alpha}_2)$ are weakly equivalent. Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are both of infinite type, the pairs $(\mathcal{R}_1, \widetilde{\alpha}_1)$ and $(\mathcal{R}_2, \widetilde{\alpha}_2)$ are also weakly equivalent. ■

Proof of Theorem 0.1. Let $\mathcal{R}_i$ be the orbit equivalence for Γ_i, $G := \mathbb{R}$, $G_0 := R_0$ and $\alpha_i := \varrho_{\mu_i}$. It is obvious that then Γ_1 and Γ_2 are orbit equivalent iff the pairs $(\mathcal{R}_1, \alpha_1)$ and $(\mathcal{R}_2, \alpha_2)$ are weakly equivalent. Since $\mathcal{R}_i$ is of type III_0, we have $\overline{r}(\alpha_i) = \{0, \infty\}$. It is well known that the Radon–Nikodym cocycle of an ergodic equivalence relation is never transient [Sc2]. Moreover, $\varrho_{\mu_i} \otimes 1$ is just the Radon–Nikodym cocycle for $\mathcal{F}_i$—recall that we furnish $X \times G$ with the product measure $\mu \times \lambda_{\mathbb{R}}$. Hence (c) follows immediately from (a). Notice that the condition (ii) of Theorem 0.1 is equivalent to (a) plus (b). Therefore Theorem 0.1 follows from Theorem 0.2. ■

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